

## Extraction of dynamical equations from chaotic data

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A method is described for extracting from a chaotic time series a system of equations whose solution reproduces the general features of the original data even when these are contaminated with noise. The equations facilitate calculation of fractal dimension, Lyapunov exponents and short-term predictions. The method is applied to data derived from numerical solutions of the logistic equation, the Hénon equations with added noise, the Lorenz equations and the Rössler equations.

### 1. Introduction

In many fields of science one measures quantities that fluctuate in time or space with no discernible pattern. Examples include magnetic and electric fields in plasmas, weather and climatological data, variation of biological populations, and stock prices. It has been generally assumed that such situations could be described by a large number of deterministic equations or by stochastic ones. More recently it has been appreciated that ordinary, but nonlinear, differential equations with as few as three degrees of freedom or difference equations with a single degree of freedom can have pseudo-random (chaotic) solutions. This has led to the hope that such simple systems can model the real world.

Ideally, one would like to be able to extract the equations from a fluctuating time series. In the absence of additional information, this goal is unrealistic. The variable observed may not be simply related to the fundamental dynamical variables of the system. The measurement will be contaminated by noise and round-off errors and limited by sample rate and duration. How-

ever, it may be possible to find a system of equations which mimic the general features such as the topology in a suitable phase space, and these equations might shed insight into the behavior of the system.

Here we present a method for extracting from a fluctuating time series such a set of equations. These model equations may be used to predict not so much the details of the time evolution, which is limited by sensitivity to initial conditions in chaotic systems, but topological changes such as the change of periodic behavior through a series of period-doubling bifurcations. Furthermore, because the model equations provide in principle an unlimited amount of data, the calculation of fractal dimension [1] and Lyapunov exponents [2] is much simplified, although the relationship between such calculated quantities and the true values remains an intriguing and open question. It is also much simpler to calculate the Lyapunov exponents directly from the equations rather than from the data [3, 4].

Of course there have been previous attempts to extract from a time series a simple set of coupled equations whose solution gives an ap-

proximation in some sense to the original data. Difference equations [5–7] corresponding to maps, and differential equations [8–10] corresponding to flows have been deduced. Although the method described here has features common to this earlier work, the novel element is the use of singular value decomposition to choose the appropriate dependent variables that appear in the dynamical equations rather than just using the data and its derivatives. An added advantage of singular value decomposition is that it provides an efficient filter for the noise that is always present in experimental data.

## 2. Numerical procedure

Until fairly recently the main method of analysis has been to express the time series  $T(t)$  in a set of Fourier modes. Then peaks in the associated power spectrum are identified with normal modes of the system. This approach breaks down if there is a large noisy contribution to the measurement or if the underlying system cannot be described in terms of a few modes.

An alternative is to use the method of singular value decomposition [11, 12]. In this method  $T(t)$  is expanded in a complete set of modes  $\psi_m(t)$ , not necessarily Fourier modes, but a set obtained from an analysis of the data rather than imposed from outside. The modes are normalized according to

$$\lambda_m = \frac{1}{N} \sum_{n=1}^N |\psi_m|^2(t = n\tau), \quad (1)$$

where the original data are assumed given at discrete times  $n\tau$  with  $1 \leq n \leq N$ . We then approximate  $T(t)$  by

$$T(t) = \sum_{m=1}^d \psi_m(t), \quad (2)$$

where the  $\psi_m$ 's are chosen to correspond to the  $d$  largest values of  $\lambda_m$ .

In practice, from the data  $T_n (= T(t = n\tau))$  one constructs a set of  $M$ -dimensional vectors  $V_l$  defined such that  $V_l = \{T_l, T_{l+1}, \dots, T_{M+l-1}\}$  and the auto-correlation function  $C(n)$  defined such that

$$C(n) = \sum_{l=1}^N T_l T_{l+n}. \quad (3)$$

Using these  $C(n)$ 's one constructs the symmetric  $M \times M$  correlation matrix  $\mathbf{M}$  with elements  $M_{lp} = C(|l - p|)$ . The eigenvalues of this matrix are in fact just the normalization constants introduced in eq. (1), that is  $\lambda_m$ . The corresponding eigenfunctions ( $\alpha_m$ ) of this matrix give the modes  $\psi_m(t)$  according to  $\psi_m(t = n\tau) = \alpha_m \cdot V_n$ .

