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Chaotic hyperjerk systems

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Abstract

A hyperjerk system is a dynamical system governed by an nth order ordinary differential equation with $n > 3$ describing the time evolution of a single scalar variable. Such systems are surprisingly general and are prototypical examples of complex dynamical systems in a high-dimensional phase space. This paper describes a numerical study of a simple subclass of such systems and shows that they provide a means to extend the extensive study of chaotic systems with $n = 3$. We present some simple chaotic hyperjerks of 4th and 5th order. Two cases are examined that are apparently the simplest possible chaotic flows for $n = 4$, together with several hyperchaotic cases for $n = 4$ and 5.

1. Introduction

The study of chaos in low-dimensional dynamical systems is now relatively mature [1], and interest is turning more to understanding the dynamics of high-dimensional systems where chaos, hyperchaos, self-organization, pattern formation, and other related processes are common. According to the Poincaré–Bendixson theorem [2], systems of the form $d\mathbf{x}/dt = f(\mathbf{x})$, where $\mathbf{x}$ is an n-dimensional vector and $f(\mathbf{x})$ is a smooth function, can exhibit chaos only for $n > 2$. The case $n = 3$ has been widely studied, and many examples of chaos have been identified in such systems [3]. Most of these systems can be cast in the form of an explicit 3rd-order ordinary differential equation [4,5] describing the time evolution of a single scalar variable x according to:

$$\frac{d^3x}{dt^3} = f\left(\frac{d^2x}{dt^2}, \frac{dx}{dt}, x\right) \tag{1}$$

Such systems have been called "jerk systems" because the successive time derivatives of the displacement in a mechanical system are the velocity, acceleration, and jerk [6].

One particularly simple example of such a system is [7,8]

$$\frac{d^3x}{dt^3} + a\frac{d^2x}{dt^2} + \frac{dx}{dt} = g(x) \tag{2}$$

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where $g(x)$ is a nonlinear function such as $g(x) = b(x^2 - 1)$, which exhibits chaos for $a = 0.6$ and $b = 0.58$. Integrating each term reveals that this equation is a damped harmonic oscillator driven by a nonlinear memory term that involves the integral of $g(x)$

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + x = \int g(x)\,dt \tag{3}$$

An obvious generalization of Eq. (2) to higher dimension is

$$\frac{d^{(n)}x}{dt^{(n)}} + a\frac{d^{(n-1)}x}{dt^{(n-1)}} + \frac{d^{(n-2)}x}{dt^{(n-2)}} = g(x) \tag{4}$$

for which Eq. (3) becomes

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + x = \int_{(n-2)} \cdots \int g(x)\,dt \tag{5}$$

This paper will examine the properties of Eq. (4) for $n > 3$ and its generalizations. Such systems are examples of what we will call a "hyperjerk system," which is a system of the form $d^{(n)}x/dt^{(n)} = f(d^{(n-1)}x/dt^{(n-1)}, \ldots, x)$, since it involves time derivatives of a jerk function. We argue that such systems for $n > 3$ warrant study because of their wide generality and elegant simplicity. In this paper we will present several hyperjerk flows of 4th and 5th order. Among these, we investigate the simplest chaotic flows for $n = 4$ together with several hyperchaotic cases for $n = 4$ and 5, one of these apparently being in its simplest possible form.

In the next section, we will present some hyperjerk cases with $n = 4$. Each case is optimized relative to the largest Lyapunov exponent (LLE) via an exhaustive numerical search. Several Poincaré sections will be plotted, and the Kaplan–Yorke dimension D_L (or else called Lyapunov dimension) is calculated for each case.

2. Hyperjerk chaotic flows with $n = 4$

In this section we will present some cases of chaotic hyperjerk systems with $n = 4$ of the form:

$$\frac{d^4x}{dt^4} + a_0\frac{d^3x}{dt^3} + a_1\frac{d^2x}{dt^2} + a_2\frac{dx}{dt} = g\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right) \tag{6}$$

where $g(x, \frac{dx}{dt}, \frac{d^2x}{dt^2})$ is the nonlinear function. Note that setting the damping term a_0 in (6) is equal to 1 gives a constant state-space contraction, and therefore the sum of the LEs is equal to -1. Then finding the largest LLE is much like maximizing the Lyapunov dimension D_L. The latter though is not always true as has been shown for many numerical examples in [9] for a maximally chaotic three-dimensional nonlinear flow.

