HIGH FREQUENCY REPRESENTATION OF A PLASMA SHEATH

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INTRODUCTION

When an electrostatic probe is used to measure fluctuations in plasma parameters which occur at frequencies close to or above the ion plasma frequency, there is a serious question about the applicability of the equations derived by Langmuir[1] to describe the steady state current to the probe. In this paper the plasma sheath is approximated by an electrical equivalent circuit and numerical results are displayed graphically in a form which should provide a convenient reference for the experimenter to check the validity of his probe data.

To simplify the mathematics, the following assumptions have been made:

1) Planar geometry (or, equivalently, sheath thickness much less than probe radius),

2) Maxwellian velocity distribution with ions and electrons at different temperatures,

3) Probe at floating potential (i.e., no net current to the probe),

4) No magnetic field or collisions in the sheath.

RESONANT BEHAVIOR

As a starting point consider the expression for the high frequency dielectric constant of a plasma:

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$$

If we assume for the moment that the sheath resembles a parallel plate capacitor with the probe and the plasma forming the plates and containing a dielectric with the above dielectric constant, we can write an expression for the sheath capacitance:

$$C = \frac{\epsilon A}{d} = \frac{A}{d}\left[1 - \frac{\omega_p^2}{\omega^2} \right]$$

where A is the area and d is the sheath thickness, which is on the order of a Debye length. The exact value of d will be determined later.

The impedance of a capacitor having the above capacitance is

$$Z = \frac{1}{j\omega c} = \frac{d}{j\omega A[1-\omega_p^2/\omega^2]} = \frac{j\omega d}{A[\omega_p^2 - \omega^2]}$$

This is the same form as the expression for the impedance of a parallel LC circuit provided we let

$$C = \frac{A}{d}, \text{ and } \frac{1}{LC} = \omega_p^2 = \frac{4\pi n_s e^2}{m} .$$

The capacitance is seen to be just the capacitance that one would calculate if the sheath were filled with a vacuum. ω_p is the resonant frequency of the circuit and can be identified with the electron plasma frequency in the sheath.

Since for a probe at the floating potential, the electron density in the sheath, n_s, is usually less than the plasma density n, this resonance will occur below the real electron plasma frequency. The electron density at a point of potential V is given by

$$n\ (V) = n \exp \left[\frac{e\ (V-V_p)}{kT_e}\right]$$

where T_e is the electron temperature and V_p is the plasma potential. At the probe we have

$$n\ (V_f) = n \exp \left[\frac{e(V_f-V_p)}{kT_e} \right]$$

where V_f is the floating potential. Since the density n_s is some kind of average density within the sheath, we probably will not be far wrong if we take the geometric mean of n (V_f) and n:

$$n_s = n \exp \left[\frac{e(V_f-V_p)}{2kT_e} \right] .$$

The quantity $V_f - V_p$ can be expressed as follows:

$$V_f - V_p = \frac{kT_e}{2e} \log \left[\frac{T^* m}{T_e M} \right]$$

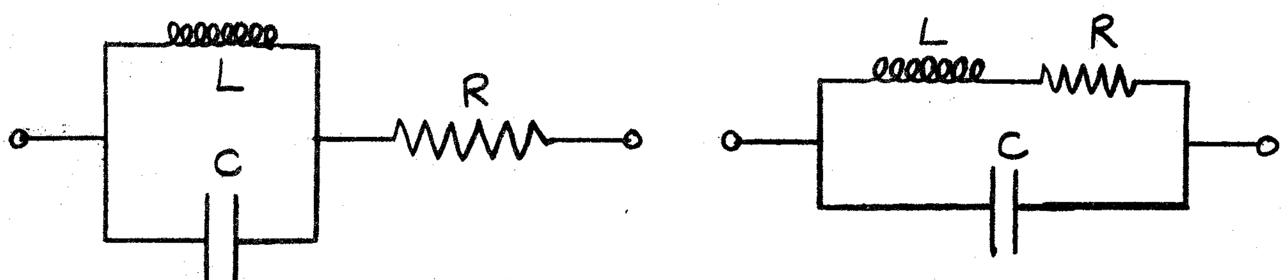
where $T^* = \begin{cases} T_i, & T_i > T_e \\ T_e, & T_i < T_e \end{cases}$.

Substituting into the above equation gives:

$$n_s = n\,\left[\frac{T^*m}{T_e M}\right]^{\frac{1}{4}}$$

The resonant behavior of the sheath predicted by this model possibly accounts for one of the hitherto unexplained peaks obtained in studies of resonant probes.[2] However, as we shall shortly see, this resonance is highly damped by the presence of a large sheath resistance. A more pronounced resonance occurs as a result of the combination of the sheath capacitance and plasma inductance outside the sheath.[3] This is a series resonance and produces an impedance minimum between two probes. This phenomenon falls outside the scope of the present paper.

We know that at the floating potential there is a D.C. resistance associated with the sheath. This resistance can be included in the LC circuit in either of the following two ways:



(1) (2)

Since the capacitance is due to a displacement current across the sheath, circuit (2) would more nearly represent the physical situation. (2) is also preferred because it is more highly damped.

The value of the sheath resistance of a floating probe has been calculated[4] and is given by

$$R = \frac{kT_e}{ne^2 A} \left[\frac{2\pi M}{kT^*} \right]^{\frac{1}{2}}$$

If one calculates the time constant of the RL part of the circuit it is

$$\tau = \frac{L}{R} = \frac{1}{RC\omega_p^2} = \frac{d}{4\pi} \left[\frac{m^3 \quad T^*}{4\pi^2 Mk^2 T_e^{\,3}} \right]^{\frac{1}{4}} .$$

This time can be thought of as the time required for a particle of velocity

$$v = 4\pi \left[\frac{4\pi^2 Mk^2 T_e^{\,3}}{m^3 T^*} \right]^{\frac{1}{4}}$$

to cross the sheath. In at least one experiment,[5] the voltage on a probe was abruptly changed and the current exponentially approached a new equilibrium value in a time roughly given by the above equation.

To complete the theory, all that remains is to evaluate the sheath thickness d. Following a derivation given by Crawford and Grard,[6] the density of electrons is integrated over the sheath thickness in order to find the charge on the probe. From Poisson's equation,

$$\frac{d^2V}{dr^2} = -\ 4\pi ne\left[\left(\frac{V_i}{V_i - V}\right)^{\frac{1}{2}} - \exp\left(\frac{V}{V_e}\right) \right],$$

where $V_i = \frac{kT_i}{e}$ and $V_e = \frac{kT_e}{e}$.

Integrating the above equation subject to the boundary condition $(\frac{dV}{dr}) = V_p$ at the sheath/plasma edge, the electric field at the probe is found:

$$E(o) = \frac{2V_e}{\lambda_D} \left[\left(\frac{V_i}{V_e} \right) \left(\frac{V_f - V_p + V_i}{V_i} \right)^{\frac{1}{2}} - \left(\frac{V_i}{V_e} + \tfrac{1}{2} \right) \right]^{\frac{1}{2}} .$$

where λ_D is the Debye length:

$$\lambda_D = \sqrt{\frac{kT}{4\pi ne^2}}, \qquad T = \frac{T_e\,T_i}{T_e + T_i} .$$

The sheath capacitance can then be calculated:

$$C = A \left(\frac{\rho}{E} \right)_{r=o} = \frac{A}{d}$$

$$\text{where } d = 2\lambda_D \left[\frac{\left[\left(\frac{V_i}{V_e} \right) \left(\frac{V_f - V_p + V_i}{V_i} \right)^{\frac{1}{2}} - \left(\frac{V_i}{V_e} + \frac{1}{2} \right) \right]^{\frac{1}{2}}}{\left(\frac{V_i}{V_f - V_p + V_i} \right)^{\frac{1}{2}} - \frac{1}{2} \exp\left(- \frac{V_f - V_p}{V_e} \right)} \right]$$

Substituting the value of $V_f - V_p$ gives:

