CLASSICAL DIFFUSION

IN A CYLINDRICAL PLASMA

by

J. C. Sprott

March 1973

PLP 508

Plasma Studies

University of Wisconsin

One frequently wishes to know the spatial and temporal evolution of the plasma density, electron temperature and ion temperature for a particular diffusion mechanism in a particular magnetic field geometry. We calculate here one of the simplest such cases in which the magnetic field is straight and uniform and the plasma is cylindrically symmetric. Classical diffusion is assumed, and collisions with neutrals as well as electron-ion collisions are considered. The particle density n and energy densities U obey continuity equations:

$$\frac{\partial n(r,t)}{\partial t} = - \nabla \cdot \vec{S}\,(r,t) = - \frac{1}{r}\frac{\partial}{\partial r}\,(rS)$$

$$\frac{\partial U_e(r,t)}{\partial t} = - \nabla \cdot \vec{P}_e\,(r,t) = - \frac{1}{r}\frac{\partial}{\partial r}\,(rP_e)$$

$$\frac{\partial U_i(r,t)}{\partial t} = - \nabla \cdot \vec{P}_i\,(r,t) = - \frac{1}{r}\frac{\partial}{\partial r}\,(rP_i)$$

where $\vec{S}$ is the particle flux and $\vec{P}$ is the energy flux crossing a surface of constant r. For classical diffusion, the radial component of the fluxes is

$$S_e = - \frac{\nu_{en}}{m_e\omega^2_{ce}}\,(nT_e'+n'T_e+neE_r) - \frac{\nu_{ei}}{m_e\omega^2_{ce}}\,[n(T_e'+T_i') + n'(T_e+T_i)]$$

$$S_i = - \frac{\nu_{in}}{m_i\omega^2_{ci}}\,(nT_i'+n'T_i-neE_r) - \frac{\nu_{ei}}{m_e\omega^2_{ce}}\,[n(T_e'+T_i') + n'(T_e+T_i)]$$

$$P_e = \frac{5}{2}\,[T_eS_e - \frac{\nu_{en}+\nu_{ee}+\nu_{ei}}{m_e\omega^2_{ce}}\,nT_eT_e']$$

$$P_i = \frac{5}{2}\,[T_iS_i - \frac{\nu_{in}+\nu_{ii}+\nu_{ie}}{m_i\omega^2_{ci}}\,nT_iT_i']$$

where E_r is the radial ambipolar electric field, and the prime denotes differentiation with respect to r. For charge neutrality, we require $S_e = S_i$, or

$$\frac{\nu_{en}}{m_e \omega^2_{ce}} (nT'_e + n'T_e + neE_r) = \frac{\nu_{in}}{m_i \omega^2_{ci}} (nT'_i + n'T_i - neE_r).$$

Solving for neE_r gives

$$neE_r = \frac{m_e \omega^2_{ce} \nu_{in} (nT'_i + n'T_i) - m_i \omega^2_{ci} \nu_{en} (nT'_e + n'T_e)}{m_i \omega^2_{ci} \nu_{en} + m_e \omega^2_{ce} \nu_{in}} .$$

For a hydrogen plasma in a neutral hydrogen background,

$$\nu_{en} \cong 21.5\ \nu_{in} \cong \alpha_1 n_o,$$

where

$$\alpha_1 \cong 1.77 \times 10^{-7}\ \text{cm}^3/\text{sec},$$

and the electric field is approximately

$$neE_r \cong nT'_i + n'T_i - \frac{m_e}{m_i} \frac{\nu_{en}}{\nu_{in}} (nT'_e + n'T_e).$$

The flux (either electrons or ions) then becomes

$$S = - \frac{\nu_{en}}{m_e \omega^2_{ce}} (nT'_e + n'T_e) - \frac{\nu_{ei}}{m_e \omega^2_{ce}} [n(T'_e + T'_i) + n'(T_e + T_i)].$$