Consider first the simple quadratic case

$$\frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + 5.2\frac{d^2x}{dt^2} + 2.7\frac{dx}{dt} = 4.5(x^2 - 1) \tag{7}$$

The Lyapunov exponent (LE) spectrum is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.185, 0, -0.483, -0.7)$, resulting in a Lyapunov dimension $D_L = 2.38$ for initial conditions of $(\frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x)_{t=0} = (0.01, 0.01, 0.01, 4)$.

A cubic case was found with a maximized LLE for all polynomial hyperjerks (we searched up to a 5th order polynomial), and it is given below:

$$\frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + 10.4\frac{d^2x}{dt^2} + 8.4\frac{dx}{dt} = -9.3x^3 + 2.4x^2 + 13.6x - 1 \tag{8}$$

The spectrum of LEs was found to be $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.4, 0, -0.23, -1.18)$, and the dimension $D_L = 3.15$, and its dynamics are very fast. The initial conditions used were $(\frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x)_{t=0} = (0, 0, 0, 5)$.

A case with the arctan(x) nonlinearity is given by

$$\frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + 2.6\frac{d^2x}{dt^2} + 2.4\frac{dx}{dt} = 1.9x - \tan^{-1}(200x) \tag{9}$$

This kind of nonlinearity is important because such a hyperjerk system can be easily implemented electronically using operational amplifiers whose open-loop gain approximates the arctangent. Its LEs spectrum is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) =$

(0.2, 0, −0.165, −1.03) resulting in a dimension $D_L = 3.03$ with initial conditions used $(\frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x)_{t=0} = (0.02, -0.33, -0.27, 0.25)$. Its Poincaré sections are given in Fig. 1.

Continuing the numerical investigation of chaotic hyperjerk systems with $n = 4$ leads to examples with more complicated nonlinearities such as

$$\frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + 1.87\frac{d^2x}{dt^2} + 2.34\frac{dx}{dt} = 5.45x + x^2\frac{d^2x}{dt^2} \tag{10a}$$

Its LE spectrum is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.37, 0, -0.53, -0.84)$, and the dimension $D_L = 2.7$ with initial conditions $(\frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x)_{t=0} = (-0.32, 0.15, -0.39, -0.36)$. Its Poincaré section is given in Fig. 2. An optimized case (relative to D_L this time and not to the LLE) similar to (10a) is given by

$$\frac{d^4x}{dt^4} + 0.25\frac{d^3x}{dt^3} + 2.2\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} = -1.9x - 3.2x^2\frac{d^2x}{dt^2} \tag{10b}$$

In this case the damping term a_0 is equal to 1/4 instead of unity, and hence its dimension is higher. The spectrum of LEs is (0.284, 0, −0.108, −0.425), and the dimension is $D_L = 3.42$. In this example it is clearly shown how the LLE is independent of the Lyapunov dimension D_L and that a maximally chaotic flow does not guarantee its maximal complexity.

Lastly, we present the maximally chaotic hyperjerk system

$$\frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 2.95\frac{dx}{dt} = -3.93\sin(x)\frac{d^2x}{dt^2} - 1.47 \tag{11}$$

with an LLE = 1.94. Its LE spectrum is (1.94, 0, −0.4, −2.54), resulting in $D_L = 3.6$. Initial conditions were $(\frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x)_{t=0} = (-0.37, -0.11, 0, 0.9)$. From all numerical examples of hyperjerks searched, Eq. (11) was found to be the most chaotic. The nonlinearity sin(x) was found to be one of the strongest of all nonlinearities. Many other cases were found with the sin(x) nonlinearity with a similarly large LLE.

