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On the synchronization of a class of electronic circuits that exhibit chaos

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Abstract

The synchronization of two nonlinear electronic circuits that exhibit chaos is numerically demonstrated using techniques from modern control theory. These circuits have been used to model a "jerk" equation and can either be identical or not identical. The technique is initially described using linear circuits. © 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

There is considerable interest in developing and analyzing various mathematical systems that exhibit chaos. In addition, there is interest in designing experiments that will verify the predicted mathematical results. It has been found that electronic circuits that consist of possibly one or two nonlinear elements can be used to verify several theoretical predictions. For example, nonlinear Duffing oscillators have been experimentally studied [1]. The nonlinear Chua diode along with its accompanying circuit has been proposed, built, and experimentally examined [2].

The electronic circuits that are described above are fairly complicated in that they may require a fairly large number of elements in their construction. In addition, they require an inductance in their realization. This inductance, with its inherent resistance, may limit the applicability and performance of the electronic circuit. Note that skin effects often lead to frequency-dependent losses and nonlinearities such as hysteresis and saturation. An inductor-less Chua's circuit has been examined [3].

Recently, a family of nonlinear electronic circuits containing just capacitors, resistors, diodes, and op-amps was proposed and experimentally demonstrated [4]. These circuits represented a "jerk" equation which is a third-order ordinary differential equation. The jerk equation determines the time derivative of the acceleration of an object and it can be described with the following set of three first-order simultaneous ordinary differential equations. The dependent variables in this set of equations are: the position x, the velocity v, and the acceleration a

$$\begin{aligned}\frac{dx}{dt} &= v,\\ \frac{dv}{dt} &= a,\\ \frac{da}{dt} &= -Aa - v + G(x),\end{aligned} \tag{1}$$

where $G(x)$ is a nonlinear element. The parameter A is a numerical constant. It was found that the value of this constant ($A \approx 0.6$) would lead to a chaotic solution for the set of equations with different nonlinear elements described below.

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With different combinations of resistors, capacitors, and operational amplifiers, a family of nonlinear elements was investigated. In particular, these combinations lead to four different $G(x)$ described by the following equations:

$$G(x) = \begin{cases} |x| - 2 & \text{(a)} \\ -6\max(x,0) + 0.5 & \text{(b)} \\ 1.2x - 4.5\,\mathrm{sgn}(x) & \text{(c)} \\ -1.2x + 2\,\mathrm{sgn}(x) & \text{(d)} \end{cases} \tag{2}$$

Additional combinations of elements were also described. However, the above set was treated in full detail.

In the present paper, we describe a series of calculations directed to the synchronization of the above circuits using methods of modern control theory. These methods have recently been employed to synchronize two Lorenz systems [5,6], and two logistic systems [7], such that the frequency of oscillation of both systems would become identical. There is an extensive list of references to related works in these papers. In preparation for this examination, the chaotic response of the four circuits will be detailed in Section 2. The phase plane solution for each system will be presented. This will be compared with previously obtained numerical and experimental results. In Section 3, we summarize the synchronization technique and apply it to the synchronization of two linear systems that have different natural frequencies of oscillation. The synchronization of the nonlinear systems is described in Section 4. Section 5 is the conclusion.

2. Nonlinear circuit

In a series of calculations, we initially examined the phase plane for each of the four systems described with Eqs. (1) and (2). The results of this simulation are shown in Fig. 1. In order to compare and contrast the behavior of the four circuits under identical conditions, the initial conditions for all systems were chosen to be identical. We chose the initial conditions: $x_0 = 0.01$, $v_0 = 0.0$, and $a_0 = 0.0$. The simulation of the phase space diagrams obtained for each of the four systems is identical with the numerical and experimental results obtained previously [4]. See [4, Fig. 3]. This agreement adds veracity to our computer program in that the nonlinear elements that we have created using the Simulink program in MATLAB agrees with the earlier results.

3. Synchronization of two linear systems

Before synchronizing the set of nonlinear equations described in (1) and (2), we synchronize two sets of linear equations. This will allow us to clearly illustrate the procedure that is to be followed. Consider a linear system #1 that is described with

$$\begin{aligned} \frac{dx_1}{dt} &= v_1, \\ \frac{dv_1}{dt} &= -mx_1. \end{aligned} \tag{3}$$

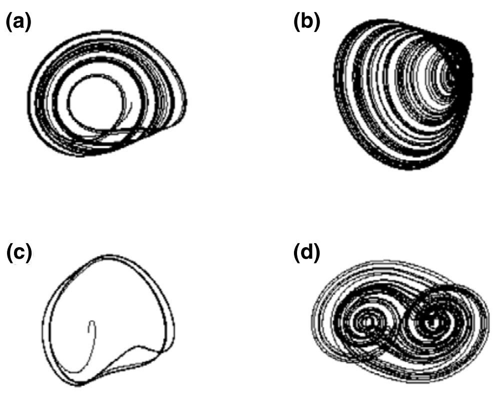

Fig. 1. Phase space diagrams using the four nonlinear elements defined in Eq. (2).

