Au: Figs. of higher resolution available?

International Journal of Bifurcation and Chaos, Vol. 21, No. 1 (2011) 1–6

DOI: 10.1142/S0218127411028325

ON THE DYNAMICS OF A NEW SIMPLE 2-D RATIONAL DISCRETE MAPPING

ZERAOULIA ELHADJ
Department of Mathematics,
University of Tébessa, 12002, Algeria
zeraoulia@mail.univ-tebessa.dz
zelhadj12@yahoo.fr

J. C. SPROTT
Department of Physics,
University of Wisconsin, Madison, WI 53706, USA
sprott@physics.wisc.edu

Received February 12, 2010; Revised

This paper is devoted to the analysis of a new simple rational map of the plane. Its dynamics are described in some detail, along with some other dynamical phenomena. In particular, the map under consideration is the first simple rational map whose fraction has no vanishing denominator that gives chaotic attractors via a quasi-periodic route to chaos.

Keywords: Rational chaotic map; quasi-periodic route to chaos; coexisting attractors.

1. Introduction

Chaos occurs in dynamical systems as a paradigm phenomenon. Several examples of simple dynamical systems display chaos. The simplicity of an equation does not affect the high complexity of its dynamics. Rational chaotic systems are rather rare in theory and practice. In this paper, we present a new simple rational chaotic map along some of its dynamical properties. In [Lu *et al.*, 2004] the following new 1-D discrete iterative system with a rational fraction was discovered in a study of evolutionary algorithms:

$$g(x) = \frac{1}{0.1 + x^2} - ax, \tag{1}$$

where a is a parameter. The map (1) describes different random evolutionary processes, and it is much more complicated than the logistic system.

In [Chang *et al.*, 2005] an extended version of the former one-dimensional discrete chaotic system given in [Lu *et al.*, 2004] to two-dimensions is given as follows:

$$h(x, y) = \begin{pmatrix} \dfrac{1}{0.1 + x^2} - ay \\ \dfrac{1}{0.1 + y^2} + bx \end{pmatrix}, \tag{2}$$

where a and b are parameters. The map (2) has more complicated dynamical behavior than the one-dimensional map (1).

Based on these studies in [Lu *et al.*, 2004; Chang *et al.*, 2005], a new and very simple 2-D map, characterized by the existence of only one rational fraction with no vanishing denominator is constructed in this paper and is given by:

$$f(x, y) = \begin{pmatrix} \dfrac{-ax}{1 + y^2} \\ x + by \end{pmatrix}, \tag{3}$$

where a and b are the bifurcation parameters. First, the new map (3) is algebraically simpler but with more complicated behavior than map (2), and second, it produces several new chaotic attractors obtained via the quasi-periodic route to chaos. Discrete maps have many applications in science and technology [Chen & Dong, 1998; Scheizer & Hasler, 1996; Abel *et al.*, 1997].

The essential motivation for this work is to provide a basic analysis of f and to give a detailed study of its dynamics. Some basic dynamical behaviors of map (3) are investigated here by both theoretical analysis and numerical simulation.

2. Some Basic Properties

The new chaotic attractors described by map (3) have several important properties such as:

(i) The map (3) is defined for all points in the plane.
(ii) The associated function $f(x, y)$ of the map (3) is of class $C^\infty(\mathbb{R}^2)$, and it has no vanishing denominator.
(iii) The new chaotic map (3) is symmetric under the coordinate transformation $(x, y) \to (-x, -y)$, and this transformation persists for all values of the map parameters.

Briefly, the fixed points of map (3) are the real solutions of the equations $-ax/(1 + y^2) = x$ and $x + by = y$. Hence, one may easily obtain the equations $(a+1+y^2)x = 0$ and $(1-b)y = x$. Assume in this paper that $-1 \le a \le 4$. Then if $b \ne 1$, the only fixed point of the map (3) is $P = (0, 0)$, and if $b = 1$, then the y-axis is invariant by the iteration of the map f. The Jacobian matrix of map (3) evaluated at a point (x, y) is given by:

$$Df(x,y) = \begin{pmatrix} \dfrac{-a}{1+y^2} & \dfrac{2axy}{(1+y^2)^2} \\ 1 & b \end{pmatrix}. \tag{4}$$

and at the fixed point $P = (0, 0)$, the Jacobian matrix is given by $Df(0,0) = \left(\begin{smallmatrix} -a & 0 \\ 1 & b \end{smallmatrix}\right)$. The local stability of P is studied by evaluating the eigenvalues of the Jacobian $Df(P)$. The eigenvalues of $Df(P)$ are $\lambda_1 = -a$ and $\lambda_2 = b$. Then one has the following results:

(1) If $|a| < 1$ and $|b| < 1$, then P is asymptotically stable.
(2) If $|a| > 1$ or $|b| > 1$, then P is an unstable fixed point.
(3) If $|a| < 1$ and $|b| > 1$, or $|a| > 1$ and $|b| < 1$, then P is a saddle point.
(4) If $|a| = 1$ or $|b| = 1$, then P is a nonhyperbolic fixed point.

