ELECTRON CYCLOTRON HEATING OF AN ANISOTROPIC PLASMA

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In this PLP, the heating model of PLP 282 will be extended to plasmas in which the electron velocity distribution is anisotropic (or, equivalently, plasmas in which the particle density is not constant on a magnetic field line: $\nabla_{\parallel} n \neq 0$). Magnetic mirror confined plasmas are necessarily anisotropic because of the loss cones, and multipole confined plasmas also exhibit a degree of anisotropy at high energies when electron cyclotron resonance heating is applied.[1] The calculated heating rate for non-relativistic electrons is completely general in the sense that, for a given density distribution $n(\ell)$, the heating rate can be determined exactly for an arbitrary magnetic field. It will be shown that the calculations of Guest[2] and the non-relativistic limit of Grawe's[3] results are a special case of this more general theory in which the magnetic field along the axis is parabolic in position and in which a special distribution function is assumed.

It was shown in PLP 282 that the power absorbed by the plasma in a flux shell $d\psi$ is given by

$$\frac{dP}{d\psi} = \frac{\pi n_o e E_{\perp o}^2}{2B_o |\nabla_{\parallel o} B|} , \tag{1}$$

where $E_{\perp}$ is the rms perpendicular rf electric field, and the subscript o refers to the value of a quantity at the resonance zone. The number of electrons in $d\psi$ is given by

$$\frac{dN}{d\psi} = \int \frac{n(\ell) d\ell}{B(\ell)} ,$$

so that the average heating rate of electrons in $d\psi$ is

$$\frac{d\overline{W}}{dt} = \frac{dP}{d\psi}\Big/\frac{dN}{d\psi} = \frac{\pi n_o e E_{\perp o}^2}{2B_o \int \frac{n(\ell)\,d\ell}{B(\ell)} \, |\nabla_{\parallel o} B|} \; . \tag{2}$$

For $n(\ell)$ = const, equation (2) reduces to

$$\frac{d\overline{W}}{dt} = \frac{\pi e E_{\perp o}^2}{2B_o V' |\nabla_{\parallel o} B|} \; , \tag{3}$$

which is the special case previously used to calculate heating rates.

The only approximations used to obtain equation (2) are the following:

1) $v << c$

2) $v_{\parallel o} << \omega B_o / |\nabla_{\parallel o} B|$

3) $|\nabla_{\parallel o} E_{\perp}^2| << E_{\perp o}^2 \sqrt{\frac{\omega |\nabla_{\parallel o} B|}{v_{\parallel o} B_o}}$

4) $|\nabla_{\parallel o} n| << n_o \sqrt{\frac{\omega |\nabla_{\parallel o} B|}{v_{\parallel o} B_o}}$.

Conditions 1), 2), and 3) are easily satisfied by non-relativistic plasmas in most experimental devices. Condition 4) is satisfied except for a very special distribution function that will be described later.

In order to compare equation (2) with other calculations in the literature, we will consider the special case of a magnetic field given by

$$B(Z) = B(0)(1+\beta Z^2).$$

For $n(\ell)$ = const,

$$V' = \int_o^{\infty} \frac{dZ}{B(Z)} = \frac{1}{B(0)} \int_o^{\infty} \frac{dZ}{1+\beta Z^2} = \frac{\pi}{2\sqrt{\beta} B(0)} \; ,$$

and

$$|\nabla_{\parallel o} B| = 2B(0)\beta Z_o .$$

The heating rate is then

$$\frac{d\overline{W}}{dt} = \frac{e E_{\perp o}^2}{2\sqrt{\beta} Z_o B_o} = \frac{e E_{\perp o}^2}{2B_o} \frac{1}{\sqrt{r-1}} \; , \tag{4}$$

where $r = B_o/B(0)$.

As a second example, we will consider a distribution function in which all the particles mirror at the same distance from the midplane, Z_M. In the absence of parallel electric fields, the density distribution for this case is given by

$$n(Z) = \begin{cases} \alpha_1 B(Z)/v_{\parallel}(Z) & \text{for } Z \le Z_M \\ 0 & \text{for } Z > Z_M \end{cases},$$

where $v_{\parallel}(Z) = v_{\perp}(0)\sqrt{\beta}\sqrt{Z_M^2 - Z^2}$.

