A Unified Theory of rf Plasma Heating

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INTRODUCTION

In this paper, the major results of PLP's 186 and 207 will be derived in a more concise and rigorous way, and the treatment of electron cyclotron resonance heating will be extended to higher densities.

The basic method is to calculate the power dP absorbed by a volume of plasma dV in the presence of an oscillating electric field $\vec{E} = \vec{E}_o e^{i\omega t}$ in terms of a real conductivity, σ:

$$\frac{dP}{dV} = \sigma\overline{E^2}$$

where $\overline{E^2}$ is the mean square electric field:

$$\overline{E^2} = \frac{1}{2} EE^* = \frac{1}{2} E_o{}^2,$$

and σ is found from a complex permittivity ε by

$$\sigma = -\omega \; \mathrm{Im}\, \varepsilon .$$

For an azimuthally symmetric poloidal magnetic field, the power can be written as

$$\frac{dP}{d\psi} = \oint \frac{\sigma\overline{E^2} d\ell}{B} . \tag{1}$$

Since σ is generally frequency dependent, it is useful to consider three separate cases which span the entire frequency spectrum:

1) $\omega \lesssim \omega_{ci}$ (ion cyclotron resonance heating)
2) $\omega \sim \omega_{ce}$ (electron cyclotron resonance heating)
3) $\omega >> \omega_{ce}$ (resistive heating)

ION CYCLOTRON RESONANCE HEATING

For $\omega \lesssim \omega_{ci}$ we expect ψ lines to be equipotentials: $\Phi = \Phi(\psi)$. If, in addition, we apply the electric field in an azimuthally symmetric way such that $E_\theta = 0$, the electric field in the plasma can be written as

$$E = |\nabla\Phi| = \frac{d\Phi}{d\psi}|\nabla\psi| = 2\pi RB\frac{d\Phi}{d\psi},$$

and Eq. (1) becomes

$$\frac{dP}{d\psi} = 4\pi^2\left(\frac{d\Phi}{d\psi}\right)^2 \oint \sigma R^2 B d\ell.$$

The permittivity appropriate to the prescribed conditions is

$$\varepsilon = \varepsilon_o\left[1 - \frac{\omega_{pi}{}^2}{\omega}\ \frac{\omega - i\nu_i}{(\omega^2 - \omega_{ci}{}^2 - \nu_i{}^2) - 2i\omega\nu_i}\right],$$

and the corresponding conductivity is

$$\sigma = \varepsilon_o\omega_{pi}{}^2\nu_i\ \frac{\omega^2 + \omega_{ci}{}^2 + \nu_i{}^2}{(\omega^2 - \omega_{ci}{}^2 - \nu_i{}^2)^2 + 4\omega^2\nu_i{}^2}\ .$$

Note that the collision frequency ν_i shifts the resonance off the ion cyclotron frequency. For the usual case of $\nu_i << \omega$ the conductivity becomes

$$\sigma = \varepsilon_o\omega_{pi}{}^2\nu_i\left[\frac{\omega^2 + \omega_{ci}{}^2}{(\omega^2 - \omega_{ci}{}^2)^2 + 4\omega^2\nu_i{}^2}\right],$$

and the power absorbed is given by

$$\frac{dP}{d\psi} = 4\pi^2 ne\left(\frac{d\Phi}{d\psi}\right)^2 \oint \frac{R^2B(B_o{}^2 + B^2)\Delta B}{(B_o^2 - B^2)^2 + 4B_o{}^2\ (\Delta B)^2}\ d\ell\ ,$$

where $B = \frac{M}{e}\ \omega_{ci}$, $B_o = \frac{M}{e}\ \omega$, and $\Delta B = \frac{M}{e}\ \nu_i$. For ΔB sufficiently small, most of the contribution to the intergral comes from $B \cong B_o$, and B can be expanded in a Taylor series about B_o,

$$B(\ell) \cong B_o + \ell\left.\frac{\partial B}{\partial \ell}\right|_{B_o}.$$

The integral can then be evaluated explicitly:

$$\frac{dP}{d\psi} = 2\pi^2 ne \left(\frac{d\Phi}{d\psi}\right)^2 \int_{-\infty}^{\infty} \frac{R^2 B_o \Delta B}{\ell^2 \left(\frac{\partial B}{\partial \ell}\right)^2 + (\Delta B)^2} \, d\ell$$

$$= 2\pi^3 ne \left(\frac{d\Phi}{d\psi}\right)^2 \sum R^2 B_o \Big/ \left.\frac{\partial B}{\partial \ell}\right|_{B_o} . \qquad (2)$$

