Steady State Solution of the Kuramoto-Sivashinsky PDE

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The Kuramoto-Sivashinsky equation is a simple one-dimensional partial differential equation (PDE) that exhibits chaos under some conditions. In its simplest form, the equation is given by

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0$$

where the subscripts denote differentiation of the state variable u with respect to time and space, respectively. Here we seek steady state standing wave solutions ($u_t = 0$) to the equation in an infinite spatial domain using Fourier analysis. The stability of such solutions is a separate matter to be examined later.

The simplest model consists of a single sine wave:

$$u = a \sin kx$$

Derivatives:

$$u_x = ak \cos kx$$

$$u_{xx} = -ak^2 \sin kx$$

$$u_{xxx} = -ak^3 \cos kx$$

$$u_{xxxx} = ak^4 \sin kx$$

Nonlinear term:

$$uu_x = a^2 k \sin kx \cos kx$$

Simplify using the following trigonometric identity:

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

To obtain the following:

$$uu_x = \frac{1}{2} a^2 k \sin 2kx$$

Steady state of Kuramoto-Sivashinski equation:

$$uu_x + u_{xx} + u_{xxxx} = 0$$

Equating term-by-term:

$$-ak^2 + ak^4 = 0$$

$$\frac{1}{2} a^2 k = 0$$

Clearly there is no solution except for $a = 0$ or $k = 0$. However, the first equation has a second solution given by $k = 1$, which is not very different from the value observed numerically at $k = 26\pi/100 = 0.816814$.

A slightly more realistic model, motivated by a numerical solution of the KS equation is:
$u = a\sin kx + b\sin 2kx$

Derivatives:
$u_x = ak\cos kx + 2bk\cos 2kx$

$u_{xx} = -ak^2\sin kx - 4bk^2\sin 2kx$

$u_{xxx} = -ak^3\cos kx - 8bk^3\cos 2kx$

$u_{xxxx} = ak^4\sin kx + 16bk^4\sin 2kx$

Nonlinear term:
$uu_x = a^2k\sin kx\cos kx + 2abk\sin kx\cos 2kx + abk\sin 2kx\cos kx + 2b^2k\sin 2kx\cos 2kx$

Simplify using the following trigonometric identity:
$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

To obtain the following:
$$uu_x = \frac{1}{2}a^2k\sin 2kx + abk[\sin 3kx - \sin kx] + \frac{1}{2}abk[\sin 3kx + \sin kx] + b^2k\sin 4kx$$

Steady state of Kuramoto-Sivashinski equation:
$uu_x + u_{xx} + u_{xxxx} = 0$

Equating term-by-term:
$$-abk + \frac{1}{2}abk - ak^2 + ak^4 = 0$$
$$\frac{1}{2}a^2k - 4bk^2 + 16bk^4 = 0$$
$$abk + \frac{1}{2}abk = 0$$
$$b^2k = 0$$
This system is over-specified since there are four equations for three unknowns. However, the last two equations are only approximations since they are inconsistent with the assumption that only terms in sin *kx* and sin 2*kx* are present.

The first two equations are exact and can be simplified (for *k*, *a*, and *b* nonzero) to:
$$2k(k^2 - 1) = b$$
$$8bk(1 - 4k^2) = a^2$$

From the numerical solution of the KS equation, we have $k = 26\pi/100 = 0.816814$, from which we can determine *a* and *b*:

$b = 2k(k^2 - 1) = -0.5436957$

$a = \sqrt{8bk(1-4k^2)} = 2.4348874$

These values are in reasonable agreement with numerical results.

Hence:

$u = 2.4348874 \sin 0.816814x - 0.5436957 \sin 0.816814x$

Assume instead a more general model:

$$u = a + b\sin kx + c\sin 2kx + d\cos 2kx$$

(There is no loss of generality in ignoring the cos *kx* term.)

