UDC 517.9

ABOUT THE BOUNDEDNESS OF 3D CONTINUOUS-TIME QUADRATIC SYSTEMS

ПРО ОБМЕЖЕНІСТЬ 3D КВАДРАТИЧНИХ СИСТЕМ З НЕПЕРЕРВНИМ ЧАСОМ

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In this paper, we generalize all the existing results in the current literature for the upper bound of a general 3-D quadratic continuous-time system. In particular, we find large regions in the bifurcation parameters space of this system where it is bounded.

Наведено узагальнення всіх відомих результатів про верхню для загальної 3D *квадратичної системи з неперервним часом. Зокрема, знайдено великі області у просторі біфуркаційних параметрів системи, де система є обмеженою.*

1. Introduction. Chaos in 3-D quadratic continuous-time autonomous systems was discovered in 1963 by E. Lorenz [1], where he proposed a simple mathematical model of a weather system which was made up of three linked nonlinear differential equations. Some surprising results clearly showed the complex behavior from supposedly simple equations. Also, it has been noted that the behavior of this system of equations was sensitively dependent on the initial conditions of the model, i.e., it implied that if there were any errors in observing the initial state of the system which is inevitable in any real system, therefore, the prediction of a future state of the system was impossible. Later, Ruelle in 1979, when computer simulation came along, the first fractal shape identified took the form of a butterfly; it arose from graphing the changes in weather systems modelled by Lorenz. The Lorenz's attractor shows just how and why weather prognostication is so involved and notoriously wrong because of the butterfly effect. This amusing name reflects the possibility that a "*butterfly in the Amazon might, in principal, ultimately alter the weather in Kansas*". A new chaotic attractor of three-dimensional system is coined by Chen and Ueta [2], in the pursuit of anti controlling chaos for Lorenz model [1]. This new chaotic model reassembles the Lorenz and Rôssler systems [1–3]. However the Chen model appears to be more complex and more sophisticated [2]. They both are three-dimensional autonomous with only two quadratic terms, but it is topologically not equivalent to the Lorenz equation. Many other systems of three smooth autonomous ordinary differential equations with two quadratic nonlinear terms have been found in addition of the Chen system, aiming to generate chaotic attractors; these include the Rôssler system [3], recently, Liu & Chen [4] found another relatively simple 3-D smooth autonomous chaotic system with only quadratic nonlinearities; and some other systems captured by Sprott via computer found in [5]. Note that the finding of simple 3-D quadratic chaotic systems is quite a very interesting field in the study of dynamical systems, since many technological applications such as communication, encryption, information storage, uses these chaotic attractors [6].

The boundedness of these systems was the subject of many works. Bounded chaotic systems and the estimate of their bounds is important in chaos control, chaos synchronization, and their applications. The estimation of the upper bound of a chaotic system is quite difficult to achieve technically. In this work, we generalize all the relevant results of the literature and describe some of these bounds using multivariable function analysis.

The most general 3-D quadratic continuous-time autonomous system is given by

$$\begin{aligned} x' &= a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 y^2 + a_6 z^2 + a_7 xy + a_8 xz + a_9 yz, \\ y' &= b_0 + b_1 x + b_2 y + b_3 z + b_4 x^2 + b_5 y^2 + b_6 z^2 + b_7 xy + b_8 xz + b_9 yz, \\ z' &= c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 y^2 + c_6 z^2 + c_7 xy + c_8 xz + c_9 yz, \end{aligned} \tag{1}$$

where $(a_i, b_i, c_i)_{0 \leq i \leq 9} \in \mathbb{R}^{30}$ are the bifurcation parameters. Several researchers have defined and studied quadratic 3-D chaotic systems as described in the references. The generalized Lorenz-like canonical form introduced in [7 – 12] gives a unique and unified classification for a large class of 3-D quadratic chaotic systems. This system contains all the well known quadratic systems given in [1 – 4, 7, 13]. In chaos control, chaos synchronization, and their applications, the estimation of an upper bound of the system under consideration is an important task. For example, in [14] the boundedness of the Lorenz system [1] was investigated and in [9] the boundedness of the Chen system [2] was investigated. Recently, a better upper bound for the Lorenz system for all positive values of its parameters was derived in [15], and it is the best result in the current literature because the estimation overcomes some problems related to the existence of singularities arising in the value of the upper bound given in [14].