For this $n(Z)$, we obtain

$$\int_o^{Z_M} \frac{n(Z)dZ}{B(Z)} = \frac{\alpha_1}{v_{\perp}(0)\sqrt{\beta}} \int_o^{Z_M} \frac{dZ}{\sqrt{Z_M^2 - Z^2}} = \frac{\pi\alpha_1}{2v_{\perp}(0)\sqrt{\beta}},$$

and

$$n_o = \frac{\alpha_1 B_o}{v_{\perp}(0)\sqrt{\beta} Z_o \sqrt{\delta}},$$

where

$$\delta = \frac{Z_M^2}{Z_o^2} - 1 .$$

Substituting the above results into equation (2) gives a heating rate of

$$\frac{d\overline{W}}{dt} = \frac{eE_{\perp o}^2}{2\sqrt{\delta}\ B_o} \frac{r}{r-1} . \qquad (5)$$

In a recent paper on stochastic heating, Grawe[3] treats the same problem by solving the equation of motion of a relativistic particle that executes sinusoidal oscillations along the magnetic field:

$$Z(t) = Z_M \sin\omega_\beta t.$$

He makes the following approximations:

1) homogeneous rf electric field

2) random phases

3) $B(Z) = B(0)(1+\beta Z^2)$

4) $E_{\parallel} = 0$

5) no collisions or other scattering processes

6) $\Delta W_{\perp} << W_{\perp}$

7) $\omega_\beta << \omega_c$

8) $\alpha = \frac{\omega}{\omega_\beta} \frac{r-1}{4r} >> 1.$

Grawe's result in the non-relativistic limit is

$$\frac{dW}{dt} = \frac{\pi e E^2}{2B_o} \left(\frac{\omega}{\omega_\beta}\right) J^2_{\alpha(1-\delta)}[\alpha(1+\delta)], \tag{6}$$

where J is a Bessel function of order $\alpha(1-\delta)$.

In order to compare the two results, consider the case $\delta=1$, for which the Bessel function can be approximated by

$$J_o^2\left[\frac{\omega}{\omega_\beta} \frac{r-1}{2r}\right] \approx \frac{2}{\pi} \frac{\omega_\beta}{\omega} \frac{r}{r-1} \cos^2\left[\frac{\omega}{\omega_\beta} \frac{r-1}{2r} - \frac{\pi}{4}\right].$$

Since the argument of the $\cos^2$ term is large as required by condition 8) above, the solution oscillates rapidly, and we can take the average value:

$$\overline{\cos^2\left[\frac{\omega}{\omega_\beta} \frac{r-1}{2r} - \frac{\pi}{4}\right]} = \frac{1}{2} .$$

Substituting into equation (6) gives

$$\frac{dW}{dt} = \frac{eE^2}{2B_o} \frac{r}{r-1} ,$$

which is identical to equation (5) with $\delta=1$.

The equivalence of the two solutions is a satisfying result since the approaches are considerably different. The advantage of the present treatment over most of those considered in the literature is that the dynamics of the particle motion is hidden in the conductivity used to derive equation (1), and so the technique is not restricted to geometries for which the particle trajectory can be analytically evaluated. Furthermore, for non-relativistic electrons, the assumptions required to derive equation (5) are less restrictive than those required for (6), and the algebra required to produce the answer is simpler.

The same result can also be obtained by a simple phenomenological argument. The average energy gained by an electron during one transit through the resonance can be written as

$$\overline{\Delta W} = \frac{1}{2} m \overline{\Delta v_\perp^2} = \frac{e^2 E_{\perp o}^2}{2m} T^2 ,$$

where T is the time during which the electron stays in phase with the electric field, or the time required for the electron to cross the resonance region. Kawamura and Terashima[4] used the above expression to calculate the heating rate in the TP-M machine at Nagoya University (Japan). Lichtenberg[5] et al. have proposed a similar expression, differing only by a numerical factor of order unity, by solving the Fokker-Planck equation for a Markovian process. The transit time of the particle through the resonance can be approximated by

$$T = \sqrt{\frac{\pi B_o}{v_{\parallel o} \omega |\nabla_{\parallel o} B|}}$$

as suggested by various authors.[1,2,6,7] The resulting energy gain is

$$\overline{\Delta W} = \frac{\pi e E_{\perp o}^2}{2 v_{\parallel o} |\nabla_{\parallel o} B|} .$$