Derivatives:

$$u_x = bk\cos kx + 2ck\cos 2kx - 2dk\sin 2kx$$

$$u_{xx} = -bk^2\sin kx - 4ck^2\sin 2kx - 4dk^2\cos 2kx$$

$$u_{xxx} = -bk^3\cos kx - 8ck^3\cos 2kx + 8dk^3\sin 2kx$$

$$u_{xxxx} = bk^4\sin kx + 16ck^4\sin 2kx + 16dk^4\cos 2kx$$

Nonlinear term:

$$uu_x = abk\cos kx + 2ack\cos 2kx - 2adk\sin 2kx$$

$$+ b^2k\sin kx\cos kx + 2bck\sin kx\cos 2kx - 2bdk\sin kx\sin 2kx$$

$$+ bck\sin 2kx\cos kx + 2c^2k\sin 2kx\cos 2kx - 2cdk\sin^2 2kx$$

$$+ bdk\cos kx\cos 2kx + 2cdk\cos^2 2kx - 2d^2k\sin 2kx\cos 2kx$$

Simplify using the following trigonometric identities:

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

To obtain the following:

$$uu_x = abk\cos kx + 2ack\cos 2kx - 2adk\sin 2kx$$

$$+\frac{1}{2}b^2k\sin 2kx + bck[\sin 3kx - \sin kx] - bdk[\cos kx - \cos 3kx]$$

$$+\frac{1}{2}bck[\sin 3kx + \sin kx] + c^2k\sin 4kx - cdk[1 - \cos 4kx]$$

$$+\frac{1}{2}bdk[\cos 3kx + \cos kx] + cdk[1 + \cos 4kx] - d^2k\sin 4kx$$

Steady state of Kuramoto-Sivashinski equation:

$$uu_x + u_{xx} + u_{xxxx} = 0$$

Equating term-by-term:

$$-\frac{1}{2}bck - bk^2 + bk^4 = 0$$

$$abk - \frac{1}{2}bdk = 0$$

$$-2adk + \frac{1}{2}b^2k - 4ck^2 + 16ck^4 = 0$$

$$2ack - 4dk^2 + 16dk^4 = 0$$

$$c^2k - d^2k = 0$$

Simplify:

$$c = 2k(k^2 - 1)$$

$$a = d/2$$

$$b^2 - 4ad = 8ck(1 - 4k^2)$$

$$ac = 2dk(1 - 4k^2)$$

$$c = \pm d$$

Eliminate a (= d/2):

$$c = 2k(k^2 - 1)$$

$$b^2 - 2d^2 = 8ck(1 - 4k^2)$$

$$c = 4k(1 - 4k^2)$$

$$c = \pm d$$

Eliminate c (= $2k(k^2 - 1)$):

$$b^2 - 2d^2 = 16k^2(k^2 - 1)(1 - 4k^2)$$

$$k^2 - 1 = 2(1 - 4k^2)$$

$$d = \pm 2k(k^2 - 1)$$

Solve the second equation above for k:

$$k^2 = 1/3$$

$$k = \pm 0.5773502$$

From k, calculate c, d, a, and b:

$$c = 2k(k^2 - 1) = \pm 0.7698004$$

$$d = \pm c = \pm 0.7698004$$

$$a = d/2 = \pm 0.3849002$$

$$b = \pm\sqrt{16k^2(k^2 - 1)(1 - 4k^2) + 2d^2} = \pm 1.5396007$$

Hence:

$$u = 0.3849002 + 1.5396007\sin 0.5773502x - 0.7698004\sin 1.15471x + 0.7698004\cos 1.15471x$$

Start over but with two extra terms (third harmonic):
$u = a + b\sin kx + c\sin 2kx + d\cos 2kx + e\sin 3kx + f\cos 3kx$

Derivatives:
$u_x = bk\cos kx + 2ck\cos 2kx - 2dk\sin 2kx + 3ek\cos 3kx - 3fk\sin 3kx$
$u_{xx} = -bk^2\sin kx - 4ck^2\sin 2kx - 4dk^2\cos 2kx - 9ek^2\sin 3kx - 9fk^2\cos 3kx$
$u_{xxx} = -bk^3\cos kx - 8ck^3\cos 2kx + 8dk^3\sin 2kx - 27ek^3\cos 3kx + 27fk^3\sin 3kx$
$u_{xxxx} = bk^4\sin kx + 16ck^4\sin 2kx + 16dk^4\cos 2kx + 81ek^4\sin 3kx + 81fk^4\cos 3kx$