This system will be called the *master*. In addition, there is a second linear system #2 that is described with

$$\frac{dx_2}{dt} = v_2 + \alpha(t),$$
$$\frac{dv_2}{dt} = -nx_2 + \beta(t). \tag{4}$$

The system #2 will be called the *slave*. The object of the control is to synchronize the slave to the master with the control variables $\alpha(t)$ and $\beta(t)$. These variables have not yet been specified.

In order to determine the control variables, we examine the difference between the two distances and the two velocities

$$x_3 = x_2 - x_1,$$
$$v_3 = v_2 - v_1. \tag{5}$$

If these two variables x_3 and v_3 can be made to approach zero as time increases, we can say that the two systems are *synchronized* in that the slave will follow the master.

Using (5) and the two sets of equations defining the master and the slave, we write

$$\frac{dx_3}{dt} = v_3 + \alpha(t),$$
$$\frac{dv_3}{dt} = -nx_2 + mx_1 + \beta(t). \tag{6}$$

There are several possible choices for the control variables that will cause the differences to approach zero as time increases. One possible set is given by

$$\alpha(t) = 0,$$
$$\beta(t) = nx_2 - mx_1 - a_1(x_2 - x_1) - a_2(v_2 - v_1). \tag{7}$$

With these control variables, (6) can be rewritten in the following notation:

$$\begin{pmatrix} \frac{dx_3}{dt} \\ \frac{dv_3}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix} \begin{pmatrix} x_3 \\ v_3 \end{pmatrix}. \tag{8}$$

Both the differences will approach zero if the roots of the following determinant have negative real parts

$$\begin{vmatrix} \lambda & -1 \\ a_1 & \lambda + a_2 \end{vmatrix} = \lambda^2 + a_2\lambda + a_1 = 0, \tag{9}$$

which is guaranteed if the constants $a_1 > 0$ and $a_2 > 0$.

The synchronization of a linear slave to a linear master is shown in Fig. 2. The initial amplitude of the master is less than the initial amplitude of the slave. The frequencies of oscillation of the two systems are also different. The controller is applied at a time $t = 50$. We observe that the slave quickly follows the master in both amplitude and frequency. The response time is governed by the choice of the constants a_1 and a_2.

Having demonstrated the procedure of synchronizing two linear systems, we now investigate the synchronization of two nonlinear systems that describe the above nonlinear circuits.

4. Synchronization of two nonlinear circuits

The chaotic circuit that has the subscript 1 will again be called the "master" and the chaotic circuit that has the subscript 2 will be called the "slave". The equations that describe the master are:

$$\frac{dx_1}{dt} = v_1,$$
$$\frac{dv_1}{dt} = a_1,$$
$$\frac{da_1}{dt} = -Aa_1 - v_1 + G(x_1). \tag{10}$$

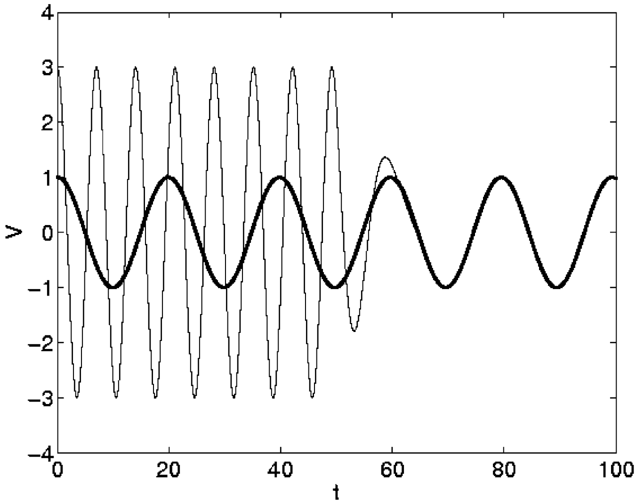

Fig. 2. Synchronization of two linear systems with a controller applied at the time $t = 50$.

The equations that describe the slave are:

$$\frac{dx_2}{dt} = v_2 + \alpha(t),$$
$$\frac{dv_2}{dt} = a_2 + \beta(t), \tag{11}$$
$$\frac{da_2}{dt} = -Aa_2 - v_2 + H(x_2) + \gamma(t).$$

The nonlinear system of the slave does not have to be the same as that of the master. We indicate this by using the representation $G(x)$ and $H(x)$. The variables $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are the nonlinear control variables that have to be determined.