3. Numerical Simulations

3.1. *Observation of chaotic attractors*

There are several possible ways for a discrete dynamical system to make a transition from regular behavior to chaos. Bifurcation diagrams display these routes and allow one to identify the chaotic regions in ab-space from which the chaotic attractors can be determined. In this subsection, we will illustrate some observed chaotic attractors, along with some other dynamical phenomena.

3.2. *Route to chaos*

In [Chang *et al.*, 2005] the chaotic attractors are obtained via a period-doubling bifurcation route to chaos as shown in Fig. 3(a). Possibly, the map (3) is the first simple rational map whose fraction has no vanishing denominator that gives chaotic attractors via a quasi-periodic route to chaos.

3.3. *Dynamical behavior with parameter variation*

In this subsection, the dynamical behavior of the map (3) is investigated numerically. Figure 3(b) shows regions of unbounded (white), fixed point (gray), periodic (blue), quasi-periodic (green), and chaotic (red) solutions in the ab-plane for the map (3), where we use $|LE| < 0.0001$ as the criterion for quasi-periodic orbits with 10^6 iterations for each point. Figure 3(a) shows a similar plot for the rational map (2) studied in [Chang *et al.*, 2005].

On the other hand, if we fix parameter $b = 0.6$ and vary $-1 \le a \le 4$, the map (3) exhibits the following dynamical behaviors as shown in Fig. 2(a): In the interval $-1 \le a \le 1$, the map (3) converges to the fixed point $(0, 0)$. For $1 < a \le 2$, it converges to a period-2 attractor followed by a quasi-periodic orbit for $2 < a \le 3$ as shown in Fig. 4(a). In the interval $3 < a \le 4$, it converges to a chaotic attractor shown in Fig. 4(b) via a quasi-periodic route to chaos except for a number of periodic windows. We remark also the appearance of a singularity in the LEs at $a = 1.25$, and $b = 0.6$.