The sum is taken over every intersection of the ψ line with the constant B surface $B = B_o$.

The ion heating produced by this power can be calculated from

$$nk\Delta T_i V' d\psi = t dP$$

where

$$V' = \frac{dV}{d\psi} = \oint \frac{d\ell}{B} .$$

The ion temperature increase is then

$$k\Delta T_i/e \ (\psi,t) = 2\pi^3 \frac{t}{V'} \left(\frac{d\Phi}{d\psi}\right)^2 \sum R^2 B_o \Big/ \frac{\partial B}{\partial \ell} . \qquad (3)$$

In a real experiment one would probably measure $\Phi(\psi)$. However, an equation for $\Phi(\psi)$ can be derived from Poisson's equation written in the form

$$\oint (\nabla \cdot \varepsilon \vec{E}) \frac{d\ell}{B} = 0.$$

Substitution of the appropriate permittivity (with $\nu_i = 0$) leads to the differential equation (see PLP 207):

$$\frac{d^2\Phi}{d\psi^2} \oint \frac{R^2B}{B^2-B_o^{\ 2}}\, d\ell + \frac{d\Phi}{d\psi}\left[\oint \frac{RB}{B^2-B_o^{\ 2}} \frac{\partial R}{\partial\psi}\, d\ell + \oint \frac{R^2}{B^2-B_o^{\ 2}} \frac{\partial B}{\partial\psi}\, d\ell + \frac{1}{n}\frac{\partial n}{\partial\psi} \oint \frac{R^2B}{B^2-B_o^{\ 2}}\, d\ell \right.$$

$$\left. - 2 \oint \frac{R^2B^2}{(B^2-B_o^{\ 2})^2} \frac{\partial B}{\partial\psi}\, d\ell \right] = 0.$$

ELECTRON CYCLOTRON RESONANCE HEATING

For $\omega \sim \omega_{ce}$ we cannot expect ψ lines to be equipotentials and the electric field $\vec{E}$ must be retained explicitly. $\vec{E}$ can be decomposed into components perpendicular and parallel to $\vec{B}$:

$$\vec{E} = \vec{E}_\perp + \vec{E}_\parallel .$$

The perpendicular permittivity is similar to that for the ion case:

$$\varepsilon_\perp = \varepsilon_o \left[1 - \frac{\omega_{pe}^{\ 2}}{\omega} \frac{\omega - i\nu_e}{(\omega^2 - \omega_{ce}^{\ 2} - \nu_e^{\ 2}) - 2i\omega\nu_e}\right] .$$

The parallel permittivity is somewhat simpler:

$$\varepsilon_\parallel = \varepsilon_o \left[1 - \frac{\omega_{pe}^{\ 2}}{\omega} \frac{\omega + i\nu_e}{\omega^2 + \nu_e^{\ 2}}\right] .$$

For $\nu_e << \omega_{ce}$ the respective conductivities are

$$\sigma_\perp = \varepsilon_o\, \omega_{pe}^{\ 2}\nu_e \left[\frac{\omega^2 + \omega_{ce}^{\ 2}}{(\omega^2 - \omega_{ce}^{\ 2})^2 + 4\omega^2\nu_e^{\ 2}}\right]$$

and

$$\sigma_\parallel = \varepsilon_o \frac{\omega_{pe}^{\ 2}}{\omega^2} \nu_e .$$

The power absorbed can then be written as

$$\frac{dP}{d\psi} = \oint [\sigma_\perp \overline{E_\perp^2} + \sigma_\parallel \overline{E_\parallel^2}] \frac{d\ell}{B}$$

$$= ne \oint \left[\frac{(B_o{}^2+B^2)\ \Delta B}{(B_o{}^2-B^2)^2 + 4B_o\ (\Delta B)^2} \overline{E_\perp^2} + \frac{\Delta B}{B_o^2} \overline{E_\parallel^2} \right] \frac{d\ell}{B} .$$