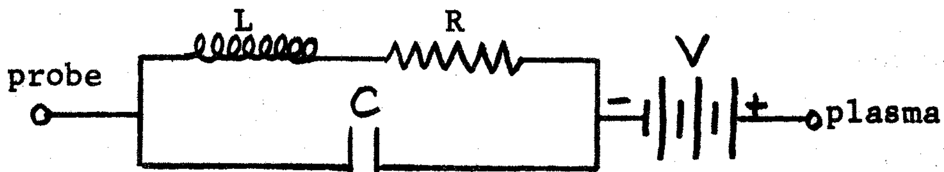
$$d = 2\lambda_D \frac{\left[\left(\frac{T_i}{T_e} \right) \left(\frac{\gamma T_e + T_i}{T_i} \right)^{\frac{1}{2}} - \left(\frac{T_i}{T_e} + \frac{1}{2} \right) \right]^{\frac{1}{2}}}{\left(\frac{T_i}{\gamma T_e + T_i} \right)^{\frac{1}{2}} - \frac{1}{2} \left(\frac{T^* m}{T_e M} \right)^{\frac{1}{2}}}$$

where $\gamma = \frac{1}{2} \log \left(\frac{T_e M}{T^* m} \right)$

This derivation is strictly valid only for frequencies which are sufficiently low that both ions and electrons respond without appreciable delay. However an alternate treatment by Crawford and Grard[6] for the case of frequencies too high for the ions to follow gives results which differ from the above by about a factor of two.

The results of the preceding derivations are summarized as follows:

ELECTRICAL REPRESENTATION
OF A PLASMA SHEATH



$$V = \frac{kT_e}{2e} \log \left[\frac{T_e M}{T^* m} \right]$$

$$R = \frac{kT_e}{ne^2 A} \left[\frac{2\pi M}{kT^*} \right]^{\frac{1}{2}}$$

$$L = \frac{d}{A} \frac{m}{4\pi ne^2} \left[\frac{T_e M}{T^* m} \right]^{\frac{1}{4}}$$

$$C = \frac{A}{d}$$

$$\omega_o = \frac{1}{\sqrt{LC}} = \left[\frac{T^* m}{T_e M}\right]^{\frac{1}{8}} \left[\frac{4\pi ne^2}{m} \right]^{\frac{1}{2}}$$

where $d = \left[\frac{kT}{\pi ne^2}\right]^{\frac{1}{2}} \frac{\left[\left(\frac{T_i}{T_e}\right)\left(\frac{\gamma T_e + T_i}{T_i}\right)^{\frac{1}{2}} - \left(\frac{T_i}{T_e} + \frac{1}{2}\right)\right]^{\frac{1}{2}}}{\left(\frac{T_i}{\gamma T_e + T_i}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\frac{T^*m}{T_e M}\right)^{\frac{1}{2}}}$

$$\gamma = \tfrac{1}{2} \log \frac{T_e M}{T^* m}$$

$$T = \frac{T_e T_i}{T_e + T_i}, \qquad T^* = \begin{cases} T_i, & T_i > T_e \\ T_e, & T_i < T_e \end{cases}$$

To facilitate plotting the data, we specialize the previous results to the case of a hydrogen plasma with $T_i = T_e$:

$$V = \frac{kT}{2e} \log (1837) = 3.7 \left(\frac{kT}{e}\right)$$

$$R = \frac{\sqrt{2\pi MkT}}{ne^2 A}$$

$$L = \frac{d}{A} \frac{m}{4\pi ne^2} \left(\frac{M}{m}\right)^{\frac{1}{4}} = 0.52 \frac{d}{A} \frac{m}{ne^2}$$

$$C = \frac{A}{d}$$

$$\omega_o = 0.39 \left[\frac{4\pi ne^2}{m}\right]^{\frac{1}{2}} = 0.39\,\omega_p$$

where $d = 1.8 \left[\frac{kT}{\pi ne^2}\right]^{\frac{1}{2}} = 3.6\,\lambda_D$

$$\gamma = 3.7$$

Numerically, the above equations become:

$$V\ (\text{volts}) = 3.7\ E\ (\text{ev})$$

$$R\ (\text{ohms}) = 1.6 \times 10^{13}\ \frac{\sqrt{E\ (\text{ev})}}{n(\text{cm}^{-3})\ A(\text{cm}^2)}$$

$$L\ (\mu\text{hy}) = 5.0 \times 10^{12}\ \frac{\sqrt{E(\text{ev})}}{A(\text{cm}^2)\ n\ (\text{cm}^{-3})\sqrt{n\ (\text{cm}^{-3})}}$$

$$C\ (\text{pf}) = 4.1 \times 10^{-4}\ \frac{A\ (\text{cm}^2)\sqrt{n\ (\text{cm}^{-3})}}{\sqrt{E\ (\text{ev})}}$$

$$RC(\mu\text{sec}) = 6.6 \times 10^3\ \frac{1}{\sqrt{n(\text{cm}^{-3})}}$$

$$\frac{L}{R}\ (\mu\text{sec}) = .31\ \frac{1}{\sqrt{n(\text{cm}^{-3})}}$$

$$\frac{1}{f_p}(\mu\text{sec}) = 110\ \frac{1}{\sqrt{n(\text{cm}^{-3})}}$$

These quantities are plotted as a function of plasma density in the graphs at the end of the paper.

Finally we can compute the Q of the sheath resonance:

$$Q = \frac{\omega_p L}{R} = \frac{1}{2\pi f_p RC} = 2.7 \times 10^{-3}$$

With such a low Q, the resonance would not be seen and to a very good approximation the inductance can be neglected.

OCTUPOLE PLASMA

As a final example, consider the case of a hydrogen plasma with $kT_e = 10$ eV and $kT_i = 100$ eV with a Maxwellian distribution for each species. This choice very nearly represents plasmas in our octupole. Specializing the general equations gives:

$$V = \frac{kT_e}{2e} \log (1837) = 26 \text{ volts}$$

$$R\text{ (ohms)} = \frac{1.6 \times 10^{13}}{n(\text{cm}^{-3})\ A(\text{cm}^2)}$$

$$L(\mu\text{hy}) = \frac{2.2 \times 10^{12}}{A(\text{cm}^2) n(\text{cm}^{-3})\sqrt{n(\text{cm}^{-3})}}$$

$$C(\text{pf}) = 4.6 \times 10^{-4}\ A(\text{cm}^2)\sqrt{n(\text{cm}^{-3})}$$

$$\omega_o = 0.58\,\omega_p$$

$$RC(\mu\text{sec}) = \frac{7.4 \times 10^3}{\sqrt{n(\text{cm}^{-3})}}$$

$$\frac{L}{R}(\mu\text{sec}) = \frac{.137}{\sqrt{n(\text{cm}^{-3})}}$$

$$Q = 4.6 \times 10^{-3}$$

$$d = .93\,\lambda_D$$

$$\gamma = 2.6$$

The above data are plotted in dotted lines in the figures.

Finally, we consider the quantity which is of prime interest in interpreting probe results - the sheath impedance. For this calculation we will neglect the inductive term which was previously shown to be small. The complex impedance of the sheath is

$$Z = \frac{R}{j\omega C(R + \frac{1}{j\omega c})} = \frac{R}{j\omega RC + 1}$$

with magnitude

$$|Z| = \sqrt{\frac{R^2}{1+4\pi^2 f^2 R^2 C^2}}$$

and phase

$$\phi = \tan^{-1} (2\pi f RC).$$

For the octupole case, these become

$$|Z| = \frac{1}{A}\sqrt{\frac{2.56 \times 10^{26}}{n^2 + 2.14 \times 10^{-4} f^2 n}}$$

$$\phi = \tan^{-1} \left(\frac{4.6 \times 10^{-2} f}{\sqrt{n}} \right)$$

The magnitude and phase of the sheath impedance is plotted for $n = 10^9$ and $n = 10^{11}$ cm^{-3}.