In this paper, we generalize all these results concerning an upper bound for the general 3-D quadratic continuous-time autonomous system. In particular, we find large regions in the bifurcation parameter space of this system where it is bounded. The method is based on multivariable function analysis.

2. Estimate of the bound for the general system. To estimate the bound for the general system (1), we consider the function $V(x, y, z)$ defined by

$$V(x,y,z) = \frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{2} \tag{2}$$

where $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ is any set of real constants for which the derivative of (2) along the solutions of (1) is given by

$$\frac{dV}{dt} = -\omega(x-\alpha_1)^2 - \varphi(y-\beta_1)^2 - \phi(z-\gamma_1)^2 + d \tag{3}$$

where

$$
\begin{aligned}
d &= \omega\alpha_1^2 + \varphi\beta_1^2 + \phi\gamma_1^2 - \beta b_0 - \gamma c_0 - \alpha a_0,\\
\omega &= \alpha a_4 - a_1 + \beta b_4 + \gamma c_4,\\
\varphi &= \alpha a_5 - b_2 + \beta b_5 + \gamma c_5,\\
\phi &= \alpha a_6 - c_3 + \beta b_6 + \gamma c_6,\\
\alpha_1 &= \frac{a_0 - \alpha a_1 - \beta b_1 - \gamma c_1}{2\omega}, \quad \text{if} \quad \omega \neq 0,\\
\beta_1 &= \frac{b_0 - \alpha a_2 - \beta b_2 - \gamma c_2}{2\varphi}, \quad \text{if} \quad \varphi \neq 0,\\
\gamma_1 &= \frac{c_0 - \alpha a_3 - \beta b_3 - \gamma c_3}{2\phi}, \quad \text{if} \quad \phi \neq 0.
\end{aligned}
\tag{4}
$$

Note that if $\omega = 0$ or $\varphi = 0$ or $\phi = 0$, then there is no need to calculate α_1, β_1 and γ_1, respectively, and a condition relating $(a_i, b_i, c_i)_{0\le i\le 9} \in \mathbb{R}^{30}$ to $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ is obtained. If not, we have the formulas given by the last three equalities of (4). The form of the function $\frac{dV}{dt}$ in (3) is possible if the following conditions on the coefficients $(a_i, b_i, c_i)_{0\le i\le 9} \in \mathbb{R}^{30}$ hold:

$$
\begin{aligned}
& a_4 = 0, \quad b_4 = -a_7, \quad b_5 = 0, \quad b_7 = -a_5, \quad c_4 = -a_8, \quad c_5 = -b_9,\\
& c_6 = 0, \quad c_7 = -a_9 - b_8, \quad c_8 = -a_6, \quad c_9 = -b_6,\\
& b_1 = \alpha a_7 - \beta a_5 - a_2 + \gamma c_7,\\
& c_1 = -a_3 - \gamma a_6 + \alpha a_8 + \beta b_8,\\
& c_2 = \alpha a_9 - b_3 - \gamma b_6 + \beta b_9,
\end{aligned}
\tag{5}
$$

i.e., the system (1) becomes

$$
\begin{aligned}
x' &= a_0 + a_1 x + a_2 y + a_3 z + a_5 y^2 + a_6 z^2 + a_7 xy + a_8 xz + a_9 yz,\\
y' &= b_0 + b_1 x + b_2 y + b_3 z - a_7 x^2 + b_6 z^2 - a_5 xy + b_8 xz + b_9 yz,\\
z' &= c_0 + c_1 x + c_2 y + c_3 z - a_8 x^2 - b_9 y^2 - (a_9 + b_8) xy - a_6 xz - b_6 yz,
\end{aligned}
\tag{6}
$$

with the formulas for b_1, c_1, and c_2 given by the last three equations of (5).