Nonlinear term:
$uu_x = abk\cos kx + 2ack\cos 2kx - 2adk\sin 2kx + 3aek\cos 3kx - 3afk\sin 3kx$
$+ b^2k\sin kx\cos kx + 2bck\sin kx\cos 2kx - 2bdk\sin 2kx\sin kx + 3bek\sin kx\cos 3kx - 3bfk\sin 3kx\sin kx$
$+ bck\sin 2kx\cos kx + 2c^2k\sin 2kx\cos 2kx - 2cdk\sin^2 2kx + 3cek\sin 2kx\cos 3kx - 3cfk\sin 3kx\sin 2kx$
$+ bdk\cos 2kx\cos kx + 2cdk\cos^2 2kx - 2d^2k\sin 2kx\cos 2kx + 3dek\cos 3kx\cos 2kx - 3dfk\sin 3kx\cos 2kx$
$+ bek\sin 3kx\cos kx + 2cek\sin 3kx\cos 2kx - 2dek\sin 3kx\sin 2kx + 3e^2k\sin 3kx\cos 3kx - 3efk\sin^2 3kx$
$+ bfk\cos 3kx\cos kx + 2cfk\cos 3kx\cos 2kx - 2dfk\sin 2kx\cos 3kx + 3efk\cos^2 3kx - 3f^2k\sin 3kx\cos 3kx$

Simplify using the following trigonometric identities:
$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$
$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

To obtain the following:
$uu_x = abk\cos kx + 2ack\cos 2kx - 2adk\sin 2kx + 3aek\cos 3kx - 3afk\sin 3kx$
$+\frac{1}{2}b^2k\sin 2kx + bck[\sin 3kx - \sin kx] - bdk[\cos kx - \cos 3kx] + \frac{3}{2}bek[\sin 4kx - \sin 2kx] - \frac{3}{2}bfk[\cos 2kx - \cos 4kx]$
$+\frac{1}{2}bck[\sin 3kx + \sin kx] + c^2k\sin 4kx - cdk[1 - \cos 4kx] + \frac{3}{2}cek[\sin 5kx - \sin kx] - \frac{3}{2}cfk[\cos kx - \cos 5kx]$
$+\frac{1}{2}bdk[\cos 3kx + \cos kx] + cdk[\cos 4kx + 1] - d^2k\sin 4kx + \frac{3}{2}dek[\cos 5kx + \cos kx] - \frac{3}{2}dfk[\sin 5kx + \sin kx]$
$+\frac{1}{2}bek[\sin 4kx + \sin 2kx] + cek[\sin 5kx + \sin kx] - dek[\cos kx - \cos 5kx] + \frac{3}{2}e^2k\sin 6kx - \frac{3}{2}efk[1 - \cos 6kx]$
$+\frac{1}{2}bfk[\cos 4kx + \cos 2kx] + cfk[\cos 5kx + \cos kx] - dfk[\sin 5kx - \sin kx] + \frac{3}{2}efk[\cos 6kx + 1] - \frac{3}{2}f^2k\sin 6kx$

Steady state of Kuramoto-Sivashinski equation:
$uu_x + u_{xx} + u_{xxxx} = 0$

Equating term-by-term:

$$\frac{1}{2}bck - \frac{1}{2}cek - \frac{1}{2}dfk - bk^2 + bk^4 = 0$$

$$-2adk + \frac{1}{2}b^2k - bek - 4ck^2 + 16ck^4 = 0$$

$$2ack - bfk - 4dk^2 + 16dk^4 = 0$$

$$-3afk + \frac{3}{2}bck - 9ek^2 + 81ek^4 = 0$$

$$3aek + \frac{3}{2}bdk - 9fk^2 + 81fk^4 = 0$$

$$2bek + c^2k^2 - d^2k^2 = 0$$

$$2bfk + 2cdk = 0$$

$$\frac{5}{2}cek - \frac{5}{2}dfk = 0$$

$$\frac{5}{2}cfk + \frac{5}{2}dek$$

The sin 6kx and cos 6kx terms are ignored since they would require $e = f = 0$.