It should be remembered that because of the crudeness of the derivations the results should not be trusted to within more than a factor of two or three. However, since the original assumptions of a thin sheath and Maxwellian distribution are only approximately valid for most experiments, greater accuracy would not improve the usefulness of this paper.

SUMMARY

The main results of the above derivation are summarized below:

1) The plasma sheath can be represented as a highly damped parallel LC circuit resonant at $\sim 0.39\ f_{pe}$.

2) The Q is typically a few times 10^{-3}, and thus the inductance can be neglected.

3) The sheath capacitance is approximately the capacitance that one obtains if the sheath is considered to have a thickness of one Debye length and a relative dielectric constant, $\epsilon = 1$.

4) The effect of the capacitance is to lower the impedance and shift the phase of the sheath at frequencies near or above the ion plasma frequency.

5) In plasmas having ions hotter than electrons, such as the octupole, the sheath resistance and capacitance depend more nearly on T_e^2/T_i, than on T_e.

References

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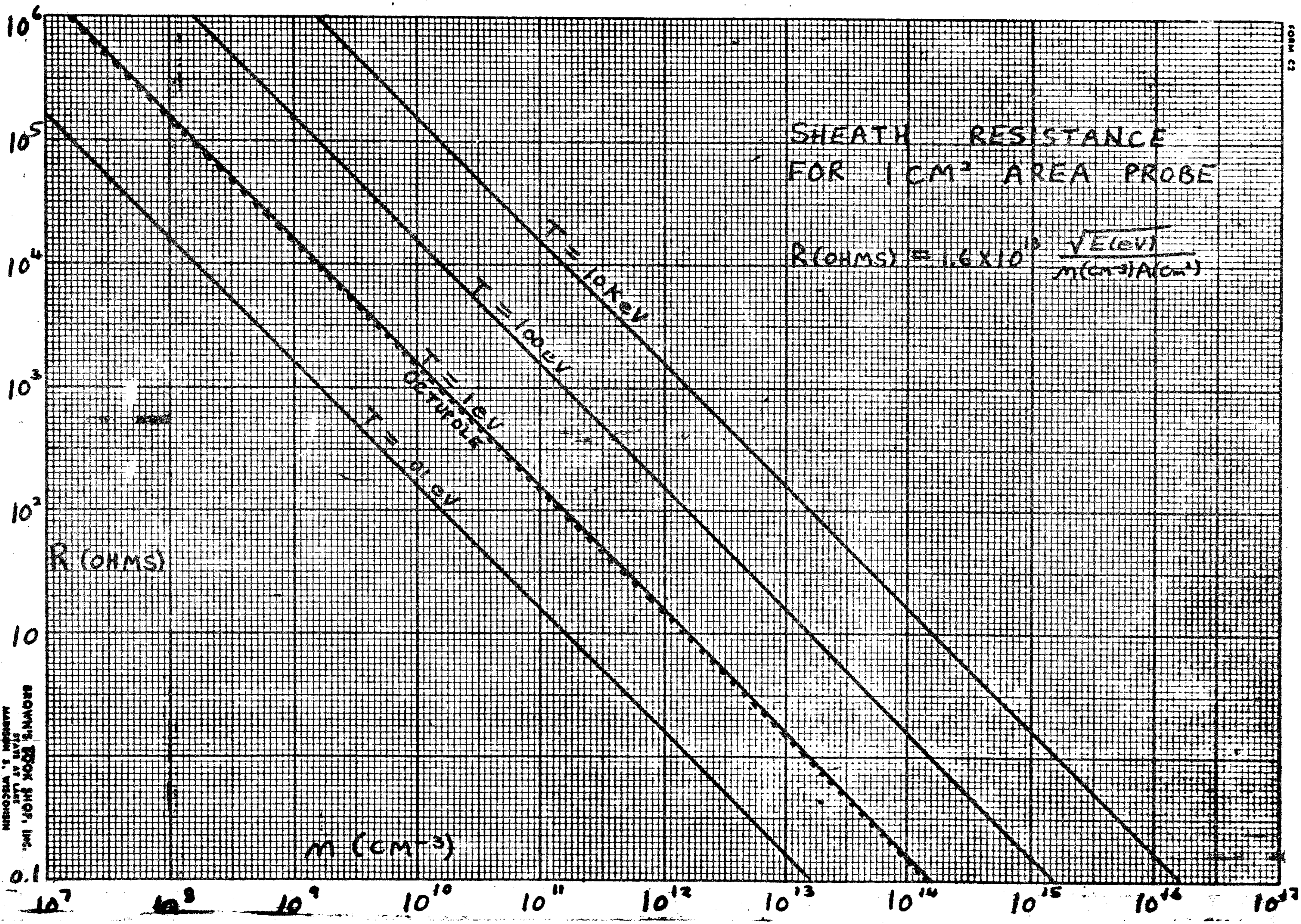
2. S. M. Levitskii, I. P. Shashwin - Sov. Phys. Tech. Phys. 8, 319 (1963)

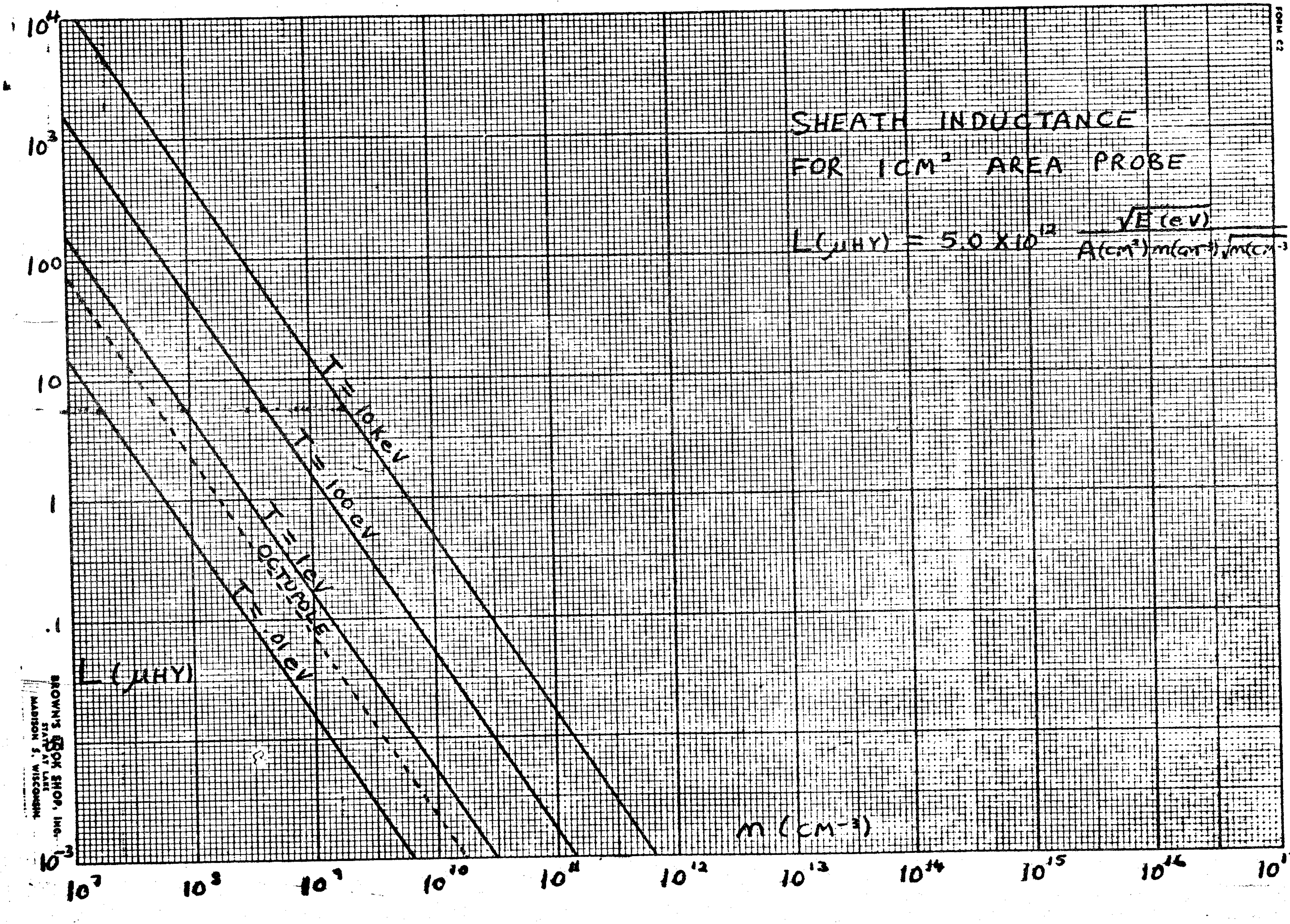
3. R. S. Harp and F. W. Crawford - Stanford Univ. Microwave Lab Report no. 1176 (1964).