Besides giving the best set of orthogonal modes in the sense mentioned above, this method involves some smoothing of the original data. A purely random time series gives  $C(n) = C\delta_{n,0}$  so that all the eigenvalues are equal to  $C$ .

If when the data are analyzed a few eigenvalues are significantly larger than the rest, then the corresponding eigenfunctions are the ones used in the approximate expansion for  $T(t)$  in eq. (2). The neglect of the rest has the effect of removing some of the noise. This whole procedure is analogous to identifying peaks in a Fourier power spectrum. The partial removal of noise by singular value decomposition has been discussed in more detail by Broomhead and King [11]. The choice of which of the eigenvalues are significant is not always obvious, and a subjective judgement has been used here. A more rigorous procedure would follow the treatment of Hediger et al. [13].

From a physical point of view the  $\psi_m$ 's for  $1 \leq m \leq d$  can be interpreted as coherent structures revealed by the method of singular value decomposition. Modern dynamical systems theory suggests that even small values of  $d$  may suffice to encapsulate the essential features of the system. These features are perhaps best appreciated by examining the  $d$ -dimensional phase space constructed using the functions  $\psi_1$ ,

$\psi_2, \dots, \psi_d$ . These topological features are the same as those present using a phase space constructed using the  $V_n$ 's since the  $V$ 's are linear combinations of the  $\psi_m$ 's. Also, features that are masked in the  $V$  phase space due to noise may be revealed in the  $\psi$  space since some of the noisy component has been removed. The more subtle point of whether a time series of a single variable known just at discrete time intervals can capture the full solution of the underlying problem which exists in continuous time and involves many independent variables has been considered by Takens [14].

Using the  $d$  distinct  $\psi_m$ 's we construct a model equation of the form

$$\psi_m^{n+1} = F_m(\psi_l^n), \tag{4}$$

where  $\psi_m^n \equiv \psi_m(t = n\tau)$  and the  $F_m$ 's are as yet general functions of the  $\psi$ 's. Guided by the fact that simple forms for the model functions  $F_m$  are sufficient to produce chaotic solutions we assume that

$$F_m(\psi_l) = a_{m0} + \sum_{p=1}^d a_{mp} \psi_p + \sum_{p,q}^d b_{mpq} \psi_p \psi_q + \sum_{p,q,r}^d c_{mpqr} \psi_p \psi_q \psi_r, \tag{5}$$

where the coefficients  $a$ ,  $b$ , and  $c$  are determined by minimizing the variational function

$$J_m = \frac{1}{N} \sum_{n=1}^N [\psi_m^{n+1} - F_m(\psi_l^n)]^2 \tag{6}$$

for each value of  $m$ . If examination of the phase-space portraits reveals any symmetry then this should be incorporated into the structure of  $F_m$ . Of course in some cases a simple polynomial may not be the appropriate form for  $F_m$ . For example if the phase-space plots reveal periodic structure then the model functions  $F_m$  should be chosen to capture such a structure. A suitable choice of polynomial has been shown [15] to model data arising from the presence of a limit

cycle. An  $F_m$  expressed as the ratio of two polynomials may have a significantly wider range of application than a simple polynomial since one is then using the power of a Padé approximant [16].

The use of such rational functions has been studied by Casdagli [5] and Gouesbet [10]. However, it is important to note that there is probably no universal panacea, and the form for the model functions  $F_m$  should be chosen taking into account all available information about the system. Computer software that carries out the procedure described above as well as many other tests for chaotic time series is available [17].

Singular value decomposition methods or equivalents have been used previously to obtain model equations [18–20], but in those cases the exact equations describing the system are assumed known. The exact equations are then used to generate the model equations. Here we only use the restricted information given by the time series.

Experimental time series  $T(t)$  can often be obtained for various values of some control parameter  $\mu$ . Thus the  $\psi_m$ 's and  $F_m$ 's are also functions of  $\mu$ . Numerical simulations of equations such as those of Lorenz [21] and Rössler [22] and the study of phase transformations using phenomenological models such as the Landau-Ginsburg equation give good reason to believe that the dependence of the coefficients  $a$ ,  $b$ , and  $c$  on  $\mu$  is simple.

### 3. Numerical examples

To illustrate some of the above techniques we have studied a few selected model situations.