Kuckes[7] arrived at an identical result by solving explicitly the equation of motion of an electron that moves through the resonance region with constant parallel velocity in a field with a constant gradient parallel to $\vec{B}$. The heating rate of an electron in a mirror field of the form considered by Guest and by Grawe can be calculated from the above result as follows:

$$\frac{d\overline{W}}{dt} = \overline{\Delta W}\,\frac{2\omega_\beta}{\pi} = \frac{eE_{\perp o}^2}{2\sqrt{\delta}B_o}\,\frac{r}{r-1}\ .$$

The result is identical to equation (5).

The dependence of the heating rate on δ is reasonable since δ approaches zero as the mirror point of the particles moves closer to the resonance region. For a distribution of particles that all mirror at the resonance (δ=0), the density rises rapidly as the resonance is approached, and becomes infinite at the resonance point. Equivalently, particles that mirror at resonance spend an infinitely longer time in a region dZ at $Z=Z_M$ than in a region dZ at $Z<Z_M$, and so the particles stay in resonance for a much longer time than would be the case if they traversed the resonance with a non-zero $v_\parallel$. Nevertheless, the results of Guest[2] and Grawe[3] indicate a finite heating rate at δ=0. This apparent contradiction comes from the fact that condition 4) on page 2 is violated for this special distribution function.

The heating model can be easily extended to this case, if we replace the density at n_o with an average density within the resonance region:

$$n_o \rightarrow \frac{1}{2}\, n(Z_o - \Delta Z/2) = \frac{\alpha_1 B_o}{2\omega_\beta\sqrt{Z_o \Delta Z}}\ .$$

A more precise derivation would start with the conductivity used to derive equation (1), and allow the density to vary near the resonance at ℓ=0 according to

$$n(\ell) = \frac{\alpha_1 B_o}{\omega_\beta\sqrt{-2Z_o \ell}}\ .$$

The integral is evaluated in a straightforward manner:

$$\frac{dP}{d\psi} = \int \sigma_\perp E_{\perp o}^2\,\frac{d\ell}{B} = \frac{eE_{\perp o}^2 \nu}{\omega}\int_{-\infty}^{\infty} \frac{(\omega^2+\omega_c^2)\,n(\ell)\,d\ell}{(\omega^2-\omega_c^2)^2+4\omega^2\nu^2}$$

$$= \frac{\alpha_1 e E_{\perp o}^2}{2\omega_\beta \sqrt{2Z_o \Delta Z}\,|\nabla_{\parallel o} B|} \int_o^\infty \frac{dx}{\sqrt{x}(1+x^2)} = \frac{\pi\alpha_1 e E_{\perp o}^2}{4\omega_\beta \sqrt{Z_o \Delta Z}\,|\nabla_{\parallel o} B|} .$$

The result is the same as would have been obtained if the previously estimated average density had been substituted into equation (1). The average heating rate is calculated by dividing by the number of particles in $d\psi$,

$$\frac{dN}{d\psi} = \int \frac{n d\ell}{B} = \frac{\alpha_1}{\omega_\beta} \int_o^{Z_o} \frac{dZ}{\sqrt{Z_o^2 - Z^2}} = \frac{\pi\alpha_1}{2\omega_\beta} ,$$

to get the result:

$$\frac{d\overline{W}}{dt} = \frac{e E_{\perp o}^2}{2\sqrt{Z_o \Delta Z}\,|\nabla_{\parallel o} B|} .$$

The width of the resonance ΔZ is calculated by a method suggested by Guest:[2]

$$2 \int_o^{\Delta Z/2} (\omega - \omega_c) \frac{d\ell}{v_\parallel} = \frac{\pi}{2} ,$$

or

$$\Delta Z = \frac{Z_o}{2} \left[\frac{3\pi\omega_\beta r}{2\omega(r-1)}\right]^{2/3} .$$