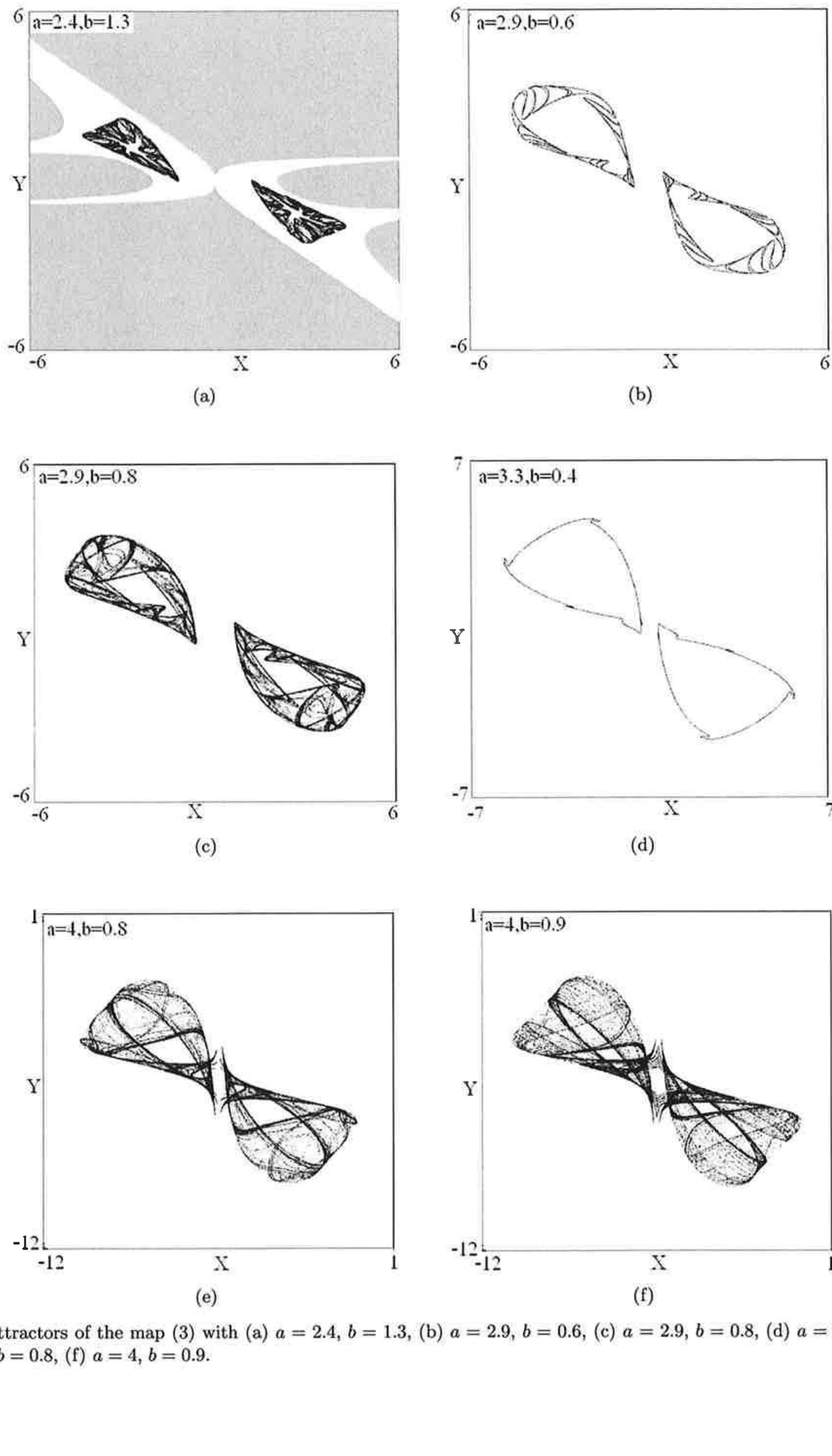


Fig. 1. Attractors of the map (3) with (a) $a = 2.4$, $b = 1.3$, (b) $a = 2.9$, $b = 0.6$, (c) $a = 2.9$, $b = 0.8$, (d) $a = 3.3$, $b = 0.4$, (e) $a = 4$, $b = 0.8$, (f) $a = 4$, $b = 0.9$.

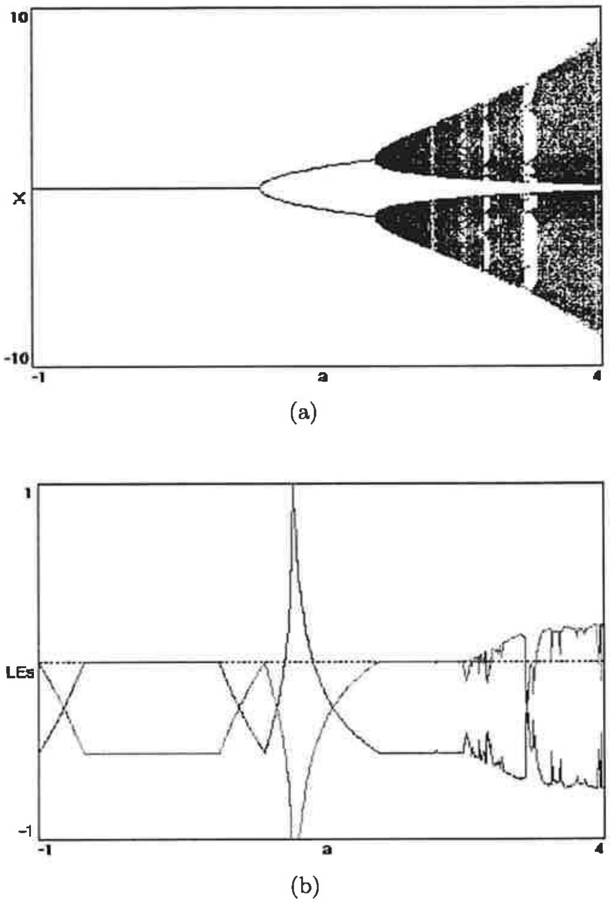


Fig. 2. (a) The quasi-periodic route to chaos for the map (3) obtained for $b = 0.6$ and $-1 < a \leq 4$. (b) Variation of the Lyapunov exponents of map (3) versus the parameter $-1 < a \leq 4$ with $b = 0.6$.