#### 3.1. Logistic equation

First the logistic equation  $x_{n+1} = \lambda x_n(1 - x_n)$  has been iterated and  $x_n$  identified with  $T_n$ . For  $\lambda = 4$  the solution is illustrated in fig. 1a. The data have been analyzed using  $M = 2$ , and the

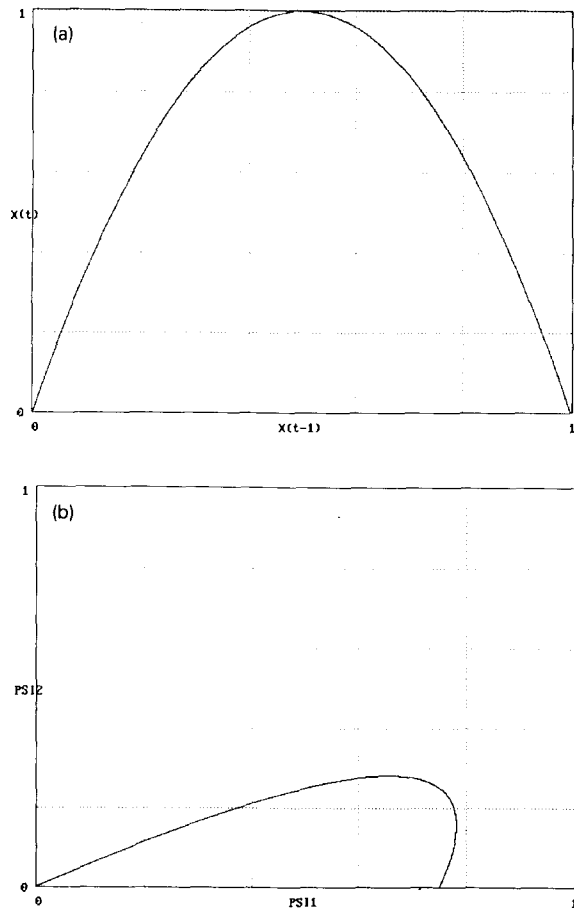


Fig. 1. Phase-space plots of the logistic equation (a) from original data and (b) after singular value decomposition.

phase plane in fig. 1b constructed from the model equations shows a single loop which is a simple distortion of fig. 1a. This simple loop structure still remains if larger values of  $M$  are used and the number of model equations is taken equal to  $M$ . Even without knowledge of the original equations that generated the data the method shows that a two-dimensional phase plane is sufficient to model the data, and the resulting equations can be linearly combined to recover the logistic equation exactly.

### 3.2. Hénon equations

The Hénon map,  $x_{n+1} = 1 - 1.4x_n^2 + 0.3y_n$ ,

$y_{n+1} = x_n$  has been treated in a similar manner. The results in fig. 2 using  $M = 2$  again show that the model equations capture the essential features of the solution, in this case a strange attractor.

A time series of 2300 values was generated by adding normally distributed deviates with zero mean and standard deviation of 0.1. These data for  $M = 2$  are shown in fig. 3a. Using these data, two coupled model equations were obtained, solved, and a new time series  $X_n = \psi_1^n + \psi_2^n$  generated. This is shown in fig. 3b and is indistinguishable from the Hénon map as given in fig. 2a.

In this case the reduction in noise is solely a

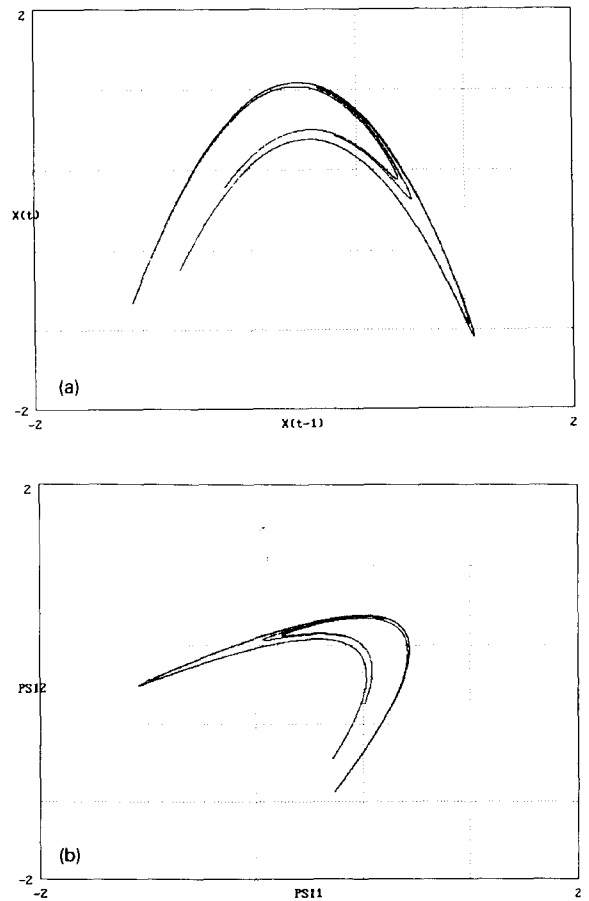


Fig. 2. Phase-space plots of the Hénon map (a) from original data and (b) after singular value decomposition.