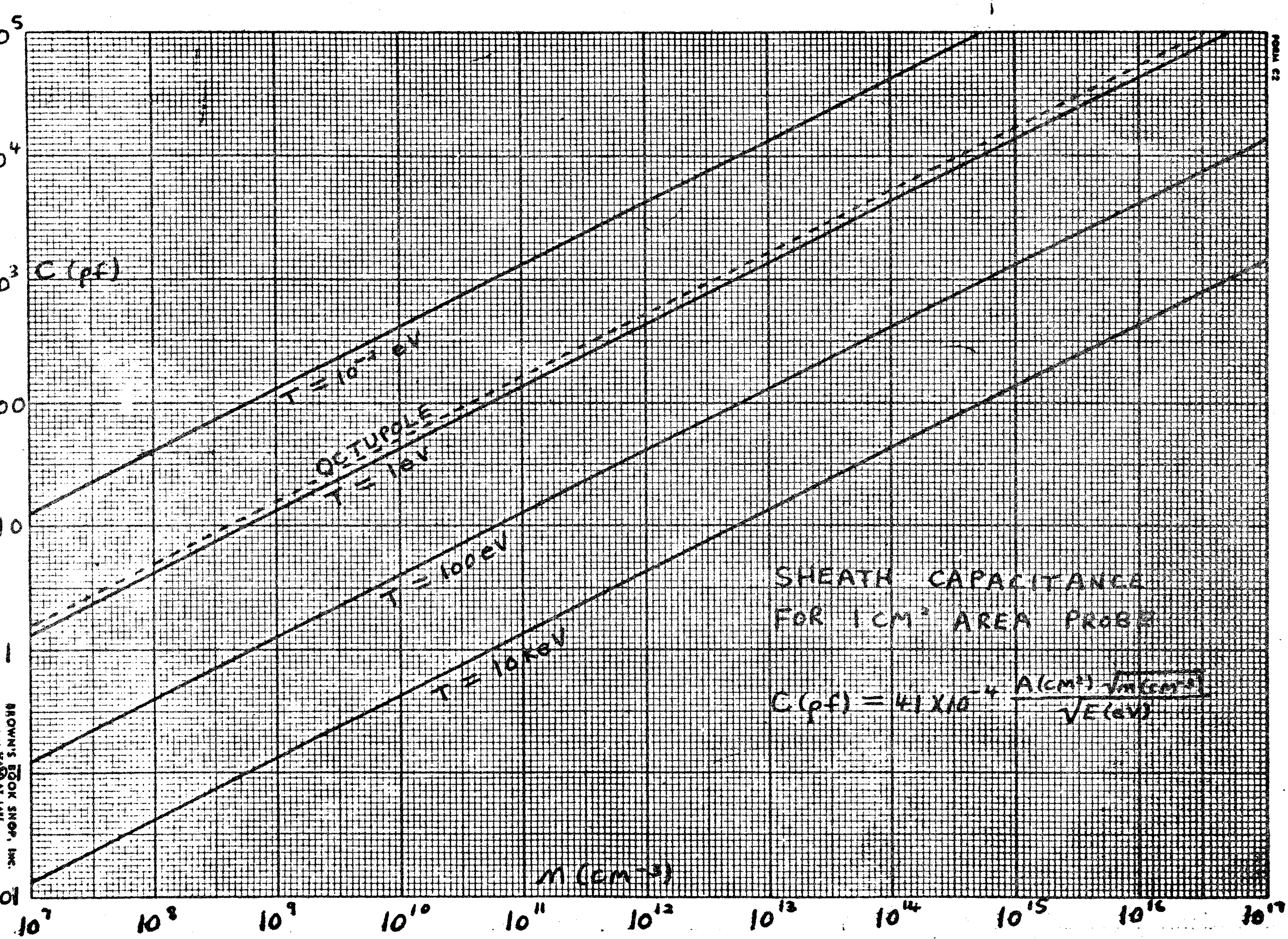
4. J. C. Sprott - Rev. Sci. Instr. (Submitted for Publication).

5. D. G. Bills, R. B. Holt, B. T. McClure - J. Appl. Phys. 33, 29 (1962).

6. F. W. Crawford and R. Grard - Journ. Appl. Phys. 37, 180 (1965).


SHEATH RESISTANCE
FOR 1 CM² AREA PROBE
R(OHMS) = 1.6 X 10^13 √E(eV) / n(cm-3) A(cm²)
T = 10 KeV
T = 100 eV
T = 1 eV
OCTUPOLE
T = .1 eV
R (OHMS)
n (CM-3)
10^6
10^5
10^4
10^3
10^2
10
0.1
10^7
10^8
10^9
10^10
10^11
10^12
10^13
10^14
10^15
10^16
10^17
FORM C2
BROWN'S BOOK SHOP, INC.
STATE AT LAKE
MADISON 5, WISCONSIN


SHEATH INDUCTANCE
FOR 1 CM² AREA PROBE
L(μHY) = 5.0 X 10^12 √E(eV) / A(cm²) m(cm⁻³) √m(cm⁻³)
T = 10 keV
T = 100 eV
T = 1 eV
OCTUPOLE
T = .01 eV
L (μHY)
m (CM⁻³)
10^4
10^3
100
10
1
.1
10^-3
10^7
10^8
10^9
10^10
10^11
10^12
10^13
10^14
10^15
10^16
10^17