To prove the boundedness of system (6), we assume that it is bounded and then we find its bound, i.e., assume that ω, φ, ϕ, and d are strictly positive, i.e.,

$$\begin{aligned} \omega\alpha_1^2 + \varphi\beta_1^2 + \phi\gamma_1^2 - \beta b_0 - \gamma c_0 - \alpha a_0 &> 0, \\ a_1 &< \alpha a_4 + \beta b_4 + \gamma c_4, \\ b_2 &< \alpha a_5 + \beta b_5 + \gamma c_5, \\ c_3 &< \alpha a_6 + \beta b_6 + \gamma c_6. \end{aligned} \tag{7}$$

Then if system (6) is bounded, the function (3) has a maximum value, and the maximum point (x_0, y_0, z_0) satisfies

$$\frac{(x_0 - \alpha_1)^2}{\frac{d}{\omega}} + \frac{(y_0 - \beta_1)^2}{\frac{d}{\varphi}} + \frac{(z_0 - \gamma_1)^2}{\frac{d}{\phi}} = 1.$$

Now consider the ellipsoid

$$\Gamma = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{(x - \alpha_1)^2}{\frac{d}{\omega}} + \frac{(y - \beta_1)^2}{\frac{d}{\varphi}} + \frac{(z - \gamma_1)^2}{\frac{d}{\phi}} = 1, \omega, \varphi, \phi, d > 0 \right\}, \tag{8}$$

and define the function

$$\begin{aligned} F(x, y, z) &= G(x, y, z) + \lambda H(x, y, z), \\ G(x, y, z) &= x^2 + y^2 + z^2, \\ H(x, y, z) &= \frac{(x - \alpha_1)^2}{\frac{d}{\omega}} + \frac{(y - \beta_1)^2}{\frac{d}{\varphi}} + \frac{(z - \gamma_1)^2}{\frac{d}{\phi}} - 1 \end{aligned}$$

where $\lambda \in \mathbb{R}$ is a finite parameter. Then we have $\max_{(x,y,z)\in\Gamma} G = \max_{(x,y,z)\in\Gamma} F$ and

$$\begin{aligned} \frac{\partial F(x, y, z)}{\partial x} &= -2d^{-1}(\omega\lambda\alpha_1 - (\omega\lambda + d)x), \\ \frac{\partial F(x, y, z)}{\partial y} &= -2d^{-1}(\varphi\lambda\beta_1 - (\varphi\lambda + d)\, y) \\ \frac{\partial F(x, y, z)}{\partial z} &= -2d^{-1}(\phi\lambda\gamma_1 - (\phi\lambda + d)z) \end{aligned}$$

and the following cases according to the value of the parameter λ with respect to the values $-\dfrac{d}{\omega}$, $-\dfrac{d}{\varphi}$, and $\lambda \neq -\dfrac{d}{\phi}$ if ω, φ, $\phi > 0$. Otherwise, a similar study can be done.

(i) If $\lambda \neq -\dfrac{d}{\omega}$, $\lambda \neq -\dfrac{d}{\varphi}$, and $\lambda \neq -\dfrac{d}{\phi}$, then

$$(x_0, y_0, z_0) = \left(\frac{\omega\lambda\alpha_1}{d + \omega\lambda}, \frac{\varphi\lambda\beta_1}{d + \varphi\lambda}, \frac{\phi\lambda\gamma_1}{d + \phi\lambda} \right)$$

and

$$\max_{(x,y,z)\in\Gamma} G = \xi_1$$

where

$$\xi_1 = \frac{\omega^2\lambda^2\alpha_1^2}{(d+\omega\lambda)^2} + \frac{\varphi^2\lambda^2\beta_1^2}{(d+\varphi\lambda)^2} + \frac{\phi^2\lambda^2\gamma_1^2}{(d+\phi\lambda)^2}. \tag{9}$$

In this case, there exists a parameterized family (in λ) of bounds given by (9) of system (6).