Simplify:

$$2[2bk(k^2 - 1) + bc - ce - df] = 0$$

$$2[8ck(4k^2 - 1) - 4ad + b^2 - 2be] = 0$$

$$4dk(4k^2 - 1) + 2ac - bf = 0$$

$$3[6ek(9k^2 - 1) - 2af + bc]/2 = 0$$

$$3[6fk(9k^2 - 1) + 2ae + bd]/2 = 0$$

$$(c^2 - d^2)k + 2be = 0$$

$$2[bf + cd] = 0$$

$$5[ce - df]/2 = 0$$

$$5[cf + de]/2 = 0$$

This system is over-determined since there are nine equations for seven unknowns.

Using only the first seven equations (ignoring the sin 5kx and cos 5kx terms) gives the following exact numerical result (may not be unique):

$k = 0.4463$

$a = 0$

$b = 0.5172$

$c = 0.5646$

$d = 0$

$e = -0.3176$

$f = 0$

Hence:
$u = 0.5172\sin 0.4463x + 0.5646\sin 0.8926x - 0.3176\sin 1.3389x$

However, we can perform a numerical least-squares fit to the entire system of nine equations with the following result (mean square error ~ 7×10^{-5}):

$k = 1.0267$

$a = 0.4469$

$b = 1.7209$

$c = -0.1111$

$d = 0.0075$

$e = 0.0036$

$f = -0.0003$

Hence:

$$u = 0.4469 + 1.7209\sin 1.0267x - 0.1111\sin 2.0534x + 0.0075\cos 2.0534x$$
$$+ 0.0036\sin 3.0801x - 0.0003\cos 3.0801x$$

The previous results suggest that we can ignore the cosine terms:
$u = a\sin kx + b\sin 2kx + c\sin 3kx$

Derivatives:
$u_x = ak\cos kx + 2bk\cos 2kx + 3ck\cos 3kx$
$u_{xx} = -ak^2\sin kx - 4bk^2\sin 2kx - 9ck^2\sin 3kx$
$u_{xxx} = -ak^3\cos kx - 8bk^3\cos 2kx - 27ck^3\cos 3kx$
$u_{xxxx} = ak^4\sin kx + 16bk^4\sin 2kx + 81ck^4\sin 3kx$

Nonlinear term:

$$uu_x = a^2k\sin kx\cos kx + 2abk\sin kx\cos 2kx + 3ack\sin kx\cos 3kx$$
$$+ abk\sin 2kx\cos kx + 2b^2k\sin 2kx\cos 2kx + 3bck\sin 2kx\cos 3kx$$
$$+ ack\sin 3kx\cos kx + 2bck\sin 3kx\cos 2kx + 3c^2k\sin 3kx\cos 3kx$$

Simplify using the following trigonometric identity:

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

To obtain the following:

$$uu_x = \frac{1}{2}a^2k\sin 2kx + abk[\sin 3kx - \sin kx] + \frac{3}{2}ack[\sin 4kx - \sin 2kx]$$
$$+ \frac{1}{2}abk[\sin 3kx + \sin kx] + b^2k\sin 4kx + \frac{3}{2}bck[\sin 5kx - \sin kx]$$
$$+ \frac{1}{2}ack[\sin 4kx + \sin 2kx] + bck[\sin 5kx + \sin kx] + \frac{3}{2}c^2k\sin 6kx$$