SHEATH CAPACITANCE
FOR 1 CM² AREA PROBE
$C(pf) = 41 \times 10^{-4} \frac{A(cm^2)\sqrt{n(cm^{-3})}}{\sqrt{E(eV)}}$
C (pf)
10^5
10^4
10^3
100
10
1
.01
$T = 10^{-2}$ eV
OCTUPOLE
T = 1eV
T = 100eV
T = 10keV
n (cm⁻³)
10^7
10^8
10^9
10^{10}
10^{11}
10^{12}
10^{13}
10^{14}
10^{15}
10^{16}
10^{17}
FORM C2
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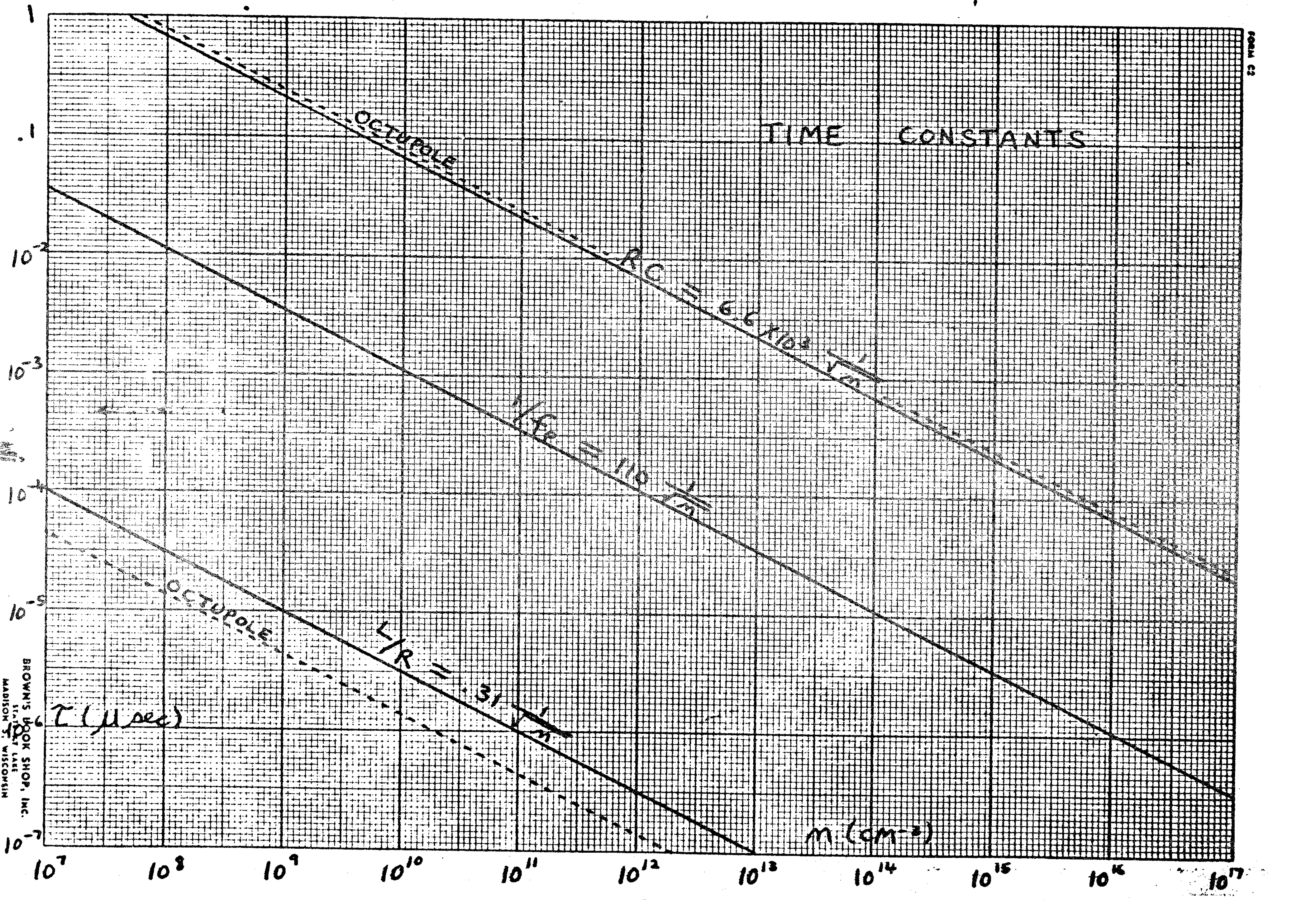
STATE AT LAKE
MADISON 5, WISCONSIN


FORM C2
TIME CONSTANTS
OCTUPOLE
RC = 6.6 x 10^3 1/√n
1/f_p = 110 1/√n
L/R = .31 1/√n
OCTUPOLE
τ (μsec)
1
.1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^7
10^8
10^9
10^10
10^11
10^12
10^13
10^14
10^15
10^16
10^17
n (cm^-3)
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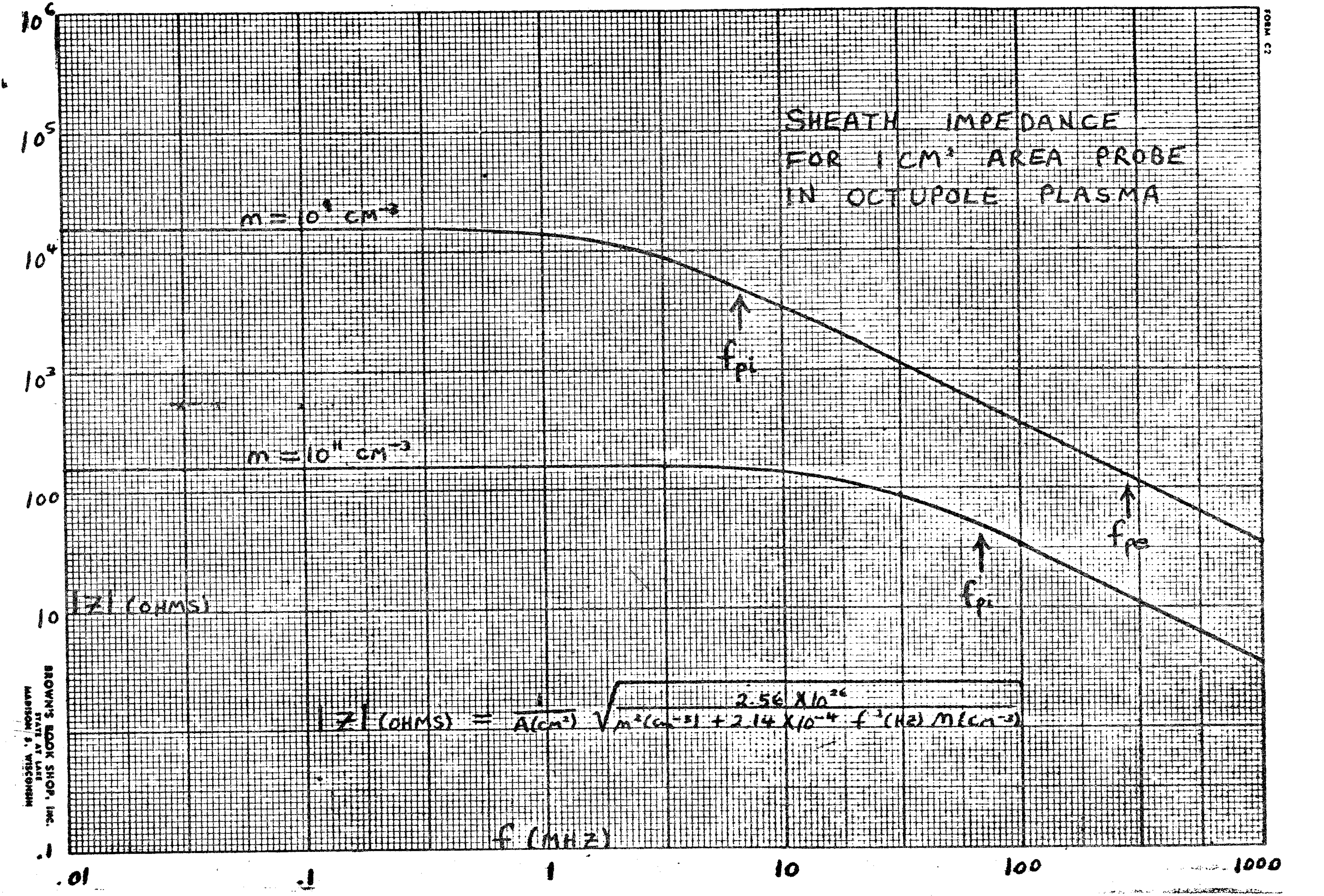
MADISON 5, WISCONSIN