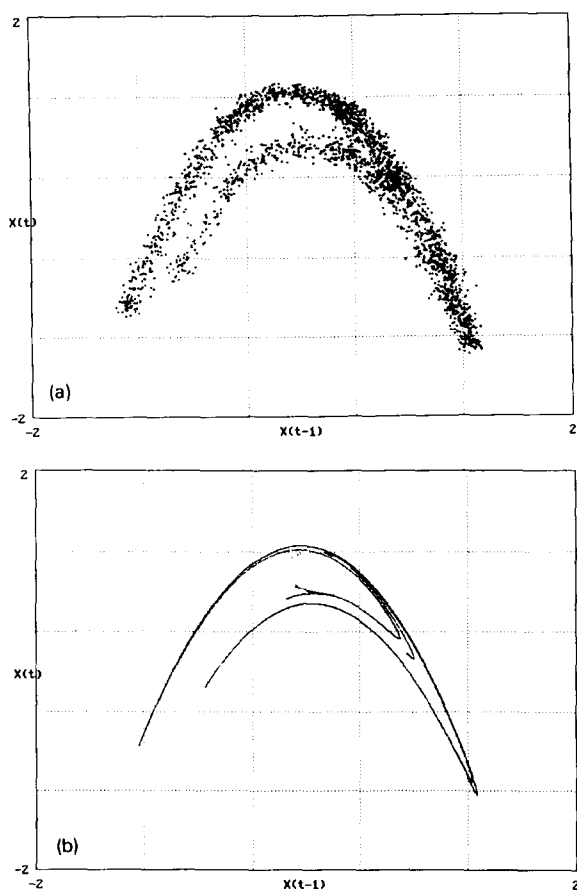


Fig. 3. Phase-space plots of the Hénon map (a) with noise added and (b) from a solution of the model equations fit to the noisy data. A comparison of (b) with fig. 2a shows that the method has completely removed the noise and restored the original data.

result of forcing the data to fit a relatively simple set of equations since only a  $2 \times 2$  correlation matrix was used and thus all the noise survives the singular value decomposition. Such a method should be used with caution since it tends to simplify the dynamics of the system.

### 3.3. Lorenz equations

The Lorenz equations

$$\begin{aligned}
 dx/dt &= \sigma(y - x), \\
 dy/dt &= rx - y - xz, \\
 dz/dt &= xy - bz,
 \end{aligned}
 \tag{7}$$

with  $\sigma = 10$ ,  $r = 28$ , and  $b = 8/3$  were solved numerically and 1000 values of  $x(t)$  at  $t = 0.05n$  taken as the input time series. These data are shown in fig. 4a. The neglect of the information contained in the variables  $y(t)$  and  $z(t)$  mirrors the experimental situation where only a limited amount of information is available. The corresponding phase space constructed using three