For the hyperjerks of Eqs. (8) and (9), the dimension barely exceeded 3. For the systems of Eqs. (10b) and (11), the dimension is large compared to these previous cases. This demonstrates that a hyperjerk flow with $n = 4$ is capable of producing attractors whose dimension can be tuned anywhere between 2 and 4 by tuning the LLE and the damping term a_0 since other more complicated examples were also found with $D_L > 3.7$. The latter is a further step in the

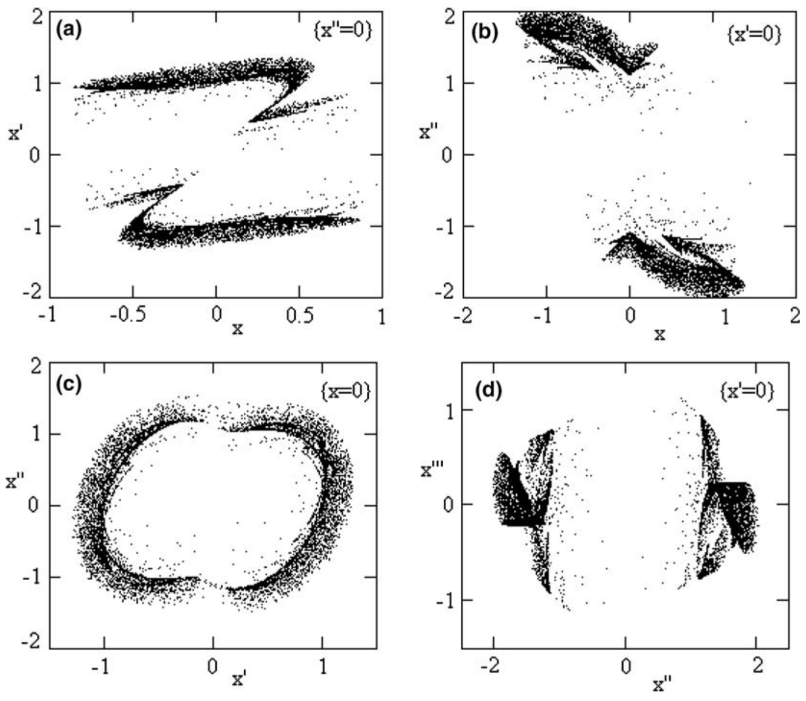

Fig. 1. Poincaré sections of Eq. (9).

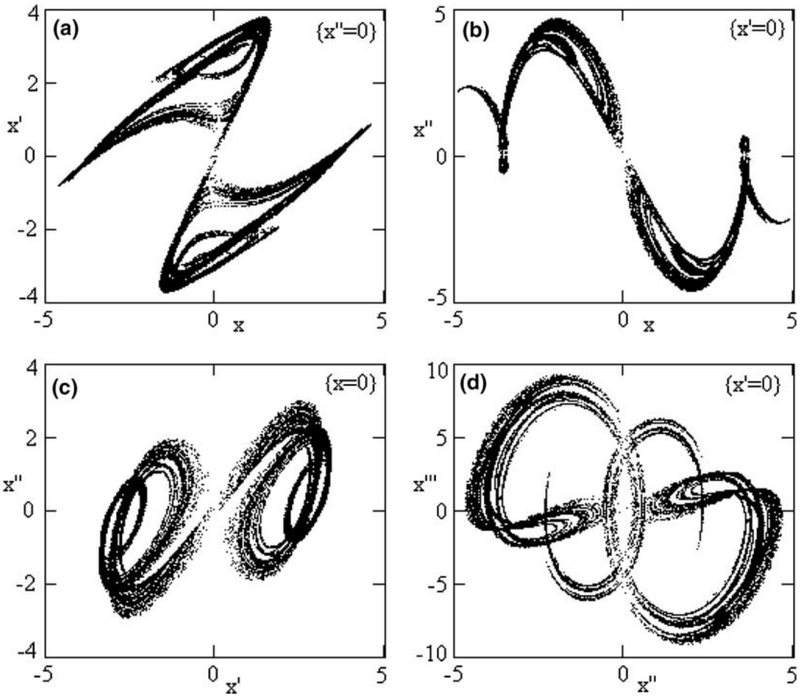

Fig. 2. Poincaré sections of Eq. (10a).

investigation of nonlinear flows like those in [10] for three-dimensional nonlinear systems that are capable of filling most of their phase-space.

In this section, we presented some chaotic hyperjerk flows with $n = 4$. Each case was optimized for the highest largest Lyapunov exponent (LLE) while the trace of the Jacobian matrix (the rate of state-space contraction) was kept constant at -1 for reasons of simplicity. We demonstrated one case that was optimized relative to its dimension D_L by varying the damping term of the hyperjerk. All these cases have an LEs spectrum of the form $(+, 0, -, -)$. In the next section, we will present two hyperchaotic cases with $n = 4$ showing that these systems can produce such complex dynamics.