For the map (3) there are dissipative as well as area-expanding regions. Numerical calculations show that the map (3) has dissipative orbits for the regions shown in black in Fig. 5 and area-expanding orbits for the regions shown in white as determined from the sign of the numerical average of $\log|(ab + 2axy + aby^2)/(y^2 + 1)^2|$ over the orbit on the attractor. If there are point attractors, both LEs must be negative, and hence there is dissipation. There are also regions of hyperchaos, for example, at $a = 2.6$, and $b = 1.2$. On the other hand, it is well known that basin boundaries arise in dissipative dynamical systems when two or more attractors are present. In such situations, each attractor has a basin of initial conditions that lead asymptotically to that attractor. The sets that separate different basins are called the basin boundaries. In some cases the basin boundaries can have very complicated fractal structure and hence pose an additional impediment to predicting long-term behavior. For the map (3) we have calculated the attractors and their basins of attraction on a grid in ab-space where the system is chaotic. There is a wide variety of possible attractors, only some of which are shown in Figs. 1 and 4. Also, most of the basin boundaries are smooth, and we note that there are basins of attraction for $b > 1$, as shown in Fig. 1(a), but evidently none for $b < 1$, i.e. the basin of attraction is apparently the whole space.

There are some regular and chaotic regions in ab-space where two coexisting attractors apparently occur as shown in the black region of Fig. 6, essentially inside the squares $(a, b) \in [2, 4] \times [0, 1]$ and $(a, b) \in [3, 4] \times [-1, 0]$. Figure 6 was obtained by using 200 different random initial conditions and

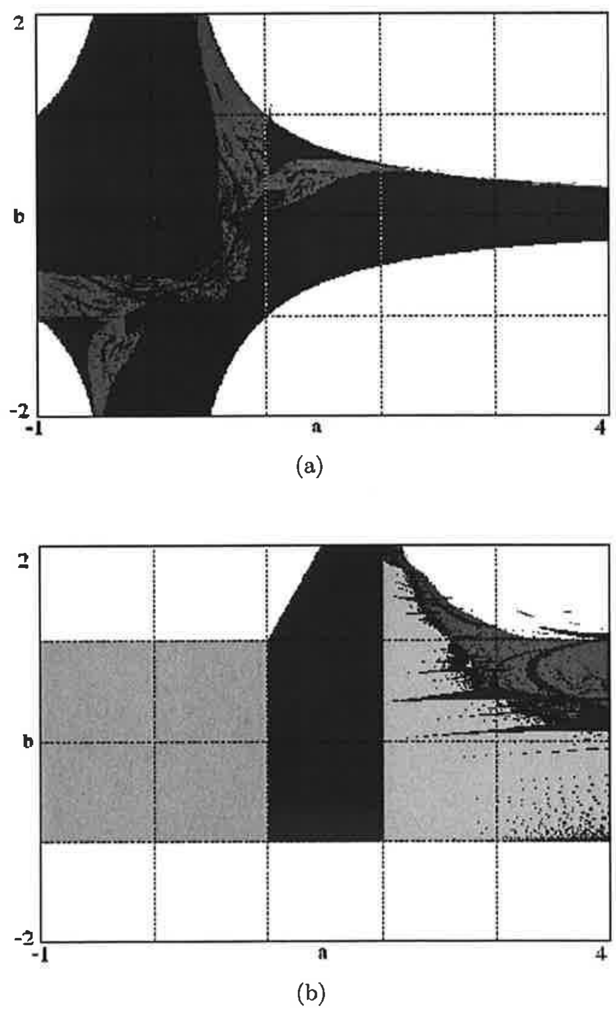
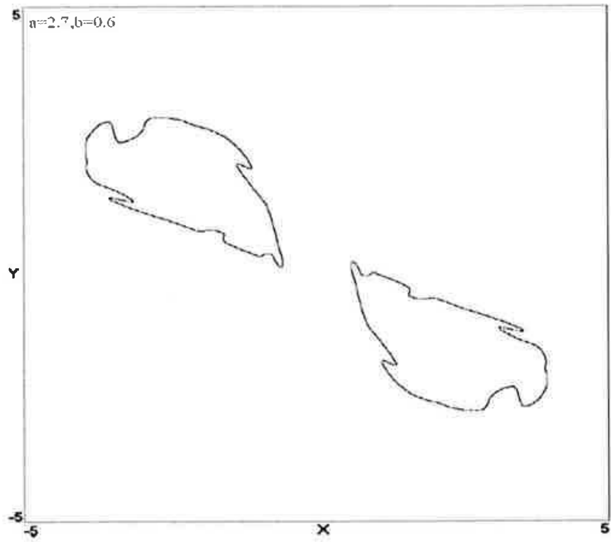
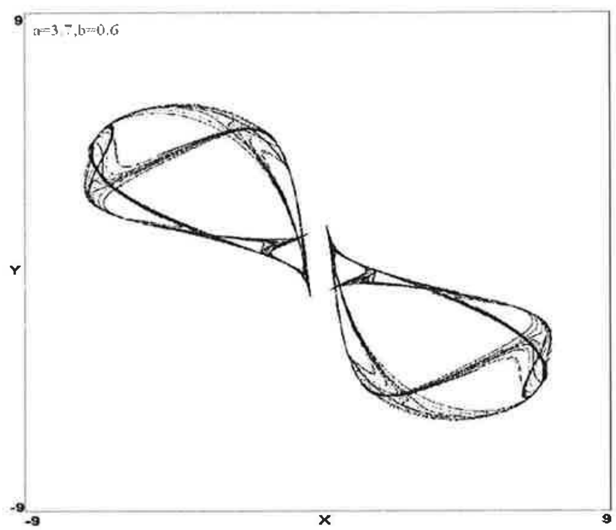