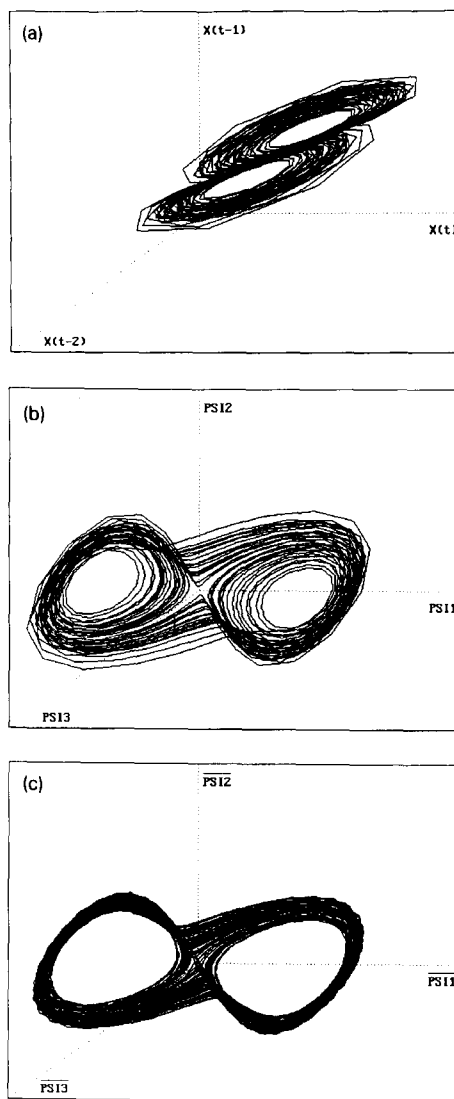


Fig. 4. Three-dimensional phase-space plots of the Lorenz attractor showing that the topology of the attractor is preserved. (a) Original input data, (b) result of singular value decomposition, and (c) solution of the model equations.

eigenfunctions corresponding to the three largest eigenvalues is shown in fig. 4b. This phase-space plot is insensitive to the value of  $M$ . A model set of three equations was then constructed using the full cubic form of  $F$  given in eq. (5). Their solution is shown in fig. 4c and is seen to capture the essentials of the time behavior. The plots obtained using the model equations are for a

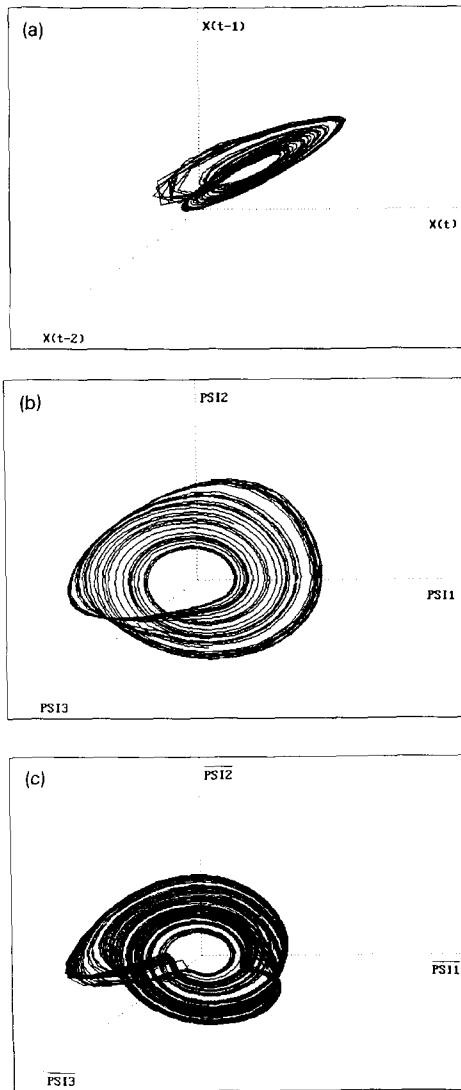


Fig. 5. Three-dimensional phase-space plots of the Rössler attractor showing that the topology of the attractor is preserved. (a) Original input data, (b) result of singular value decomposition, and (c) solution of the model equations.

much longer time than the original data were given.

The correlation dimension calculated using the method of Grassberger and Proccacia [1] with the original data set of 1000 points is  $1.97 \pm 0.18$ , and the value calculated from 13 000 values generated by solving the model equations is  $2.10 \pm 0.10$ . This is to be compared with the accepted value [1] of  $2.05 \pm 0.01$ . It has been pointed out by Ott et al. [23] that correlation dimensions are not necessarily invariant under coordinate changes. However, in the present case, since the  $\psi_m$ 's are just linear combinations of the  $T(n\tau)$ 's, the correlation dimension must be invariant.

### 3.4. Rössler equations

A similar treatment has been applied to the Rössler equations

$$\begin{aligned} dx/dt &= -(y + z), \\ dy/dt &= x + \alpha y, \\ dz/dt &= \beta + z(x - \gamma), \end{aligned} \quad (8)$$

with  $\alpha = \beta = 1/5$ ,  $\gamma = 5.7$ , and  $t = 0.2n$ . The corresponding results are shown in fig. 5. The correlation dimension for this case calculated from the original data set of 1000 points is  $1.92 \pm 0.08$ , and the value calculated from 15 000 values generated by solving the model equations is  $1.94 \pm 0.08$ . The expected value is slightly greater than 2.0.