3. Hyperchaotic hyperjerk flows with *n* = 4

In the previous section we presented some chaotic hyperjerk systems with one positive LE. Here we will demonstrate that a system written in a jerk form with $n = 4$ can also produce hyperchaotic dynamics with an LEs spectrum of the form $(+, +, 0, -)$. One such system, with a constant damping term, is given by

$$\frac{d^4x}{dt^4} + 0.1\frac{d^3x}{dt^3} - \left(3x - 4\frac{d^2x}{dt^2}\right)\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{dx}{dt}\right)^3 = -2x^3 \quad (12)$$

The LEs spectrum is $(0.141, 0.0168, 0, -0.257)$ with $D_L = 3.61$ with initial conditions $\left(\frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x\right)_{t=0} = (0.1, 0.1, 0.1, 0.1)$. Its Poincaré plots are given in Fig. 3. Note that the damping term of Eq. (12) is small and equal to 0.1. The hyperchaos is robust with respect to parameter variation except for the damping term. The hyperchaotic Rossler flow [11] can be written in hyperjerk form [12], but the result is extremely complicated and inelegant in contrast to all the cases presented here.

The hyperjerk system in Eq. (12) above has eight terms in its dynamical system representation, in contrast to the Rossler hyperchaotic flow that has nine terms, but it has four nonlinearities in contrast to the Rossler flow with only one. Generally, our search has not revealed simpler hyperchaotic flows with $n = 4$ (for example, with fewer nonlinearities), but this is beyond the scope of this paper. The hyperjerk system of Eq. (12) illustrates that such systems can produce a variety of dynamical behaviors (chaos, hyperchaos, tuning of the dimension, etc.).

In the previous sections, hyperjerk forms were chosen that have a constant state-space contraction for reasons of simplicity. Next we present a very simple and elegant hyperjerk that results in hyperchaos, has only one parameter and only seven terms with a state-space contraction that depends on the nonlinearity of the system. This case is conjectured to be the simplest hyperchaotic flow and is given by

$$\frac{d^4x}{dt^4}+\frac{d^3x}{dt^3}x^4+A\frac{d^2x}{dt^2}+\frac{dx}{dt}+x=0 \quad (13)$$

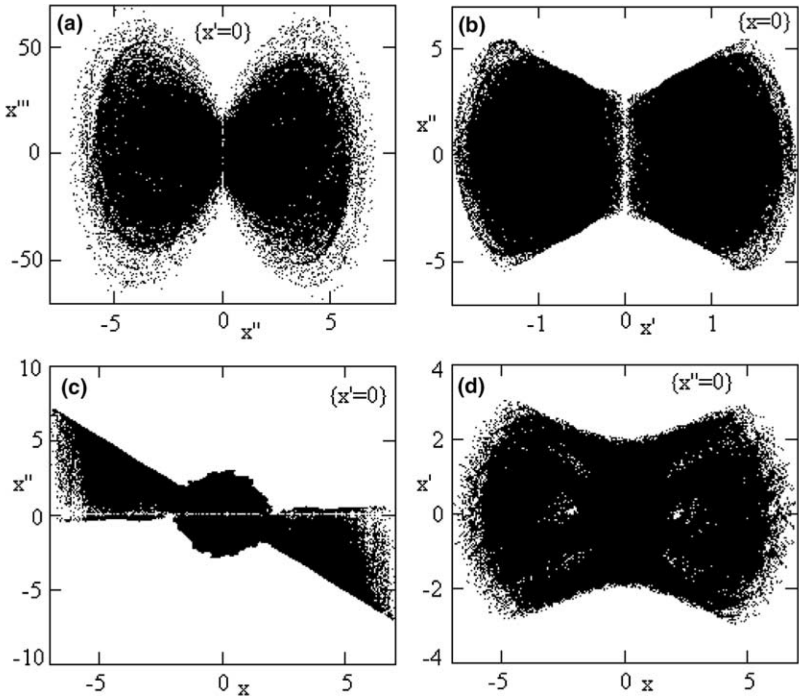

Fig. 3. Poincaré sections for the hyperchaotic case of Eq. (12).