The boundary conditions on $\vec{E}$ require that

$$\vec{E}_\parallel = \text{const.} \qquad \text{and} \qquad \varepsilon_\perp \vec{E}_\perp = \text{const.}$$

We can therefore bring $\overline{E_\parallel^2}$ out of the integral and write $\overline{E_\perp^2}$ as

$$\overline{E_\perp^2} = \frac{\varepsilon_o^2}{\varepsilon_\perp \varepsilon_\perp^*} \overline{E^2}_{\perp o} .$$

Since

$$\frac{\varepsilon_\perp \varepsilon_\perp^*}{\varepsilon_o^2} = 1 - \frac{2\alpha\omega^2(\omega^2-\omega_{ce}{}^2+2\nu_e{}^2)}{(\omega^2-\omega_{ce}{}^2)^2 + 4\omega^2\nu_e{}^2} + \alpha^2\omega^2 \left[\frac{\omega^2(\omega^2-\omega_{ce}{}^2+2\nu_e{}^2)^2+\nu_e{}^2(\omega^2+\omega_{ce}{}^2)^2}{[(\omega^2-\omega_{ce}{}^2)^2 + 4\omega^2\nu_e^2]^2} \right] ,$$

where $\alpha = \omega_{pe}{}^2/\omega^2 = n/n_c$, the magnetic field can be expanded in a Taylor series as in the ion case to give

$$\frac{\varepsilon_\perp \varepsilon_\perp^*}{\varepsilon_o{}^2} = \frac{\ell^2\ (\partial B/\partial\ell)^2\ -\ell\alpha\ B_o \partial B/\partial\ell\ +\ (1-\alpha)(\Delta B)^2 + \frac{1}{4}\ \alpha^2 B_o{}^2}{\ell^2(\partial B/\partial\ell)^2\ +\ (\Delta B)^2}$$

provided we assume that $\alpha << 1$. The power integral can now be written as

$$\frac{dP}{d\psi} = \frac{1}{2}\, ne \int_{-\infty}^{\infty} \left[\frac{\Delta B\ \overline{E^2_{\perp o}}}{\ell^2(\partial B/\partial\ell)^2 - \ell\alpha B_o \partial B/\partial\ell\ +\ (1-\alpha)(\Delta B)^2\ +\ \frac{1}{4}\alpha^2\ B_o{}^2} + \frac{\Delta B \overline{E_\parallel^2}}{B_o{}^2} \right] \frac{d\ell}{B_o}$$

For an isotropic electric field, $\overline{E_{\perp o}^{2}}$ and $\overline{E^2}$ are of the same order. In particular,

$$\overline{E_{\parallel}^{2}} = \frac{1}{2}\,\overline{E_{\perp o}^{2}} = \frac{1}{3}\,\overline{E^2}\ .$$

For $\Delta B << B_o$ and $\alpha << 1$, the parallel component of the integral can be neglected and the power becomes

$$\frac{dP}{d\psi} = \frac{1}{3}\,ne\,\overline{E^2}\,\frac{\Delta B}{B_o}\int_{-\infty}^{\infty}\frac{d\ell}{\ell^2(\partial B/\partial\ell)^2-\ell\alpha B_o\partial B/\partial\ell + (1-\alpha)(\Delta B)^2+\frac{1}{4}\alpha^2 B_o^{\,2}}$$

$$= \frac{\pi}{3}\,\frac{ne\,\overline{E^2}}{B_o\sqrt{1-\alpha}}\sum 1/\frac{\partial B}{\partial\ell}$$

$$\cong \frac{\pi}{3}\,\frac{ne\,\overline{E^2}}{B_o}\sum 1/\frac{\partial B}{\partial\ell}$$

$$= \frac{\pi}{3}\,ne\,\overline{E^2}\,\left.\frac{d^2V}{dBd\psi}\right|_{B_o}\ . \qquad (4)$$