## 4. Sensitivity to parameters

This whole procedure has been carried out for the Lorenz equations for a range of  $r$  values between 25 and 90, and in particular the coefficients  $a$ ,  $b$ , and  $c$  appearing in eq. (5) were evaluated as a function of  $r$ . The variation with  $r$  of the coefficients of the largest nine terms is shown in fig. 6a from which it is seen that the variation is reasonably smooth. From the sym-

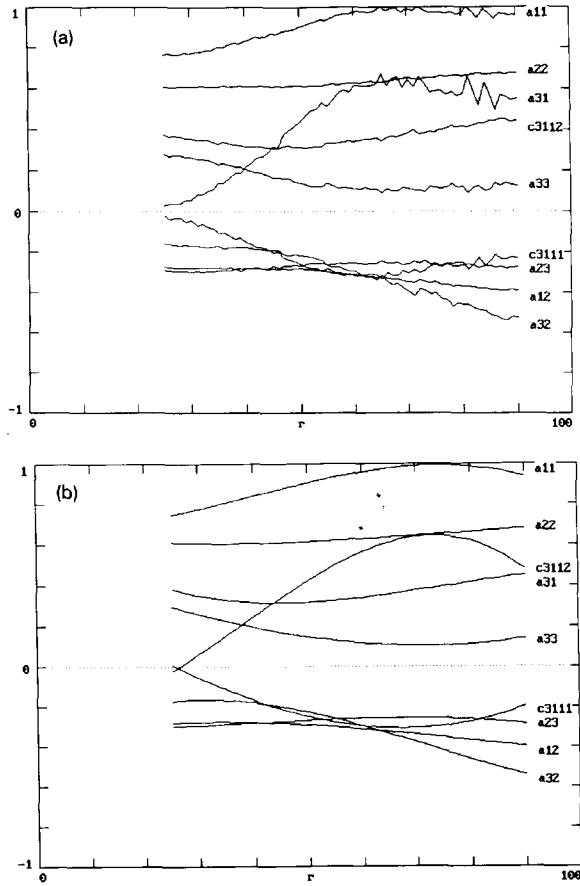


Fig. 6. (a) Variation of the nine largest coefficients of the model equations with the parameter  $r$  in the Lorenz equations, (b) along with a least squares fit of each coefficient to a cubic polynomial in  $r$ .

metry of the Lorenz equations, the terms involving even powers of  $\psi$  ( $a_{mo}$  and  $b_{mpq}$ ) are negligibly small. Using the least squares method, the coefficients are readily fitted to simple polynomials in  $r$ . A cubic fit as shown in fig. 6b is sufficient.

One now has a set of dynamical equations of the form given by eqs. (4) and (5) where the coefficients  $a$ ,  $b$ , and  $c$  are known in the form of simple polynomials in the parameter  $r$ . It is on this set of equations that one can base an interpolation or extrapolation procedure. By taking  $r$  values other than those measured, and solving the model equations, the behavior of the system

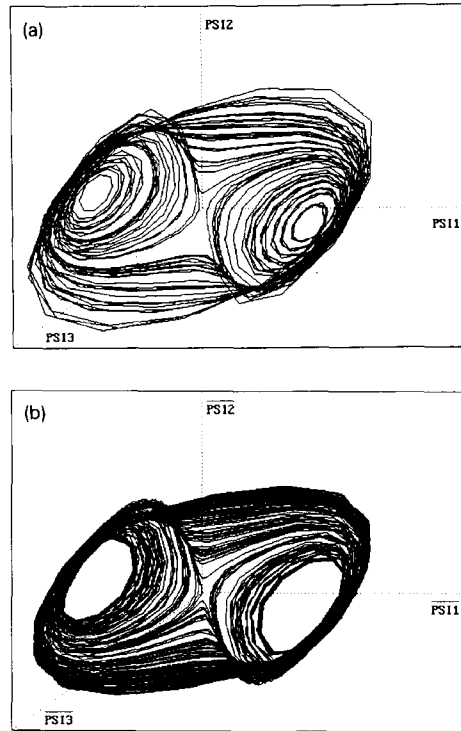


Fig. 7. Phase-space portraits for the Lorenz attractor with  $r = 57$  (a) obtained directly from singular value decomposition, and (b) resulting from solution of the model equations with coefficients calculated from least squares fits to cubic polynomials in  $r$ .

can be predicted. This can be in the form of the relevant phase-space plot or by using eq. (2) to form  $x(t)$ .