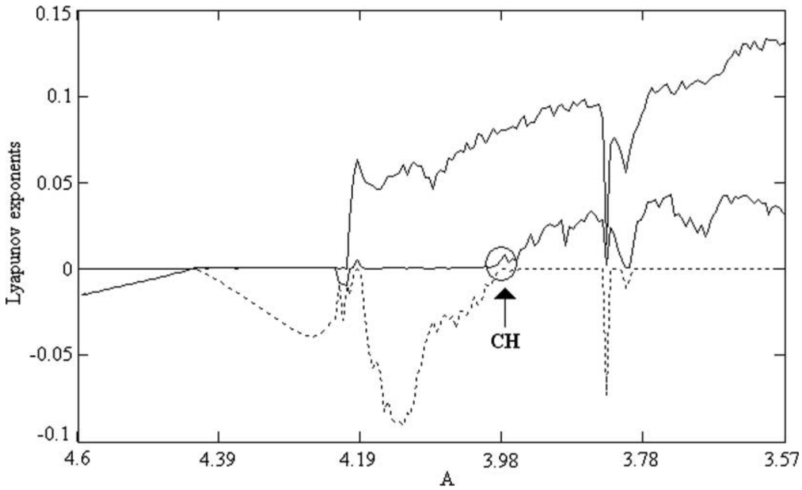

Fig. 4. Largest three Lyapunov exponents with varying A for the case of Eq. (13). The point "CH" with the arrow shows the route from chaos to hyperchaos.

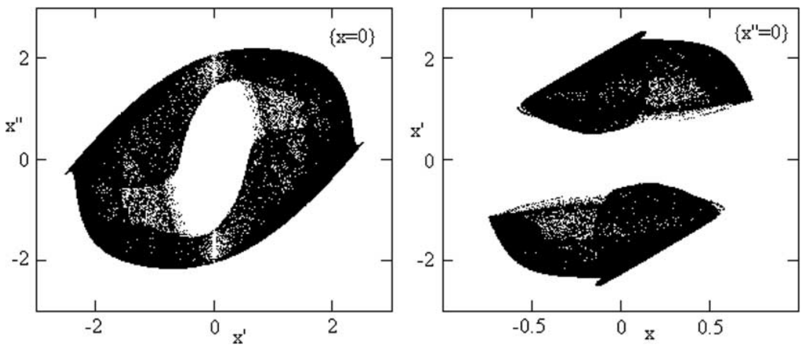

Fig. 5. Poincaré sections for the hyperchaotic case of Eq. (13) for $A = 3.6$.

Fig. 4 shows the LEs spectrum as a function of the parameter A. For reasons of simplicity and for better visualization, only the first three LEs are plotted. For $A \approx 3.98$ this hyperjerk system experiences a route from chaos to hyperchaos as shown in Fig. 4. The initial conditions were: $\left(\frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x\right)_{t=0} = (0.1, 0.1, 0.1, 0.1)$. The maximum LLE is 0.132 for $A = 3.6$ resulting in $D_L = 3.13$, and its whole LEs spectrum is $(0.132, 0.035, 0, -1.25)$. Its correlation dimension was also calculated using the method in [13] and was found to be $D_2 = 3.126 \pm 0.184$. Fig. 5 shows Poincaré plots for this case with $A = 3.6$.

In the next section, we present hyperjerk flows that are apparently the simplest chaotic flows for $n = 4$ using various nonlinearities following the work of Sprott [3].

4. Simplest chaotic flows with $n = 4$

It is already known that for a flow to exhibit chaos it must have at least three degrees of freedom [2]. The simplest chaotic flows for $n = 3$ have been known for several years now [3]. An obvious extension for the latter is to find the simplest chaotic flows with $n = 4$. Next, we present two hyperjerk systems using different nonlinearities that are in their simplest possible form.