In cylindrical coordinates, the divergence becomes

$$\frac{\partial n}{\partial t} = - \nabla \cdot \vec{S} = - \frac{1}{r} \frac{\partial}{\partial r} (rS_r) = - \frac{S_r}{r} - \frac{\partial S_r}{\partial r}$$

$$= \frac{\nu_{en}}{m_e \omega_{ce}^2 r} (nT_e' + n'T_e) + \frac{\nu_{ei}}{m_e \omega_{ce}^2 r} [n(T_e' + T_i') + n'(T_e + T_i)]$$

$$+ \frac{1}{m_e \omega_{ce}^2} [\nu_{en}' (nT_e' + n'T_e) + \nu_{en} (nT_e'' + 2n'T_e' + n''T_e)]$$

$$+ \frac{1}{m_e \omega_{ce}^2} [\nu_{ei}' n(T_e' + T_i') + \nu_{ei}' n'(T_e + T_i) + \nu_{ei} n(T_e'' + T_i'')$$

$$+ 2\nu_{ei} n'(T_e' + T_i') + \nu_{ei} n''(T_e + T_i)] .$$

From Spitzer,

$$\nu_{ei} = \alpha_2 n T_e^{-3/2} ,$$

where

$$\alpha_2 \cong 5.75 \times 10^{-5} \text{ cm}^3 \text{ eV}^{3/2}/\text{sec.}$$

Then

$$\nu_{ei}' = \frac{\alpha_2}{T_e^{5/2}} (n'T_e - \frac{3}{2} nT_e') ,$$

and the rate of change of density is

$$\frac{\partial n}{\partial t} = \frac{\alpha_1}{m_e \omega_{ce}^2} [(n_o' + \frac{n_o}{r})(nT_e' + n'T_e) + n_o(nT_e'' + 2n'T_e' + n''T_e)]$$

$$+ \frac{\alpha_2}{m_e \omega_{ce}^2 T_e^{5/2}} [(\frac{nT_e}{r} + n'T_e - \frac{3}{2} nT_e')(nT_e' + nT_i' + n'T_e + n'T_i)$$

$$+ nT_e(nT_e'' + nT_i'' + 2n'T_e' + 2n'T_i' + n''T_e + n''T_i)]$$

The classical electron thermal conduction is

$$\frac{\partial U_e}{\partial t} = -\nabla \cdot \vec{P}_e = -\frac{1}{r}\frac{\partial}{\partial r}(rP_e)$$

$$= \frac{5}{2} T_e \frac{\partial n}{\partial t} + \frac{5\alpha_1}{2m_e\omega_{ce}^2} [2n_o T_e'(nT_e' + n'T_e) + nn_o T_e(T_e'' + T_e'/r)$$

$$+ n_o' n T_e T_e'] + \frac{5\alpha_2 n}{2m_e\omega_{ce}^2 T_e^{3/2}} [5n'T_e T_e' + 2nT_e(T_e'' + T_e'/r)$$

$$+ nT_e'T_i' + n'T_e'Ti].$$

The ion thermal conduction is calculated using

$$\nu_{ii} + \nu_{ie} \cong \frac{\alpha_2 n}{43T_i^{3/2}} \quad (T_e >> T_i/43)$$

and

$$\nu_{in} \cong \frac{\alpha_1 n_o}{21.5} .$$

The result is

$$\frac{\partial U_i}{\partial t} = -\nabla \cdot \vec{P}_i = -\frac{1}{r}\frac{\partial}{\partial r}(rP_i)$$

$$= \frac{5}{2} T_i \frac{\partial n}{\partial t} + \frac{5\alpha_1}{2m_e\omega_{ce}^2} [\frac{86n_o nT_i T_i'}{r} + 86n_o' nT_i T_i' + 86n_o n'T_i T_i'$$

$$+ 86n_o nT_i'^2 + 86n_o nT_i T_i'' + \frac{n_o T_i (nT_e' + n'T_e)}{r}]$$

$$+ \frac{5\alpha_2 n}{m_e\omega_{ce}^2} [(\frac{43nT_i T_i'}{r} + 86n'T_i'T_i + 43nT_i T_i'' - 21.5nT_i^{1/2})/T_i^{3/2}$$

$$+ (nT_e' + nT_i' + n'T_e + n'T_i)/rT_e^{3/2}].$$

The complexity of these results for this simple case illustrates the difficulty of doing a rigorously correct neoclassical diffusion calculation in a realistic magnetic geometry such as a Tokamak or Bumpy Torus.