(ii) If $\lambda = -\dfrac{d}{\omega}, (\omega \neq 0, \varphi \neq 0, \phi \neq 0, \omega \neq \varphi, \omega \neq \phi), \lambda \neq -\dfrac{d}{\varphi}, \lambda \neq -\dfrac{d}{\phi}$, then

$$(x_0, y_0, z_0) = \left(\pm\sqrt{\frac{d}{\omega}\left(1-\frac{\xi_2}{\xi_3}\right)} + \alpha_1, \frac{-\beta_1\varphi}{\omega-\varphi}, \frac{-\gamma_1\phi}{\omega-\phi}\right) \tag{10}$$

where

$$\xi_2 = \frac{(\varphi\beta_1^2+\phi\gamma_1^2)(\omega-\phi)^2 d^3}{\omega^2},$$

$$\xi_3 = \frac{(\phi-\omega)^2(\omega-\varphi)^2 d^4}{\omega^4}$$

with the condition

$$\xi_3 \geq \xi_2.$$

This confirms that the value x_0 in (10) is well defined and the conditions $\omega \neq 0$, $\varphi \neq 0$, $\phi \neq 0$, $\omega \neq \varphi$, and $\omega \neq \phi$ are formulated as follows:

$$\begin{aligned} a_1 &\neq \alpha a_4 + \beta b_4 + \gamma c_4, \\ b_2 &\neq \beta b_5 + \gamma c_5 + \alpha a_5, \\ c_3 &\neq \beta b_6 + \gamma c_6 + \alpha a_6, \\ b_2 - a_1 &\neq (a_5 - a_4)\alpha + (b_5 - b_4)\beta + (c_5 - c_4)\gamma, \\ c_3 - a_1 &\neq (a_6 - a_4)\alpha + (b_6 - b_4)\beta + (c_6 - c_4)\gamma. \end{aligned} \tag{11}$$

In this case, we have

$$\max_{(x,y,z)\in\Gamma} G = \left(\sqrt{\frac{d}{\omega}\left(1-\frac{\xi_2}{\xi_3}\right)} + \alpha_1\right)^2 + \frac{\beta_1^2\varphi^2}{(\omega-\varphi)^2} + \frac{\gamma_1^2\phi^2}{(\omega-\phi)^2}.$$

(iii) If $\lambda \neq -\dfrac{d}{\omega}, \lambda = -\dfrac{d}{\varphi}\,(\omega \neq 0, \varphi \neq 0, \phi \neq 0, \omega \neq \varphi, \varphi \neq \phi), \lambda \neq -\frac{d}{\phi}$, then

$$(x_0, y_0, z_0) = \left(-\frac{\alpha_1\omega}{\varphi-\omega}, \pm\sqrt{\frac{d}{\varphi}\left(1-\frac{\xi_4}{\xi_5}\right)} + \beta_1, \frac{\gamma_1\phi}{\phi-\varphi}\right) \tag{12}$$

where

$$\xi_4 = \left(2\omega\varphi\phi\alpha_1^2 - 2\omega\varphi\phi\gamma_1^2 - \omega\varphi^2\alpha_1^2 - \omega\phi^2\alpha_1^2 + \omega^2\phi\gamma_1^2 + \varphi^2\phi\gamma_1^2\right)\varphi^2,$$

$$\xi_5 = (\phi - \varphi)^2 (\varphi - \omega)^2 d$$

with the condition

$$\xi_5 \geq \xi_4.$$

This confirms that the value y_0 in (12) is well defined and the conditions $\omega \neq 0$, $\varphi \neq 0$, $\phi \neq 0$, $\omega \neq \varphi$, and $\varphi \neq \phi$ are formulated by the first four equations of (11) and

$$c_3 - b_2 \neq (a_6 - a_5)\,\alpha + (b_6 - b_5)\,\beta + (c_6 - c_5)\,\gamma. \tag{13}$$

In this case, we have

$$\max_{(x,y,z)\in\Gamma} G = \left(\sqrt{\frac{d}{\varphi}\left(1 - \frac{\xi_4}{\xi_5}\right)} + \beta_1\right)^2 + \frac{\alpha_1^2\omega^2}{(\varphi - \omega)^2} + \frac{\gamma_1^2\phi^2}{(\varphi - \phi)^2}.$$