Following the same procedure as outlined above, we write

$$x_3 = x_2 - x_1,$$
$$v_3 = v_2 - v_1, \tag{12}$$
$$a_3 = a_2 - a_1.$$

The difference of the three variables satisfies

$$\frac{dx_3}{dt} = v_3 + \alpha(t),$$
$$\frac{dv_3}{dt} = a_3 + \beta(t), \tag{13}$$
$$\frac{da_3}{dt} = -Aa_3 - v_3 + H(x_2) - G(x_1) + \gamma(t).$$

The control variables are similarly defined as

$$\alpha(t) = V_a(t),$$
$$\beta(t) = V_b(t), \tag{14}$$
$$\gamma(t) = -H(x_2) + G(x_1) + V_c(t),$$

where

$$\begin{pmatrix} V_a(t) \\ V_b(t) \\ V_c(t) \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & +1 & -1+A \end{pmatrix} \begin{pmatrix} x_3 \\ v_3 \\ a_3 \end{pmatrix}. \tag{15}$$

Then, the differences will satisfy

$$\begin{aligned}\frac{dx_3}{dt} &= -x_3,\\ \frac{dv_3}{dt} &= -v_3,\\ \frac{da_3}{dt} &= -a_3.\end{aligned} \tag{16}$$

There are other choices that can be selected.

In Fig. 3, we simulate the synchronization of two nonlinear circuits that contain two identical nonlinear elements defined with (2). The initial conditions for the two circuits are different. The control is applied at the time $t = 50$.

The controller variables for the four nonlinear circuits are described by

$$\gamma(t) = \begin{cases} |x_1| - |x_2| & \text{(a)}\\ -6\max(x_1, 0) + 6\max(x_2, 0) & \text{(b)}\\ 1.2x_1 - 4.5\,\mathrm{sgn}(x_1) - 1.2x_2 + 4.5\,\mathrm{sgn}(x_2) & \text{(c)}\\ -1.2x_1 + 2\,\mathrm{sgn}(x_1) + 1.2x_2 - 2\,\mathrm{sgn}(x_2) & \text{(d)}\end{cases} \tag{17}$$

In Fig. 4, we simulate the synchronization of two nonlinear circuits that contain two different nonlinear elements defined with Eqs. 2(a) and (b). The controller is defined with

$$\gamma(t) = \lfloor |x_1| - 2 \rfloor - [-6\max(x_2, 0) + 0.5]. \tag{18}$$

The control is applied at the time $t = 50$.

It is also possible to synchronize a nonlinear circuit with a linear circuit. This, however, requires a redevelopment of the controller. In particular, the linear circuit described in (3) is defined with two equations and the nonlinear circuit described in (10) is defined with three equations. An ‘‘artificial’’ third equation must be introduced into the set (3). We find that the three equations are:

$$\begin{aligned}\frac{dx_1}{dt} &= v_1,\\ \frac{dv_1}{dt} &= -mx_1,\\ \frac{da_1}{dt} &= \frac{d^2v_1}{dt^2} = -m\frac{dx_1}{dt} = -mv_1.\end{aligned} \tag{19}$$

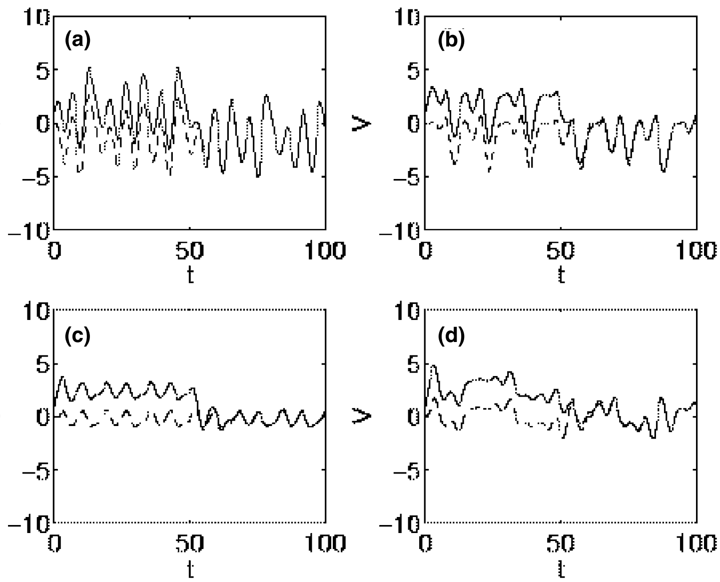

Fig. 3. Synchronization of two nonlinear circuits with the same structure using the nonlinear elements defined in Eq. (2).