SHEATH IMPEDANCE
FOR 1 CM² AREA PROBE
IN OCTUPOLE PLASMA
$m = 10^{9}\ cm^{-3}$
$m = 10^{11}\ cm^{-3}$
f_{pi}
f_{pi}
f_{pe}
$|Z|$ (OHMS)
$|Z|(\text{OHMS}) = \frac{1}{A(cm^2)}\sqrt{\frac{2.56 \times 10^{26}}{m^2(cm^{-3}) + 2.14 \times 10^{-4} f^2(Hz)\, m(cm^{-3})}}$
f (MHZ)
10^6
10^5
10^4
10^3
100
10
.1
.01
.1
1
10
100
1000
FORM C2
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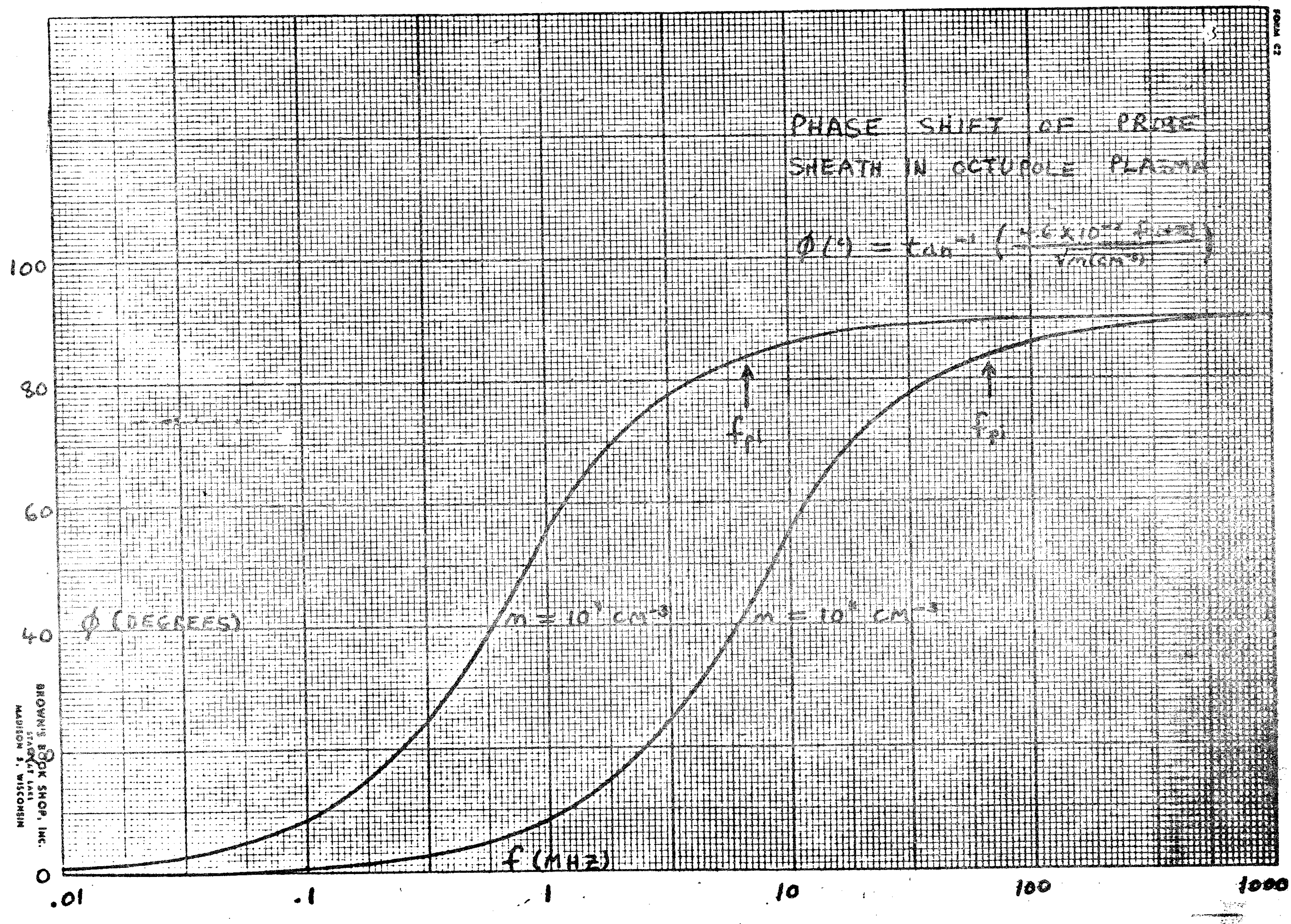
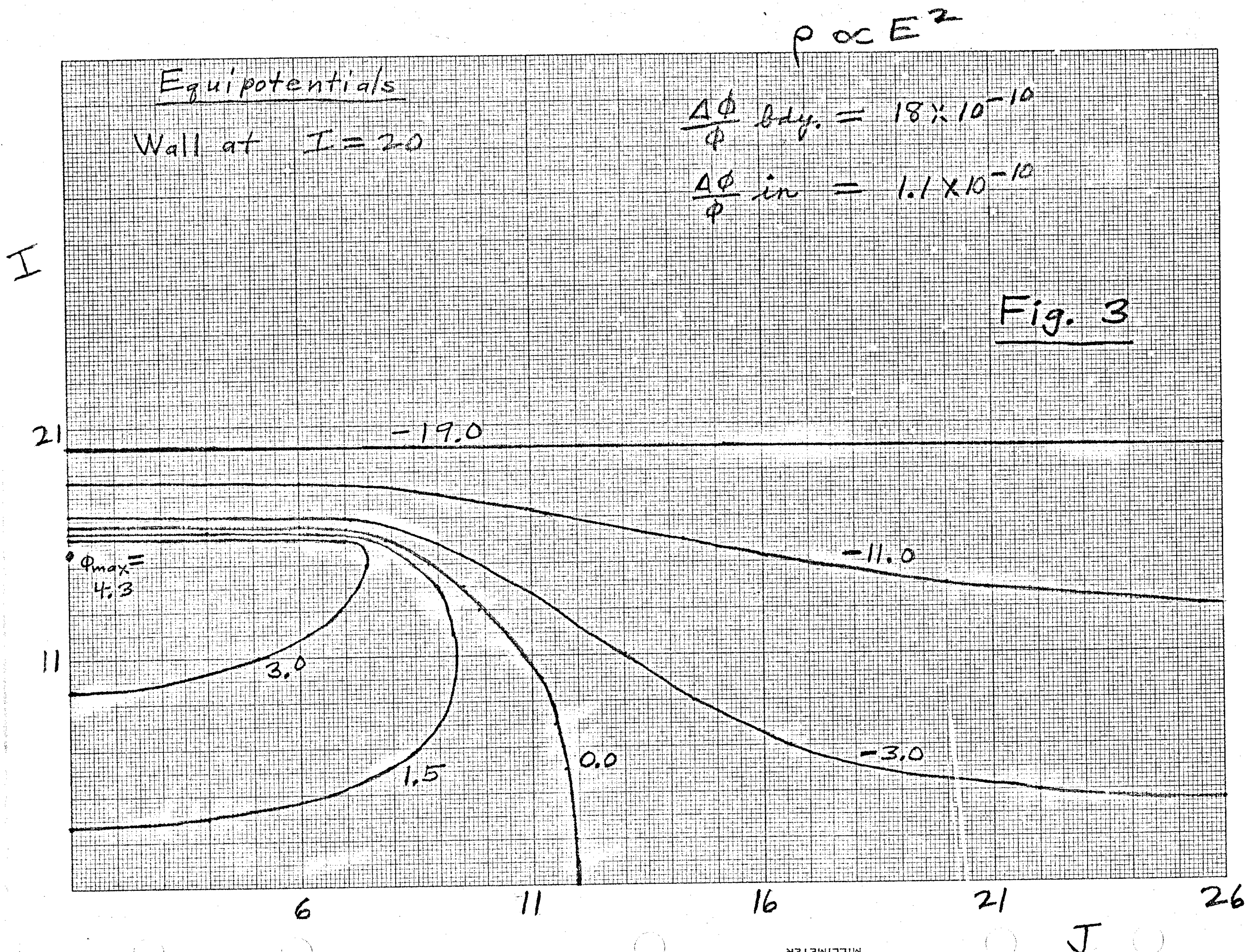
STATE AT LAKE
MADISON 5, WISCONSIN


PHASE SHIFT OF PROBE
SHEATH IN OCTUPOLE PLASMA
100
80
60
40
0
.01
.1
1
10
100
1000
f (MHZ)
ϕ (DEGREES)
f_{pi}
f_{pi}
FORM C2