The first hyperjerk system is given by

$$\frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + A\frac{d^2x}{dt^2} + \frac{dx}{dt} = g(x) \tag{14}$$

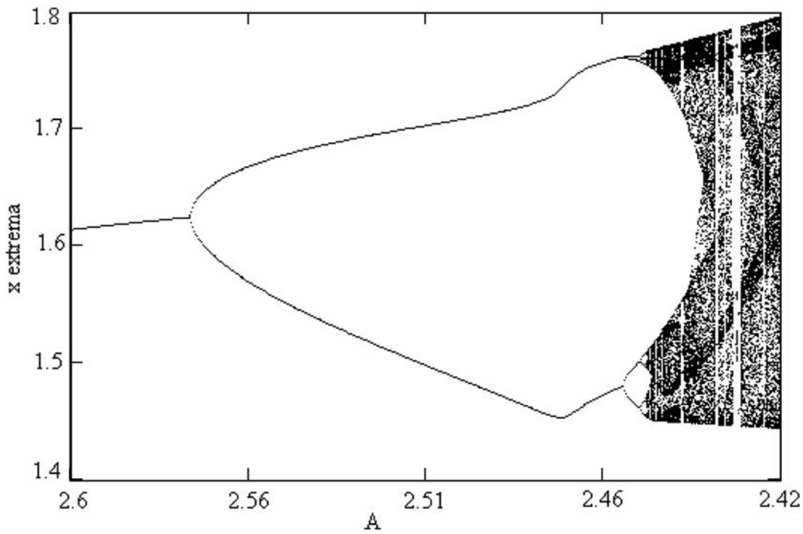

Fig. 6. Bifurcation diagram for Eq. (14).

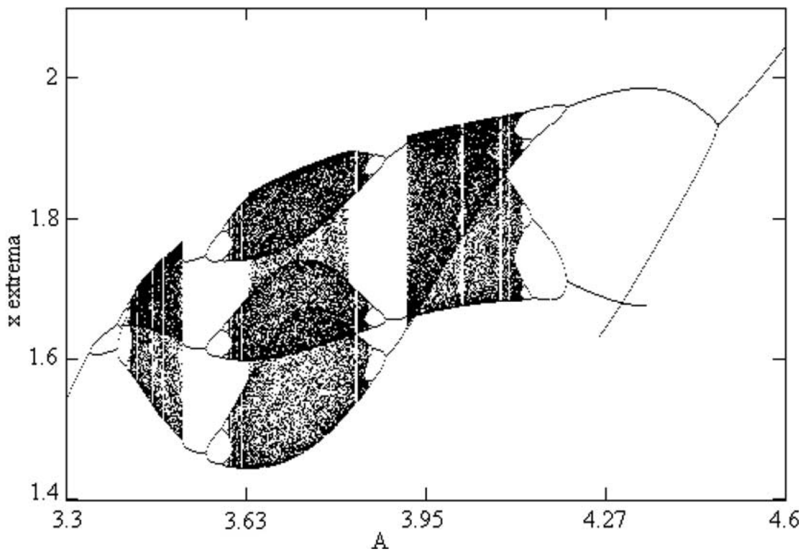

Fig. 7. Bifurcation diagram for Eq. (15).

For $2.42 < A < 2.6$ and for the nonlinearity $g(x) = x^2 - 1$, this flow exhibits a period-doubling route to chaos as shown in Fig. 6. The initial conditions used were: $(\frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x)_{t=0} = (-0.11, -0.55, 0.5, 0.01)$. Its maximum Lyapunov exponent occurs for $A = 2.42$ where LLE = 0.078 and the Kaplan–Yorke dimension is $D_L = 2.17$.

The second simplest hyperjerk with only seven terms in its dynamical system representation and with a different nonlinearity is given by

$$\frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + A\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + x = 0 \tag{15}$$

For $3.3 < A < 4.6$, the flow exhibits a very interesting and rich dynamical behavior. The bifurcation diagram of Fig. 7 shows the extrema of the x as a function of A. We note the existence of period-doubling routes to chaos and also the well-known windows of period-3. The most interesting phenomenon though is the antimonotonicity effect of the reverse period doubling cascades and the sudden transition to chaos for $A = 3.9$. The initial conditions used were: $(\frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x)_{t=0} = (-0.85, 0.26, -0.48, -0.18)$. The maximum Lyapunov exponent occurs for $A = 3.48$ where LLE = 0.031 and the Kaplan–Yorke dimension is $D_L = 2.068$.