Steady state of Kuramoto-Sivashinski equation:

$$uu_x + u_{xx} + u_{xxxx} = 0$$

Equating term-by-term:

$$-abk+\frac{1}{2}abk-\frac{3}{2}bck+bck-ak^{2}+ak^{4}=0$$

$$\frac{1}{2}a^{2}k-\frac{3}{2}ack+\frac{1}{2}ack-4bk^{2}+16bk^{4}=0$$

$$abk+\frac{1}{2}abk-9ck^{2}+81ck^{4}=0$$

$$\frac{3}{2}ack+b^{2}k+\frac{1}{2}ack=0$$

$$\frac{3}{2}bck+bck=0$$

$$\frac{3}{2}c^{2}k=0$$

Simplify:

$$-\frac{1}{2}ab-\frac{1}{2}bc-ak+ak^{3}=0$$

$$\frac{1}{2}a^{2}-ac-4bk+16bk^{3}=0$$

$$\frac{3}{2}ab-9ck+81ck^{3}=0$$

$$2ac+b^{2}=0$$

$$\frac{5}{2}bc=0$$

$$\frac{3}{2}bc=0$$

The only solutions have $a = b = c = 0$ and either $k = 0$ or $k = \pm 1$.

Let's add one more term (sin 4*kx*):

$$u = a\sin kx + b\sin 2kx + c\sin 3kx + d\sin 4kx$$

Derivatives:

$$u_x = ak\cos kx + 2bk\cos 2kx + 3ck\cos 3kx + 4dk\cos 4kx$$

$$u_{xx} = -ak^2\sin kx - 4bk^2\sin 2kx - 9ck^2\sin 3kx - 16dk^2\sin 4kx$$

$$u_{xxx} = -ak^3\cos kx - 8bk^3\cos 2kx - 27ck^3\cos 3kx - 64dk^3\cos 4kx$$

$$u_{xxxx} = ak^4\sin kx + 16bk^4\sin 2kx + 81ck^4\sin 3kx + 256dk^4\sin 4kx$$

Nonlinear term:

$$\begin{aligned} uu_x &= a^2k\sin kx\cos kx + 2abk\sin kx\cos 2kx + 3ack\sin kx\cos 3kx + 4adk\sin kx\cos 4kx \\ &+ abk\sin 2kx\cos kx + 2b^2k\sin 2kx\cos 2kx + 3bck\sin 2kx\cos 3kx + 4bdk\sin 2kx\cos 4kx \\ &+ ack\sin 3kx\cos kx + 2bck\sin 3kx\cos 2kx + 3c^2k\sin 3kx\cos 3kx + 4cdk\sin 3kx\cos 4kx \\ &+ adk\sin 4kx\cos kx + 2bdk\sin 4kx\cos 2kx + 3cdk\sin 4kx\cos 3kx + 4d^2k\sin 4kx\cos 4kx \end{aligned}$$

Simplify using the following trigonometric identity:

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

To obtain the following:

$$\begin{aligned} uu_x &= \frac{1}{2}a^2k\sin 2kx + abk[\sin 3kx - \sin kx] + \frac{3}{2}ack[\sin 4kx - \sin 2kx] + 2adk[\sin 5kx - \sin 3kx] \\ &+ \frac{1}{2}abk[\sin 3kx + \sin kx] + b^2k\sin 4kx + \frac{3}{2}bck[\sin 5kx - \sin kx] + 2bdk[\sin 6kx - \sin 2kx] \\ &+ \frac{1}{2}ack[\sin 4kx + \sin 2kx] + bck[\sin 5kx + \sin kx] + \frac{3}{2}c^2k\sin 6kx + 2cdk[\sin 7kx - \sin kx] \\ &+ \frac{1}{2}adk[\sin 5kx + \sin 3kx] + bdk[\sin 6kx + \sin 2kx] + \frac{3}{2}cdk[\sin 7kx + \sin kx] + 2d^2k\sin 8kx \end{aligned}$$

Steady state of Kuramoto-Sivashinski equation:

$$uu_x + u_{xx} + u_{xxxx} = 0$$

Equating term-by-term:

$$-abk+\frac{1}{2}abk-\frac{3}{2}bck+bck-2cdk+\frac{3}{2}cdk-ak^2+ak^4=0$$

$$\frac{1}{2}a^2k-\frac{3}{2}ack-2bdk+\frac{1}{2}ack+bdk-4bk^2+16bk^4=0$$

$$abk-2adk+\frac{1}{2}abk+\frac{1}{2}adk-9ck^2+81ck^4=0$$

$$\frac{3}{2}ack+b^2k+\frac{1}{2}ack-16dk^2+256dk^4=0$$

$$2adk+\frac{3}{2}bck+bck+\frac{1}{2}adk=0$$

$$2bdk+\frac{3}{2}c^2k+bdk=0$$

The sin $7kx$ and sin $8kx$ terms are ignored since they would require $d = 0$.