Equipotentials
Wall at I = 20
$\frac{\Delta\phi}{\phi}$ bdy. = 18×10^{-10}
$\frac{\Delta\phi}{\phi}$ in = 1.1×10^{-10}
$\rho \propto E^2$
Fig. 3
I
J
21
11
6
11
16
21
26
−19.0
−11.0
−3.0
0.0
1.5
3.0
ϕ_{max} = 4.3

Equipotentials

Wall at $I = 34$

$\rho \propto \phi$

$$\frac{\Delta\phi}{\phi}\,\text{bdy.} = 17 \times 10^{-5}$$

$$\frac{\Delta\phi}{\phi}\,\text{in} = 7.5 \times 10^{-6}$$

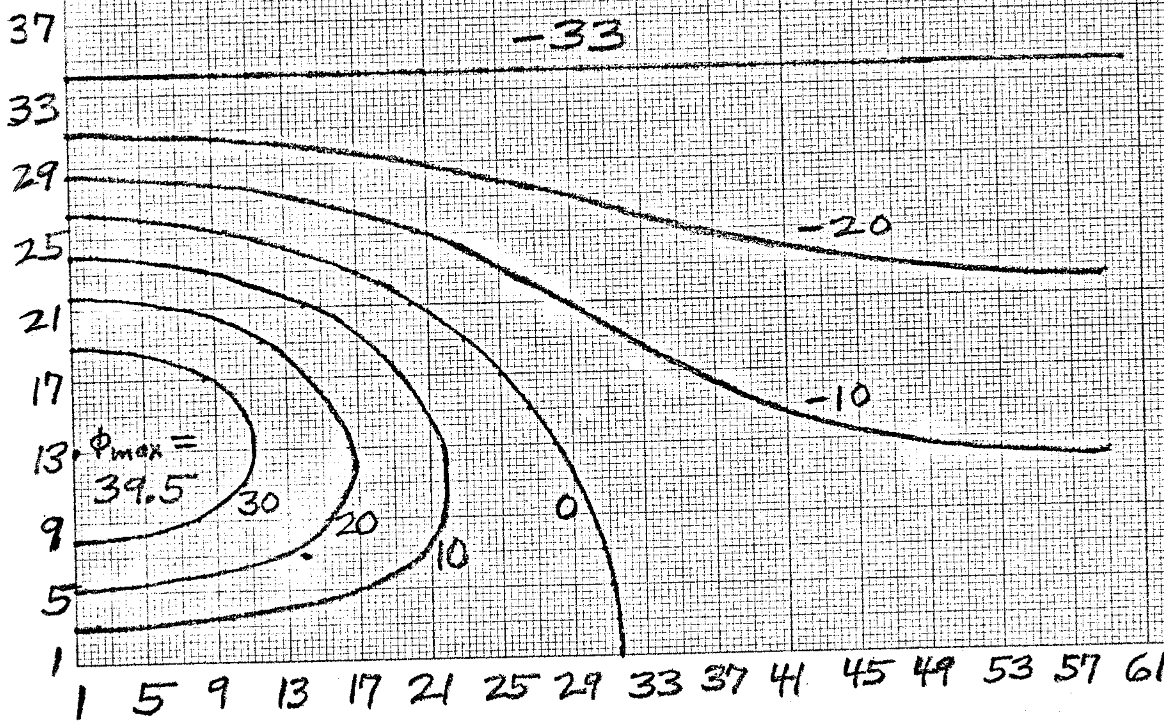
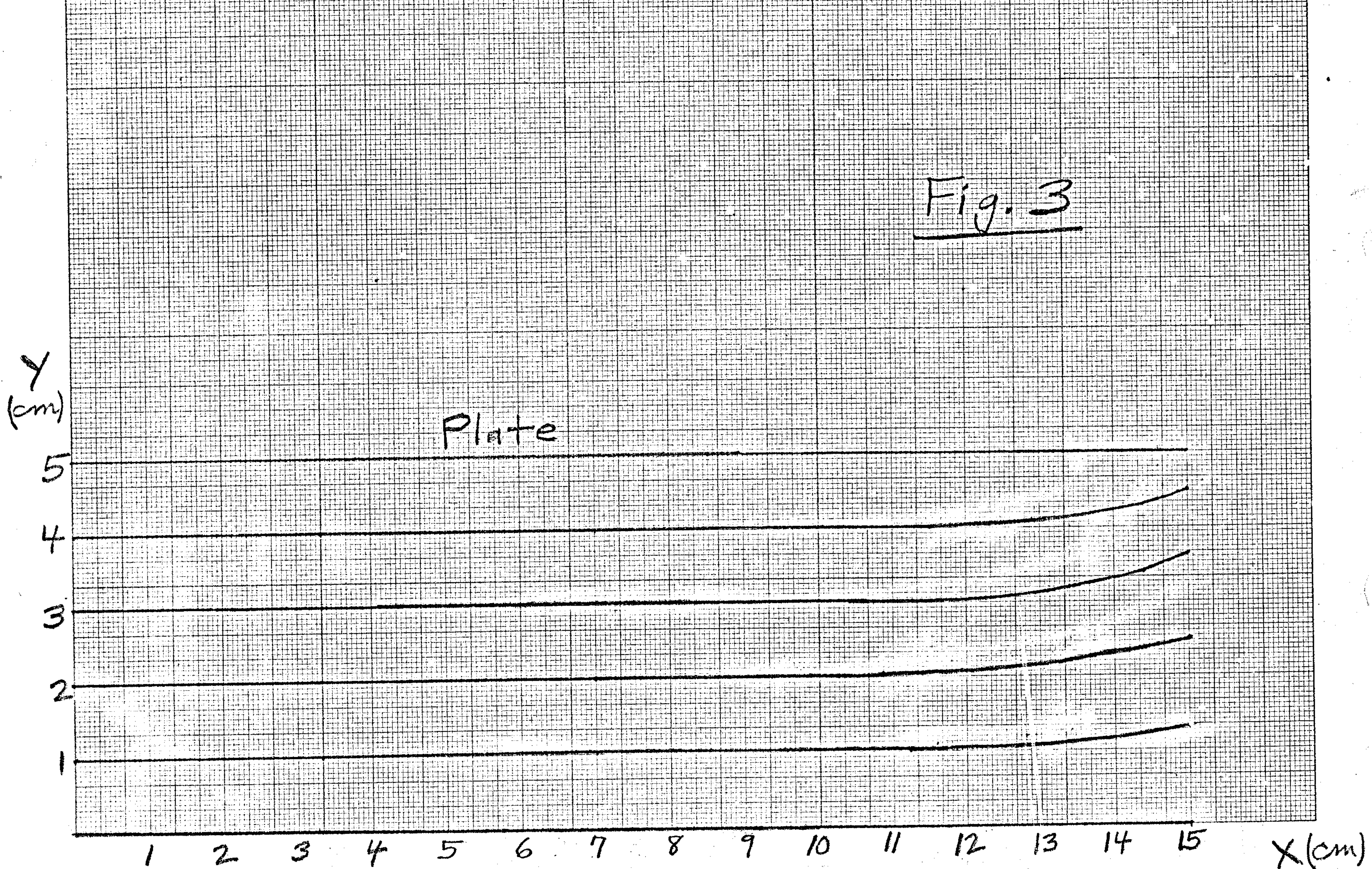
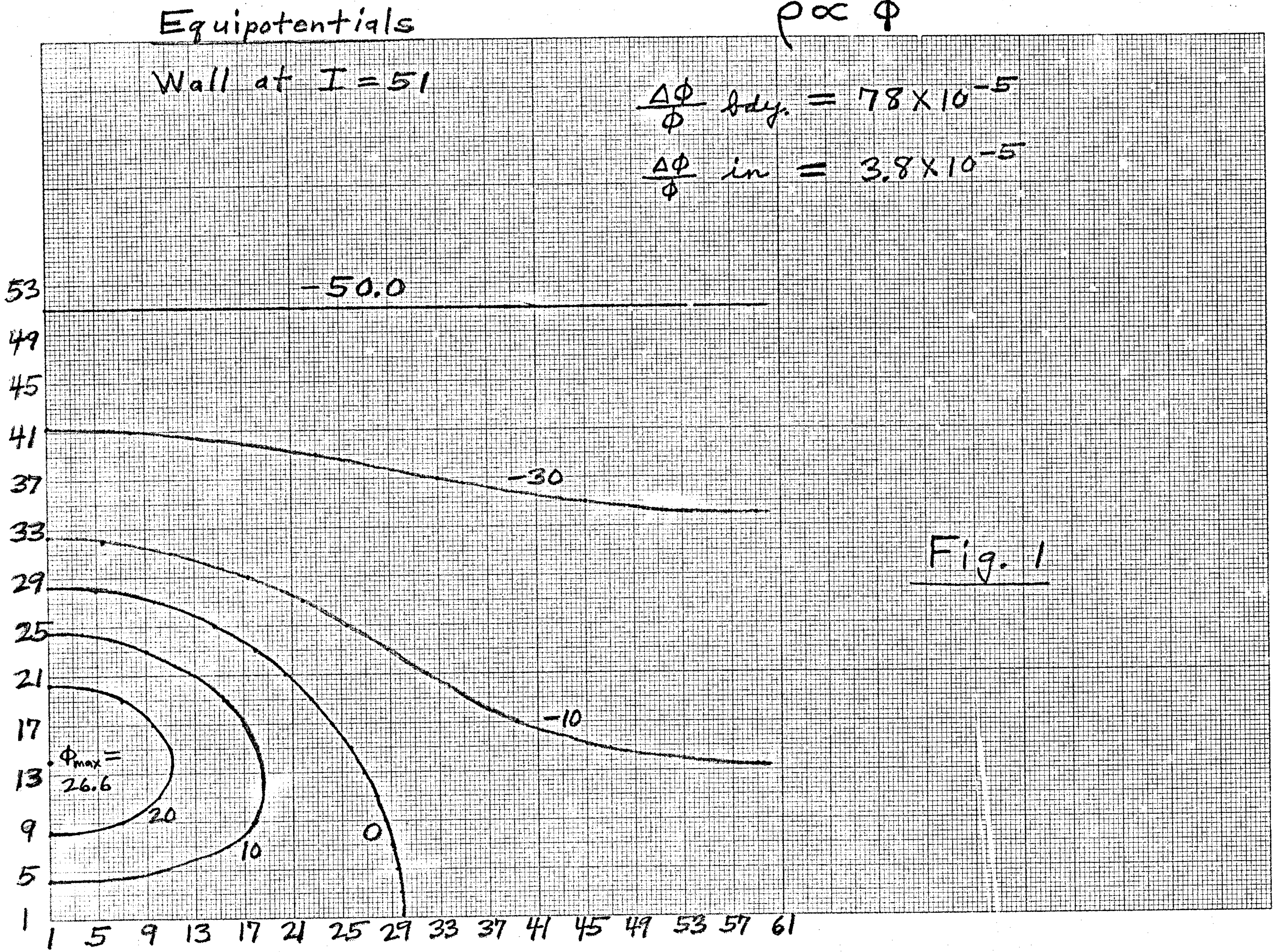