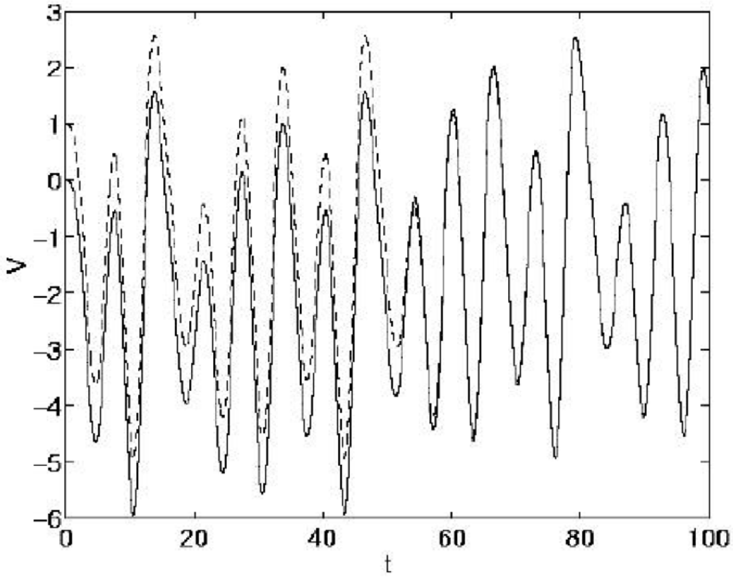

Fig. 4. Synchronization of two nonlinear circuits with different nonlinearities.

In (19), the third equation is the artificial equation since the linear circuit is clearly defined by the first two equations. To make this third-order differential equation to be equivalent to the original second-order equation, there is a constraint on the initial conditions that is given by

$$a_1(0) = \left.\frac{\mathrm{d}v_1}{\mathrm{d}t}\right|_0 = -mx_1(0). \tag{20}$$

In Fig. 5, we simulate the synchronization of the nonlinear circuit containing the nonlinear element defined with 2(a) to the linear circuit described previously. The controller is defined with

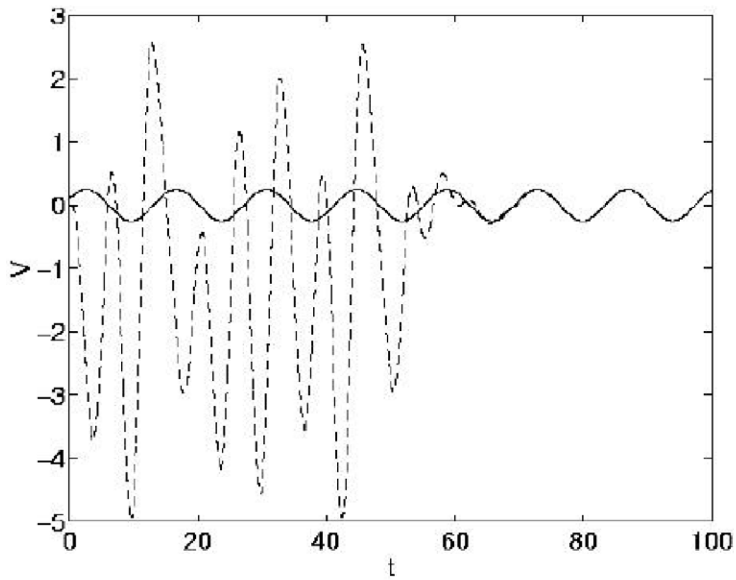

Fig. 5. Synchronization of a nonlinear circuit with a linear circuit.

$$\alpha(t) = 0,$$
$$\beta(t) = -mx_1 - a_1, \tag{21}$$
$$\gamma(t) = -H(x_2) + b_1(x_1 - x_2) + v_2(1 - b_2) + v_1(b_2 - m) + a_2(A - b_3) + b_3 a_1.$$

The difference equation is consequently given by

$$\begin{pmatrix} \frac{dx_3}{dt} \\ \frac{dv_3}{dt} \\ \frac{da_3}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -b_1 & -b_2 & -b_3 \end{pmatrix} \begin{pmatrix} x_3 \\ v_3 \\ a_3 \end{pmatrix}. \tag{22}$$

The control is applied at the time $t = 50$, where $b_1 = 1$ and $b_2 = b_3 = 2$.

5. Conclusion

Using numerical techniques, we have demonstrated that it is possible to synchronize two nonlinear electronic circuits that have been used to model a jerk equation. The technique that we have employed is straightforward and should be applicable to several systems of chaotic equations.

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