Simplify:

$$[2ak(k^2-1)-ab-bc-cd]/2=0$$

$$[8bk(4k^2-1)+a^2-2ac-2bd]/2=0$$

$$3[6ck(9k^2-1)+ab-ad]/2=0$$

$$16dk(16k^2-1)+2ac+b^2=0$$

$$5[ad+bc]/2=0$$

$$3[2bd+c^2]/2=0$$

We have six equations and five unknowns.

Ignoring the last equation (sin $6kx$) gives the following exact numerical solution (not necessarily unique):

$$k=0.9708$$

$$a=1.5520$$

$$b=-0.1114$$

$$c=0.0040$$

$$d=-0.0001$$

Hence:

$$u=1.5520\sin 0.9708x-0.1114\sin 1.9416x+0.0040\sin 2.9124x-0.0001\sin 3.8832x$$

Recall that the expected value from the numerical solution of the K-S equation is $k = 26\pi/100 = 0.8168$. The results from the Fourier expansion appear to be converging on that value but very slowly (1 → 0.5774 → 0.4463 → 0.9708). The amplitude is also converging toward a plausible value (~2.3 for the sin kx term).

Let's add one more term (sin 5kx):

$$u = a\sin kx + b\sin 2kx + c\sin 3kx + d\sin 4kx + e\sin 5kx$$

Derivatives:

$$u_x = ak\cos kx + 2bk\cos 2kx + 3ck\cos 3kx + 4dk\cos 4kx + 5ek\cos 5kx$$

$$u_{xx} = -ak^2\sin kx - 4bk^2\sin 2kx - 9ck^2\sin 3kx - 16dk^2\sin 4kx - 25ek^2\sin 5kx$$

$$u_{xxx} = -ak^3\cos kx - 8bk^3\cos 2kx - 27ck^3\cos 3kx - 64dk^3\cos 4kx - 125ek^3\cos 5kx$$

$$u_{xxxx} = ak^4\sin kx + 16bk^4\sin 2kx + 81ck^4\sin 3kx + 256dk^4\sin 4kx + 625ek^4\sin 5kx$$

Nonlinear term:

$$uu_x = a^2k\sin kx\cos kx + 2abk\sin kx\cos 2kx + 3ack\sin kx\cos 3kx$$
$$+ 4adk\sin kx\cos 4kx + 5aek\sin kx\cos 5kx$$
$$+ abk\sin 2kx\cos kx + 2b^2k\sin 2kx\cos 2kx + 3bck\sin 2kx\cos 3kx$$
$$+ 4bdk\sin 2kx\cos 4kx + 5bek\sin 2kx\cos 5kx$$
$$+ ack\sin 3kx\cos kx + 2bck\sin 3kx\cos 2kx + 3c^2k\sin 3kx\cos 3kx$$
$$+ 4cdk\sin 3kx\cos 4kx + 5cek\sin 3kx\cos 5kx$$
$$+ adk\sin 4kx\cos kx + 2bdk\sin 4kx\cos 2kx + 3cdk\sin 4kx\cos 3kx$$
$$+ 4d^2k\sin 4kx\cos 4kx + 5dek\sin 4kx\cos 5kx$$
$$+ aek\sin 5kx\cos kx + 2bek\sin 5kx\cos 2kx + 3cek\sin 5kx\cos 3kx$$
$$+ 4d3k\sin 5kx\cos 4kx + 5e^2k\sin 5kx\cos 5kx$$

Simplify using the following trigonometric identity:

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

To obtain the following:

$$uu_x = \frac{1}{2}a^2k\sin 2kx + abk[\sin 3kx - \sin kx] + \frac{3}{2}ack[\sin 4kx - \sin 2kx]$$
$$+ 2adk[\sin 5kx - \sin 3kx] + \frac{5}{2}aek[\sin 6kx - \sin 4kx]$$
$$+ \frac{1}{2}abk[\sin 3kx + \sin kx] + b^2k\sin 4kx + \frac{3}{2}bck[\sin 5kx - \sin kx]$$
$$+ 2bdk[\sin 6kx - \sin 2kx] + \frac{5}{2}bek[\sin 7kx - \sin 3kx]$$
$$+ \frac{1}{2}ack[\sin 4kx + \sin 2kx] + bck[\sin 5kx + \sin kx] + \frac{3}{2}c^2k\sin 6kx$$
$$+ 2cdk[\sin 7kx - \sin kx] + \frac{5}{2}cek[\sin 8kx - \sin 2kx]$$
$$+ \frac{1}{2}adk[\sin 5kx + \sin 3kx] + bdk[\sin 6kx + \sin 2kx] + \frac{3}{2}cdk[\sin 7kx + \sin kx]$$
$$+ 2d^2k\sin 8kx + \frac{5}{2}dek[\sin 9kx - \sin kx]$$
$$+ \frac{1}{2}aek[\sin 6kx + \sin 4kx] + bek[\sin 7kx + \sin 3kx] + \frac{3}{2}cek[\sin 8kx + \sin 2kx]$$
$$+ 2dek[\sin 9kx + \sin kx] + \frac{5}{2}e^2k\sin 10kx$$

Steady state of Kuramoto-Sivashinski equation:
$$uu_x + u_{xx} + u_{xxxx} = 0$$

Equating term-by-term:
$$-abk + \frac{1}{2}abk - \frac{3}{2}bck + bck - 2cdk + \frac{3}{2}cdk - \frac{5}{2}dek + 2dek - ak^2 + ak^4 = 0$$
$$\frac{1}{2}a^2k - \frac{3}{2}ack - 2bdk + \frac{1}{2}ack - \frac{5}{2}cek + bdk + \frac{3}{2}cek - 4bk^2 + 16bk^4 = 0$$
$$abk - 2adk + \frac{1}{2}abk - \frac{5}{2}bek + \frac{1}{2}adk + bek - 9ck^2 + 81ck^4 = 0$$
$$\frac{3}{2}ack - \frac{5}{2}aek + b^2k + \frac{1}{2}ack + \frac{1}{2}aek - 16dk^2 + 256dk^4 = 0$$
$$2adk + \frac{3}{2}bck + bck + \frac{1}{2}adk - 25ek^2 + 625ek^4 = 0$$
$$\frac{5}{2}aek + 2bdk + \frac{3}{2}c^2k + bdk + \frac{1}{2}aek = 0$$
$$\frac{5}{2}bek + 2cdk + \frac{3}{2}cdk + bek = 0$$

Simplify:

$$[2ak(k^2-1)-ab-bc-cd-de]/2=0$$

$$[8bk(4k^2-1)+a^2-2ac-2bd-2ce]/2=0$$

$$3[6ck(9k^2-1)+ab-be-ad]/2=0$$

$$16dk(16k^2-1)+2ac-2ae+b^2=0$$

$$5[10ek(25k^2-1)+ad+bc]/2=0$$

$$3[2ae+2bd+c^2]/2=0$$

$$7[be+cd]/2=0$$

We have seven equations and six unknowns.

Ignoring the last equation (sin 7*kx*) gives the following exact numerical solution (not necessarily unique):

$k=0.8909$

$a=2.4011$

$b=-0.3637$

$c=0.0265$

$d=-0.0016$

$e=0.0001$

Hence:

$$u=2.4011\sin 0.8909x-0.3637\sin 1.7818+0.0265\sin 2.6727x$$
$$-0.0016\sin 3.5636x+0.0001\sin 4.4545x$$

This is the best model yet, but convergence is very slow.