Fig. 4

Analytic solution for $\vec{B}$ lines between two current-carrying plates

Fig. 3





Equipotentials
Wall at I = 51
ρ ∝ Φ
ΔΦ/Φ bdy. = 78 × 10^-5
ΔΦ/Φ in = 3.8 × 10^-5
-50.0
-30
-10
0
10
20
Φmax = 26.6
1 5 9 13 17 21 25 29 33 37 41 45 49 53 57 61
1 5 9 13 17 21 25 29 33 37 41 45 49 53

Fig. 1

Equipotentials

$$\rho \propto \frac{C_1\phi^3 + C_2\phi^2 E^2}{1 + C_3\phi^3}$$

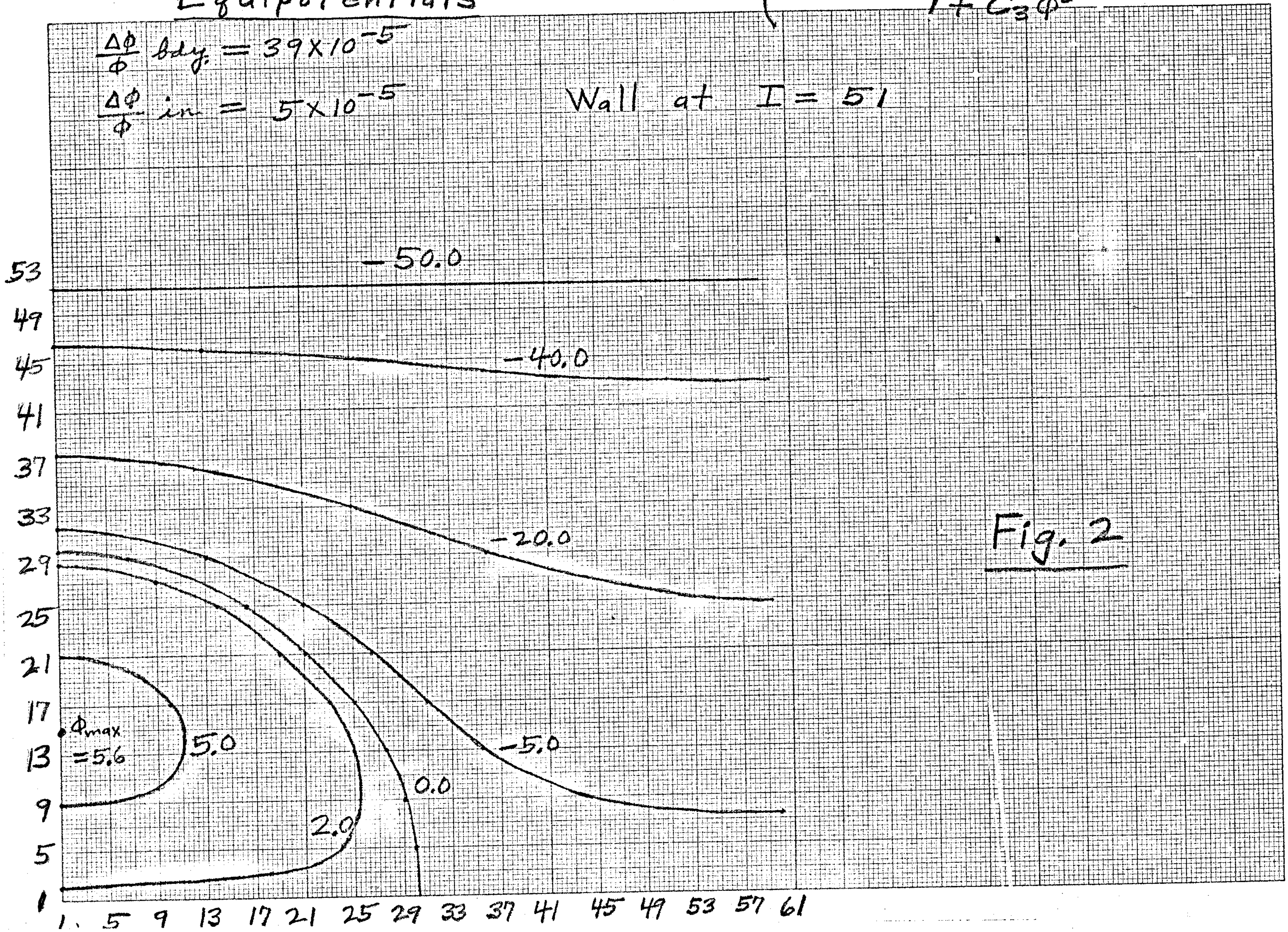


Fig. 2

Analytic solution for $\vec{B}$ lines between two current-carrying plates

Fig. 3