The phase-space portrait for  $r = 57$  obtained directly from the values of the  $\psi$ 's is shown in fig. 7a, while the form predicted using the above procedure is shown in fig. 7b. The agreement is good. Extrapolation outside the range of measured values should be applied with caution, however, since the least squares method of fitting curves to polynomials is not ideal.

### 5. Discussion

In the above method there are two quantities, namely  $M$ , the order of the correlation matrix and  $d$ , the number of significant eigenfunctions

retained. These are to be considered as parameters of the method which can be adjusted to obtain the best fit between the real system under investigation through the data  $T(t)$  and the solution of the model, eq. (4). Since we envisage applying the method to situations where the auto-correlation function shows little structure, we hope the complicated time variation can be attributed to the presence of a strange attractor. Then the parameters  $M$  and  $d$  are chosen to represent best the topological features of the attractor.

An alternative approach would be to introduce constraints into the quantity that is to be minimized. For example, if it is apparent from the phase-space plots that the phase portrait has certain symmetry properties, then a term

$$\lambda \sum_j \{ \psi_m[(s+1)\Delta t] - OF_m[\psi_j(s\Delta t)] \}^2 \quad (9)$$

could be added to eq. (6). Here  $O$  is the symmetry operator, and  $\lambda$  is a Lagrangian multiplier. Furthermore, one may impose a smoothness condition on the fit by adding a term which minimizes the average second derivative,

$$\sum_j \{ F_m[\psi_j(s\Delta t)] + F_m\psi_j[(s-2)\Delta t] - 2F_m\psi_j[(s-1)\Delta t] \}^2. \quad (10)$$

The Lagrangian multipliers can then be used to get the best fit to the coefficients  $a$ ,  $b$ , and  $c$ . However, the results given here are optimized only by changing  $M$ .

The results for the Lorenz and Rössler equations have been obtained using the value of  $x_n$  at only 1000 points. The phase-space portraits for the model equations are shown for times longer than a thousand time intervals, illustrating the stability of the equations.

However, the coefficients in the model equations and hence the solution of these equations depend sensitively on the order of the correlation matrix  $\mathbf{M}$ . Though the value of  $T(t)$  (that is  $x$ ) generated using eq. (2) with  $d = 3$  is in good

agreement (over the time where  $x(t)$  is given) with the original data, the associated model equations do not reconstruct the strange attractor. Usually after a short interval of time the solutions tend to become infinite or attract to a fixed point or limit cycle. There is an optimal choice of  $M$  for the reconstruction of the attractor. It is reasonable to expect this value to be associated with (a) the maximum difference between  $\lambda_3$  and the higher eigenvalues and (b) that the elements  $C(n)$  used in the correlation matrices span the region where the major variation of  $C$  occurs. For the results presented in the case of the Lorenz equation, a value of 9 has been found to be appropriate, while for the Rössler equation, because of the longer correlation time, it was found optimal to make  $M = 16$ .

The relative difficulty of finding chaotic solutions to the model equations raises the more general question of how common is chaos. A numerical experiment was carried out in which about  $10^6$  three-dimensional cubic maps of the form given by eqs. (4) and (5) were iterated with the 60 coefficients chosen randomly over a 60-dimensional hypercube with each side extending from  $-1.2$  to  $1.2$ . Initial conditions were chosen near the origin, and the Lyapunov exponent was calculated for each case. About 99% of the solutions were unstable. (This number increases rapidly with the size of the hypercube.) Of the roughly 10 000 stable solutions, which are candidates for modeling bounded physical processes, about two-thirds attract to a fixed point and about one-third are either limit cycles or two-tori. A small subset of about 4% are strange attractors with positive Lyapunov exponents.

Thus we conclude that for the subset of phenomena that can be represented by three-dimensional cubic maps, nature is chaotic about 4% of the time. There is some evidence to suggest that the regions of parameter space corresponding to chaotic motion are elongated. This means that as long as the parameter change is along the direction of elongation the system has a degree of robustness. The imposition of a symmetry re-



quirement tends to elongate the chaotic region. A by-product of this calculation was the generation of several hundred new examples of strange attractors, no two of which look the same.

## 6. Conclusion

A method has been described for determining a set of model equations from limited data whose global solutions resemble those of the original data. However, the method is not robust, and the existence of a strange attractor depends sensitively on the value of  $M$ . This sensitivity might be reduced by a better choice of the variational function.

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