Let's calculate the general model with *M* harmonics, all in phase:

$$u = \sum_{n=1}^{M} a_n \sin nkx$$

Derivatives:

$$u_x = k\sum_{n=1}^{M} a_n n \cos nkx$$

$$u_{xx} = -k^2 \sum_{n=1}^{M} a_n n^2 \sin nkx$$

$$u_{xxxx} = k^4 \sum_{n=1}^{M} a_n n^4 \sin nkx$$

Nonlinear term:

$$uu_x = k\sum_{p=1}^{M}\left[a_p \sin pkx \sum_{q=1}^{M} a_q q \cos qkx \right]$$

Simplify using the following trigonometric identity:

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

To obtain the following:

$$uu_x = \frac{k}{2}\sum_{p=1}^{M} a_p \left[\sum_{q=1}^{M} a_q q \sin(p+q)kx + \sum_{q=1}^{M} a_q q \sin(p-q)kx \right]$$

$$= \frac{k}{2}\sum_{p=1}^{M} a_p \left[\sum_{q=1}^{M} a_q q \sin(p+q)kx \right] + \frac{k}{2}\sum_{p=1}^{M} a_p \left[\sum_{q=1}^{M} a_q q \sin(p-q)kx \right]$$

$$= \frac{k}{2}\sum_{p=1}^{M} a_p a_{n-p}(n-p)\sin nkx - \frac{k}{2}\sum_{p=1}^{n-1} a_p a_{n-p}(n-p)\sin nkx + \frac{k}{2}\sum_{p=n+1}^{M} a_p a_{p-n}(p-n)\sin nkx$$

Steady state of Kuramoto-Sivashinski equation:

$$uu_x + u_{xx} + u_{xxxx} = 0$$

Equating term-by-term:

$$\frac{k}{2}\sum_{p=1}^{M} a_p a_{n-p}(n-p)\sin nkx - \frac{k}{2}\sum_{p=1}^{M-n} a_p a_{n+p}(n+p)\sin nkx + \frac{k}{2}\sum_{p=n+1}^{M} a_p a_{p-n}(p-n)\sin nkx$$
$$-k^2 a_n n^2 \sin nkx + k^4 a_n n^4 \sin nkx = 0$$

Simplify:

$$\frac{1}{2}\sum_{p=1}^{M} a_p a_{n-p}(n-p) - \frac{1}{2}\sum_{p=1}^{M-n} a_p a_{n+p}(n+p) + \frac{1}{2}\sum_{p=n+1}^{M} a_p a_{p-n}(p-n) + k a_n n^2 (k^2 n^2 - 1) = 0$$

We have $2M$ equations (n = 1 to $2M$) and $M + 1$ unknowns (k, a_1, … a_M). Thus for all $M > 1$ the system is potentially overdetermined. However, for such a nonlinear system, there is no guarantee of a solution, and when one exists, there is no guarantee that it is unique. Thus we adopt a numerical procedure of minimizing the value of all $2M$ functions by least squares. This amounts to demanding that the waveform of $uu_x + u_{xx} + u_{xxxx}$ resulting from the combination of the M Fourier terms (which is equal to $-u_t$) is as small as possible. Actually, the quantity minimized is $u_t^2/k^2a_1^2$ to avoid settling into the trivial solution with all quantities zero. As a check, u and u_t are plotted versus x for $0 < x < 2\pi/k$ (one wave of the fundamental wavelength). Numerical results are as follows:

M	k	a_1	a_2	a_3	a_4	a_5	a_6	a_7
1	1.0000	0						
2	1.0000	0.0111	-0					
3	0.9973	0.5038	-0.0107	0.0001				
4	0.9801	1.3141	-0.0772	0.0023	-0.0001			
5	0.9033	2.3439	-0.3294	0.0224	-0.0012	0.0001		
6	0.8214	2.4704	-0.5237	0.0514	-0.0041	0.0003	-0	
7	0.8433	2.5042	-0.4856	0.0442	-0.0033	0.0002	-0	0

Appendix

These are values obtained by Fourier analysis of the steady state numerical solution of the Kuramoto-Sivashinsky equation (courtesy of Jon Seaton):

$k = 0.8168$

n	Real Part	Imaginary Part	Amplitude	Phase
1	-0.02678	-2.913	2.913	89.5
2	0.01649	0.8968	-0.897	88.9
3	-0.004718	-0.1711	0.1711	88.4
4	0.001284	0.03491	-0.03493	87.9
5	-0.0003761	0.008161	0.008169	-87.4