This result is identical to that derived in PLP 207 under the more stringent condition of $\omega_{pe}^{\,2} << \omega\nu$. The present result is valid for $\omega_{pe}^{\,2} << \omega^2$ or $n << n_c$. The quantity $\frac{d^2V}{dBd\psi}$ lends itself to a simple physical interpretation since it represents the differential resonance volume in the flux shell $d\psi$.

As in the ion case, the temperature increase can be written as

$$k\Delta T_e/e\quad(\psi,t) = \frac{\pi}{3}\,\overline{E^2}\,t\,\frac{1}{V'}\,\frac{d^2V}{dBd\psi} \qquad (5)$$

provided diffusion and energy losses are neglected. The quantity $\frac{1}{V'}\,\frac{d^2V}{dBd\psi}$ is plotted vs. ψ for various values of B_o in PLP 142. If ionizing collisions are taken into account, the power balance can be written as

$$\frac{dP}{d\psi} = V'(kT_e + U_i)\frac{dn}{dt} + V'n\frac{d}{dt}(kT_e)$$

where U_i is the ionization energy. But $\frac{dn}{dt}$ can be written in terms of the ionization time τ_i as

$$\frac{dn}{dt} = \frac{n}{\tau_i} .$$

Substitution of this relation and Eq. (4) into the power balance equation leads to the following differential equation for kT_e:

$$\frac{d}{dt}(kT_e) + \frac{kT_e + U_i}{\tau_i} = \frac{\pi}{3} e \overline{E^2} \quad \frac{1}{V'} \frac{d^2V}{dBd\psi} . \qquad (6)$$

If the duration of the rf pulse is short compared with τ_i, the second term can be neglected and the temperature increases linearly with time according to Eq. (5) until the rf is turned off whereupon the temperature decays according to the equation

$$\frac{d}{dt}(kT_e) + \frac{kT_e + U_i}{\tau_i} = 0.$$

If the rf pulse duration is long compared with τ_i, the temperature initially increases linearly, but after a time of $\sim\tau_i$ it levels off at a value given by

$$kT_{ef} + U_i = \frac{\pi}{3} e \overline{E^2} \tau_i \frac{1}{V'} \frac{d^2V}{dBd\psi} .$$

The density then increases according to the equation

$$\frac{dn}{dt} = \frac{\pi e \overline{E^2} n}{3V'(kT_{ef} + U_i)} \frac{d^2V}{dBd\psi}$$

which has an exponentially growing solution:

$$n(\psi,t) = n_o(\psi)e^{t/\tau_i} .$$

The ionization time τ_i is unfortunately a complicated function not only of the electron temperature but also of the exact form of the electron velocity distribution. It can be formally written as

$$\frac{1}{\tau_i} = n_n \int_0^\infty \sigma_i(v) v f(v) dv$$

where $\sigma_i(v)$ is the ionization cross section and $f(v)$ is the electron distribution function. The ionization time has been calculated for the special case of a Maxwellian distribution in PLP 142. The result is sufficiently complicated that an analytic expression cannot be written, but a graphical solution is presented in PLP 142.

The efficiency of power transfer for electron cyclotron resonance heating can be written as

$$\eta = \frac{1}{P_o} \int \frac{dP}{d\psi} d\psi = \frac{\pi}{3} e \frac{\overline{E^2}}{P_o} \int n \frac{d^2V}{dB d\psi} d\psi$$

where P_o is the input power to the cavity. The mean square electric field can be expressed in terms of P_o and the perturbed Q of the cavity as

$$\overline{E^2} = \frac{P_o Q}{\varepsilon_o \omega V}.$$

For $n(\psi)$ = const., the efficiency therefore becomes

$$\eta = \frac{\pi}{3} Q \frac{n}{n_c} \frac{B_o}{V} \left.\frac{dV}{dB}\right|_{B_o} .$$

Hence for efficient heating, we want a high Q, a high density, and an rf frequency such that $\frac{B_o}{V} \left.\frac{dV}{dB}\right|_{B_o}$ is large.