(iv) If $\lambda \neq -\dfrac{d}{\omega}$, $\lambda \neq -\dfrac{d}{\varphi}$, $\lambda = -\dfrac{d}{\phi}\,(\omega \neq 0, \varphi \neq 0, \phi \neq 0, \omega \neq \phi, \phi \neq \varphi)$, then

$$(x_0, y_0, z_0) = \left(\frac{-\alpha_1\omega}{\phi - \omega}, \frac{-\beta_1\varphi}{\phi - \varphi}, \pm\sqrt{\frac{d}{\phi}\left(1 - \frac{\xi_6}{\xi_7}\right)} + \gamma_1\right) \tag{14}$$

where

$$\xi_6 = \left(\omega\varphi^2\alpha_1^2 - 2\omega\varphi\phi\beta_1^2 - 2\omega\varphi\phi\alpha_1^2 + \omega\phi^2\alpha_1^2 + \omega^2\varphi\beta_1^2 + \varphi\phi^2\beta_1^2\right)\phi^2,$$

$$\xi_7 = (\phi - \varphi)^2 (\phi - \omega)^2 d$$

with the condition

$$\xi_7 \geq \xi_6.$$

This confirms that the value z_0 in (14) is well defined and the conditions $\omega \neq 0$, $\varphi \neq 0$, $\phi \neq 0$, $\omega \neq \phi$, and $\phi \neq \varphi$ are formulated by the first four equations of (11) and (13), respectively.

The other possible cases are treated using the same logic.

Theorem 1. *Assume that conditions (4), (5), and (7) hold. Then the general 3-D quadratic continuous-time system (1) is bounded, i.e., it is contained in the ellipsoid (8).*

Similar results can be found using the cases discussed above.

3. Example. Consider the Lorenz system given by

$$\dot{x} = a\,(y - x)\,,$$

$$\dot{y} = cx - y - xz,$$

$$\dot{z} = xy - bz,$$

i.e., $a_i = 0$, $i = 0,3,4,5,6,7,8,9$, $a_1 = -a$, $a_2 = a$, $b_i = 0$, $i = 0,3,4,5,6,7,9$, $b_1 = c$, $b_2 = -1$, $b_8 = -1$, $c_i = 0$, $i = 0,1,2,4,5,6,8,9$, $c_3 = -b$, and $c_7 = 1$. We choose $\alpha = \beta = 0$ and $\gamma = a + c$ as in [15]. Thus $V(x,y,z) = \dfrac{x^2 + y^2 + (z-(a+c))^2}{2}$ and $d = b\left(\dfrac{a+c}{2}\right)^2$, $\omega = a$, $\varphi = 1$, $\phi = b$, $\alpha_1 = 0$, $\beta_1 = 0$, and $\gamma_1 = \dfrac{a+c}{2}$. Then we have $\dfrac{dV}{dt} = -ax^2 - -y^2 - b\left(z - \dfrac{a+c}{2}\right)^2 + b\left(\dfrac{a+c}{2}\right)^2$, which is the same as in [15]. Also, it is easy to verify that conditions (5) and (7) hold for this case. The ellipsoid Γ is given by

$$\Gamma = \left\{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{\frac{b}{a}\left(\frac{a+c}{2}\right)^2} + \frac{y^2}{b\left(\frac{a+c}{2}\right)^2} + \frac{\left(z - \frac{a+c}{2}\right)^2}{\left(\frac{a+c}{2}\right)^2} = 1, a,b,c > 0\right\},$$

which is also the same as in [15]. Finally, we have the result shown in [15] that confirms that if $a > 0$, $b > 0$, and $c > 0$, then the Lorenz system [1] is contained in the sphere $\Omega = = \left\{(x,y,z) \in \mathbb{R}^3 / x^2 + y^2 + (z-a-c)^2 = R^2\right\}$, where

$$R^2 = \begin{cases} \dfrac{(a+c)^2 b^2}{4(b-1)}, & \text{if} \quad a \geq 1, \quad b \geq 2, \\[2ex] (a+c)^2, & \text{if} \quad a > \dfrac{b}{2}, \quad b < 2, \\[2ex] \dfrac{(a+c)^2 b^2}{4a(b-a)}, & \text{if} \quad a < 1, \quad b \geq 2a. \end{cases}$$

4. Conclusion. Using multivariable function analysis, we generalize all the results about finding an upper bound for the general 3-D quadratic continuous-time autonomous system. In particular, we find large regions in the bifurcation parameters space of this system where it is bounded.

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Received 16.04.09,
after revision — 16.06.09