Using this value for ΔZ, the heating rate is

$$\frac{d\overline{W}}{dt} = \frac{e E_{\perp o}^2}{2\sqrt{2} B_o} \left[\frac{2\omega}{3\pi\omega_\beta}\right]^{1/3} \left[\frac{r}{r-1}\right]^{2/3} \simeq 0.21 \frac{e E_{\perp o}^2}{B_o} \left[\frac{\omega}{\omega_\beta}\right]^{1/3} \left[\frac{r}{r-1}\right]^{2/3} \qquad (7)$$

which is identical to the non-relativistic limit of Grawe's result (equation (6)) with $\delta=0$, except that his coefficient is 0.79. Guest[2] also obtained a similar result with a coefficient of 0.45. The heating efficiency of particles that mirror at the resonance zone is a factor $(\omega/\omega_\beta)^{1/3}$ greater than for particles that mirror well beyond resonance ($\delta>0$). For typical cases, this enhancement is less than an order of magnitude. The non-relativistic model predicts no heating for electrons that mirror before the resonance ($\delta<0$). Grawe[3] has shown, that at relativistic energies, heating does occur for these non-resonant particles.

In many experimental situations, the distribution required to produce the heating rate given by equation (7) probably cannot be attained. For example, in multipoles where most of the particles pass through a region where B is very small, the magnetic moment is not well conserved, and the mirror points of energetic electrons fluctuate widely.[8] At low energies, coulomb collisions cause the electrons to diffuse rapidly in velocity space. Parallel electric fields, either from the microwaves or from space charge effects, casue a broadening of the density distribution near resonance. Magnetic field errors may also smear out the particles.

In these situations, it is probably more realistic to assume a distribution function in the midplane of the form

$$f_o = Ae^{-\frac{W_\perp(0)}{kT_\perp}} e^{-\frac{W_\parallel(0)}{kT_\parallel}} .$$

Using the conservation of energy and magnetic moment, we can write

$$W_\perp(0) = \frac{B(0)}{B} W_\perp$$

and

$$W_\parallel(0) = W_\parallel + W_\perp(1 - \frac{B(0)}{B}) ,$$

so that the distribution function off the midplane is given by

$$f(W_\perp, W_\parallel, B) = Ae^{-\frac{B(0)W_\perp}{BkT_\perp}} e^{-\frac{W_\parallel + W_\perp(1-B(0)/B)}{kT_\parallel}} .$$

The density distribution is determined by a simple integration:

$$n(\ell) = \int_0^\infty \int_0^\infty f(W_\perp, W_\parallel, B(\ell))\,dW_\perp\,dW_\parallel = \frac{B(\ell)T_\parallel n(0)}{B(0)T_\parallel + [B(\ell) - B(0)]T_\perp} .$$

This distribution, incidently, suggests a simple means for estimating the anisotropy of a plasma by measuring the density gradient, since

$$\theta = T_{\perp}/T_{\parallel} = 1 - \frac{\nabla_{\parallel} n}{n} \quad \frac{B}{\nabla_{\parallel} B} \ .$$

Note that for $\theta=1$, $n(\ell) = n(0) = \text{const}$, as required for an isotropic distribution. If we assume a parabolic mirror field, we obtain

$$n_o = \frac{rn(0)}{1+\beta Z_o^2} \ ,$$

and

$$\frac{dN}{d\psi} = \int_o^{\infty} \frac{ndZ}{B} = \frac{\pi rn(0)}{2B_o\sqrt{\beta\theta}} \ .$$

Substituting into equation (2) gives the heating rate:

$$\frac{d\overline{W}}{dt} = \frac{eE_{\perp o}^2}{2B(0)(1+\beta Z_o^2\theta)}\sqrt{\frac{\theta}{r-1}} \ . \tag{8}$$

For an isotropic plasma ($\theta=1$), the heating rate reduces to the case in equation (4). The effect of the anisotropy is to enhance the heating if the resonance is near the midplane ($Z_o=0$) and to reduce the heating if it is well away from the midplane ($\beta Z_o^2>1$). This result is reasonable since a large anisotropy causes the electrons to remain near the bottom of the magnetic well. For a real magnetic mirror with a finite mirror ratio, the assumed distribution function would have to be replaced by a loss cone distribution, and the problem would have to be solved using a more realistic $B(Z)$. In order to arrive at equation (8) using the usual analytic evaluation of particle trajectories, it would be necessary to integrate Grawe's Bessel function solution (equation (6)) over a weighted distribution of values of δ. Such a procedure would almost certainly require numerical methods.

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