RESISTIVE HEATING

For $\omega >> \omega_{ce}$, heating takes place only through collisional resistivity, and the magnetic field serves only to inhibit the flow of plasma energy to the walls of the confinement device. The permittivity is isotropic and has the form

$$\varepsilon = \varepsilon_o \left[1 - \frac{\omega_{pe}{}^2}{\omega} \frac{\omega + i\nu_e}{\omega^2 + \nu_e{}^2} \right] .$$

The corresponding conductivity is

$$\sigma = \varepsilon_o \frac{\omega_{pe}{}^2 \nu_e}{\omega^2 + \nu_e{}^2} .$$

As in the previous section, the electric field can be decomposed into perpendicular and parallel components which are subject to different boundary conditions, and Eq. (1) takes the form

$$\frac{dP}{d\psi} = \oint \frac{1}{3} \sigma\overline{E^2} \left(1 + \frac{2\varepsilon_o{}^2}{\varepsilon\varepsilon^*} \right) \frac{d\ell}{B}$$

$$= \frac{1}{3} \sigma\overline{E^2} \left(1 + \frac{2\varepsilon_o{}^2}{\varepsilon\varepsilon^*} \right) V' .$$

Since

$$\frac{\varepsilon\varepsilon^*}{\varepsilon_o{}^2} = \frac{(\omega^2 - \omega_{pe}{}^2) + \omega^2 \nu_e{}^2}{\omega^2(\omega^2 + \nu_e{}^2)} ,$$

the power can be written as

$$\frac{dP}{d\psi} = \frac{1}{3} \varepsilon_o \overline{E^2} \omega_{pe}{}^2 \nu_e{}^2 \left[\frac{1}{\omega^2 - \nu_e{}^2} + \frac{2\omega^2}{(\omega^2 - \omega_{pe}{}^2)^2 + \omega^2 \nu_e{}^2} \right] V' .$$

For the usual case of $\nu_e << \omega$, the above equation becomes

$$\frac{dP}{d\psi} = \frac{1}{3} \quad \varepsilon_o \overline{E^2} \quad \nu_e \left[\alpha + \frac{2\alpha}{(1-\alpha)^2}\right] \quad V'$$

where

$$\alpha = \frac{\omega_{pe}^{\ 2}}{\omega^2} = \frac{n}{n_c} \quad .$$

It is clear from the above expression that the absorption decreases monatonically with decreasing density and in the limit $n << n_c$ becomes

$$\frac{dP}{d\psi} = \nu_e \frac{ne^2}{m\omega^2} \overline{E^2} V' \quad . \qquad (7)$$

The mass dependence indicates that the power is abosrbed almost entirely by electrons rather than ions.

Following the usual procedure, the temperature rise can be calculated:

$$k\Delta T_e/e \quad (\psi,t) = \frac{e}{m} \frac{\nu_e t}{\omega^2} \overline{E^2}. \qquad (8)$$

The temperature rise is independent of ψ if ν_e is constant in space. Ionization can be included by a method exactly analagous to that in the previous section leading to the differential equation

$$\frac{d}{dt} (kT_e) + \frac{kT_e + U_i}{\tau_i} = \nu_e \frac{e^2}{m\omega^2} \overline{E^2}. \qquad (9)$$

For $n(\psi)$ = const., the efficiency of resistive heating is given by

$$\eta = \frac{1}{P_o} \int \frac{dP}{d\psi} \, d\psi = Q \frac{n}{n_c} \frac{\nu_e}{\omega} \, .$$

Hence, resistive heating is less efficient than electron cyclotron resonance heating by the factor

$$\frac{3}{\pi} \frac{\nu_e}{\omega} \frac{V}{B_o} \frac{dB}{dV} \quad .$$