SOME PROPERTIES OF ISOTROPIC DISTRIBUTIONS

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Consider a magnetic field line along which the field strength $B(\ell)$ increases monotonically from a point $\ell = 0$ and an isotropic, monoenergetic velocity distribution at $\ell = 0$ given by $f(\vec{v},0)$. The parallel velocity distribution is easily obtained from geometrical considerations as

$$f(v_{\parallel},0) \equiv \frac{dN(0)}{dv_{\parallel}(0)} = n(0)\,\frac{2\pi v^2 \sin\theta d\theta}{4\pi v^2 dv_{\parallel}(0)}$$

$$= n(0)\,\frac{2\pi v^2 \sin\theta d\theta}{4\pi v^3 \sin\theta d\theta} = \frac{n(0)}{2v} \;,$$

i.e., all values of $v_{\parallel}$ over the range $-v < v_{\parallel} < v$ are equally probable. We can define the turning point distribution as the density of particles at $\ell = 0$ whose turning points lie within dB of B_T:

$$f(B_T,0) = \frac{dN(0)}{dB_T} = \frac{dN}{dv_{\parallel}(0)}\,\frac{dv_{\parallel}(0)}{dB_T} = \frac{n(0)}{2v} = \frac{dv_{\parallel}(0)}{dB_T} \;.$$

If the magnetic moment $(mv_{\perp}^2/2B)$ is conserved,

$$v_{\parallel}(0) = \sqrt{v^2 - v_{\perp}^2(0)} = v\sqrt{1 - B(0)/B_T} \;,$$

and the turning point distribution becomes

$$f(B_T,0) = \frac{n(0)B(0)}{2B_T\sqrt{B_T^2 - B(0)B_T}} \;.$$

The density of particles at position ℓ is

$$n(\ell) = \frac{B(\ell)}{B(0)} \int_{B(\ell)}^{\infty} \frac{v_{\parallel}(0)}{v_{\parallel}(\ell)}\, f(B_T,0)dB_T \;,$$

where the factor $B(\ell)/B(0)$ accounts for the compression of the plasma as the field lines converge, the factor $v_{\parallel}(0)/v_{\parallel}(\ell)$ causes the density to increase where the particles move slowly, and the integral counts only those particles with turning points beyond the point considered. If the magnetic moment is conserved,

$$n(\ell) = \frac{n(0)B(\ell)}{2}\int_{B(\ell)}^{\infty} \frac{dB_T}{B_T\sqrt{B_T^2 - B(\ell)B_T}} = n(0) \ .$$

Therefore, an isotropic distribution at $\ell = 0$ requires that the density be constant everywhere along the field line.

Furthermore, since the energy is conserved, the distribution at $\ell \neq 0$ is also monoenergetic with the same energy as at $\ell = 0$. Since an arbitrary distribution is composed of a sum of monoenergetic distributions, and since the density of each monoenergetic distribution is constant along the field line, it follows trivially that any isotropic distribution at $\ell = 0$ remains unchanged along the field:

$$f(\vec{v},\ell) = f(\vec{v},0).$$

It also follows that any moment of the distribution is also constant, so that, for example, if $f(\vec{v},0)$ is maxwellian, $f(\vec{v},\ell)$ is also maxwellian with the same temperature and density.

To summarize, if a distribution is isotropic at a minimum in the magnetic field, it is isotropic everywhere along the field, and the distribution function and all its moments including density and temperature are constant along the field, provided the field increases monotonically from the minimum.