Fig. 3. (a) Regions of dynamical behaviors in the ab-plane for the rational map (2). (b) Regions of dynamical behaviors in the ab-plane for the rational map (3).



(a)



(b)

Fig. 4. Attractors of the map (3) (a) Quasi-periodic orbit for $a = 2.7$, $b = 0.6$. (b) Chaotic orbit for $a = 3.7$, $b = 0.6$.

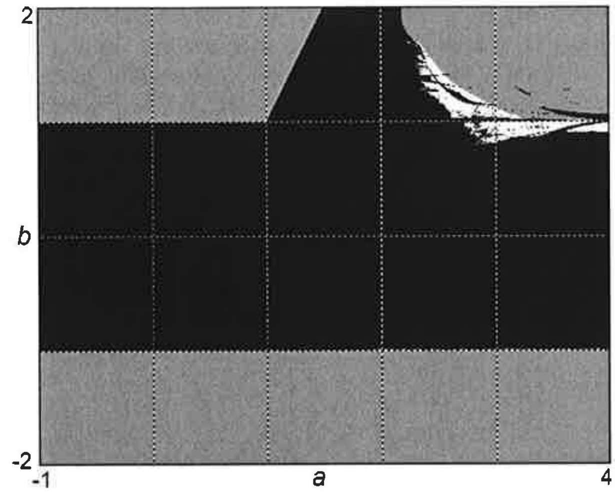


Fig. 5. The sign of the average of $\log|(ab + 2axy + aby^2)/(y^2 + 1)^2|$ over the orbit on the attractors of the system (3) in the ab-plane defines the regions of net expansion and contraction.

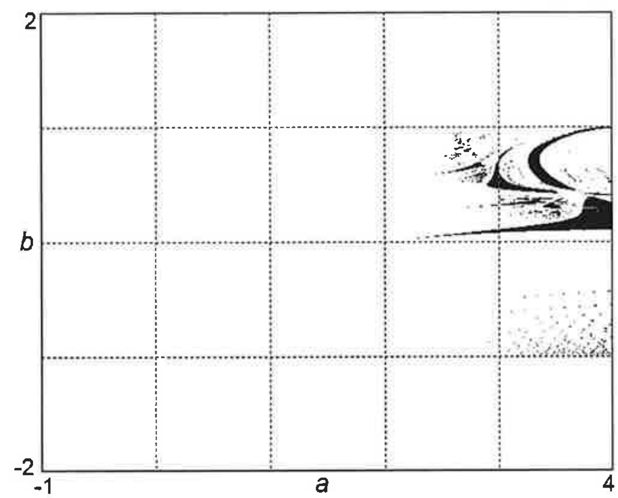


Fig. 6. The regions of ab-space with multiple attractors for the map (3).

looking for cases where the distribution of the average value of x on the attractor is bimodal. Since there is no rigorous test for bimodality, this was done by sorting the 200 values of $\langle x\rangle$ and then dividing them into two equal groups. The group with the smallest range of $\langle x\rangle$ was assumed to represent one of the attractors, and a second attractor was assumed to exist if the largest gap in the values of those in the other group was twice the range of the first group.

4. Conclusion

We have reported a new algebraically simple 2-D discrete rational chaotic map with complicated dynamics. The dynamical behavior of this map was further investigated in some detail using both theoretical analysis and numerical simulation.

References

Abel, A., Bauer, A., Kerber, K. & Schwarz, W. [1997] "Chaotic codes for CDMA application," *Proc. ECCTD'97* **1**, 306–309.