5. Hyperjerk flows for $n > 4$

In this section we extended the numerical investigation even further, including cases of hyperjerk flows for $n > 4$. Chaos was found to be common for these higher dimensions. In contrast to Eq. (4) though, it was found after an exhaustive numerical search that chaotic phenomena are more common when the hyperjerk is of the general form:

$$\frac{d^{(n)}x}{dt^{(n)}} + a_0\frac{d^{(n-1)}x}{dt^{(n-1)}} + \cdots + a_n\frac{dx}{dt} = g(x) \tag{16}$$

where $a_0, \ldots, a_n$ are constant parameters. Hence it is concluded that a linear superposition of all the time-derivatives helps to bound all the variables if $g(x)$ is finite. Perhaps this is a generalization of a simple harmonic oscillator or even of many coupled harmonic oscillators. If the linear superposition of the left-hand side of Eq. (16) is not applied and instead Eq. (4) is used, then unbounded solutions were dominate the dynamical behavior of these hyperjerk systems.

Hence, by extending Eq. (4) according to Eq. (16) we come to a 5th order hyperjerk with one only quadratic nonlinearity similar to Eq. (7) that is optimized relative to its LLE:

$$\frac{d^5x}{dt^5} + \frac{d^4x}{dt^4} + 7.068\frac{d^3x}{dt^3} + 3.94\frac{d^2x}{dt^2} + 9.17\frac{dx}{dt} = 3.9(x^2 - 1) \tag{17}$$

This hyperjerk equation results in a low-dimensional attractor. The initial conditions used were $(\frac{d^4x}{dt^4}, \frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x)_{t=0} = (0.1, 0.1, 0.1, 0.1, 0.1)$. The LE spectrum was found to be $(0.159, 0, -0.242, -0.315, -0.6)$ with $D_L = 2.65$.

Similarly to Eq. (9) we present a chaotic hyperjerk system for $n = 5$ with the arctan(x) nonlinearity:

$$\frac{d^5x}{dt^5} + \frac{d^4x}{dt^4} + 7.278\frac{d^3x}{dt^3} + 4\frac{d^2x}{dt^2} + 9.19\frac{dx}{dt} = -7.9x + 2.06\tan^{-1}(200x) \tag{18}$$

The initial conditions used were again $(\frac{d^4x}{dt^4}, \frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x)_{t=0} = (0.1, 0.1, 0.1, 0.1, 0.1)$, and the LE spectrum is $(0.257, 0, -0.217, -0.36, -0.68)$ with $D_L = 3.11$.

Continuing with the investigation of hyperjerk flows with $n = 5$, we come to a very interesting case that results in hyperchaos and that can be implemented electronically as described for the case of Eq. (9):

$$\frac{d^5x}{dt^5} + 0.46\frac{d^4x}{dt^4}G(x) + 8.5\frac{d^3x}{dt^3} + 3.4\frac{d^2x}{dt^2} + 8.9\frac{dx}{dt} + 5.2x - 1.256G(x) = 0 \tag{19}$$

The nonlinear function $G(x)$ is the Heaviside step function, which in the following numerical simulations it was well approximated by

$$G(x) \cong \frac{\tan^{-1}(Ax) + \pi/2}{\pi}$$

with $A \gg 1$. Values of A in the range $100 < A < 800$ give similar results. The case above is hyperchaotic with an LE spectrum $(0.0637, 0.0213, 0, -0.038, -0.454)$ and $D_L = 4.1$. The initial conditions used were $(\frac{d^5x}{dt^5}, \frac{d^4x}{dt^4}, \frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x)_{t=0} = (0.1, 0.1, 0.1, 0.1, 0.1)$. As with the hyperchaotic case of Eq. (13), the nonlinearity is in the damping term.

6. Conclusions

In this paper, we investigated several chaotic flows for $3 < n < 6$ called "hyperjerks" describing the time evolution of a single scalar variable. Their elegance and surprising simplicity allow one to easily investigate their dynamical properties and furthermore to construct minimal chaotic systems, some of them presented here with the use of bifurcation diagrams and Lyapunov exponent calculations. This work leads one step further in the investigation of dynamical systems and specifically of autonomous systems, since chaotic flows with $n = 3$ have been already extensively studied. Many hyperjerk systems presented here are suitable also for experimental realization such as with electrical circuits. Additionally, several hyperchaotic systems were presented for $3 < n < 6$, one of these having only one nonlinearity and seven terms, simplifying even more the famous hyperchaotic Rössler system. The numerical investigation of such hyperjerk flows in even higher dimensions is anticipated.

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