# RFP PROFILE REPRESENTATIONS

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## I. Introduction

For many purposes, it is desirable to have simple analytic expressions for the magnetic field and current profiles in a reversed field pinch (RFP). For example, numerical codes that follow field lines or particle orbits over great distances are much more efficient if the field at a given location can be calculated quickly. Also, real-time determination of experimental quantities such as plasma resistivity and inductance is facilitated by the use of such expressions. The goal of this paper is to derive physically realistic profiles that are algebraically simple, yet sufficiently accurate for such purposes.

The input parameters to the calculation are the three quantities routinely and straightforwardly measured in an RFP: total plasma current $I_p$, average toroidal field $\langle B_t \rangle$ and toroidal field at the wall $B_{tw}$. We consider only the cylindrical (large aspect ratio) approximation in which the toroidal field is the same everywhere on the (circular) wall at radius a. From the measured quantities, one can define the dimensionless parameters:

$$F = B_{tw} \ / \ \langle B_t \rangle$$

$$\theta = \mu_o I_p \ / \ 2\pi a \langle B_t \rangle$$

## II. Bessel Function Model

The classic model of the RFP begins with Ampere's law,

$$\nabla \times \underline{B} = \mu_o \underline{J}$$

Force balance requires

$$\underline{J} \times \underline{B} = \nabla p$$

where p is the plasma pressure. For a low-beta plasma, where

$$\beta = 2\mu_o p \;/\; B^2$$

the pressure vanishes, and $\underline{J} \times \underline{B} = 0$ leads to the condition

$$\underline{J} = \lambda \underline{B} \;/\; \mu_o$$

where $\lambda$ is a scalar function of position. J. B. Taylor showed[1] that helicity conservation leads to a minimum energy state of

$$\nabla \times \underline{B} = \lambda \underline{B}$$

in which $\lambda$ is a scalar constant, independent of position. In a circular cylinder, this equation has the solutions:

$$B_t = B_{to} J_o(\lambda r)$$

$$B_p = B_{to} J_1(\lambda r)$$

where $J_o$ and $J_1$ are Bessel functions of zero and first order, respectively, and $B_{to}$ is the toroidal field on axis (r=0). In terms of $\lambda$, $\theta$ is given by

$$\theta = \lambda a / 2$$

and F is related to $\theta$ through

$$F = \theta J_o(2\theta) / J_1(2\theta)$$

Field reversal occurs when $J_o(\lambda a) = 0$ or $\theta = 1.20241$. Above this value of $\theta$, the field reversal surface is at a radius of $r_R = 1.20241a/\theta$.

In the usual RFP state, $\theta$ is less than the value of 1.91586 at which the first zero of $J_1(\lambda a)$ occurs. Since $\lambda r < \lambda a = 2\theta$, it is possible to expand the Bessel functions in power series that converge reasonably rapidly for the range of interest. Such an expansion for $J_o(2\theta)$ is given by

$$J_o(2\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(k!)^2}$$

the first six terms of which are

$$J_o(2\theta) = 1 - \theta^2 + \theta^4/4 - \theta^6/36 + \theta^8/576 - \theta^{10}/14400 + \cdots$$

yielding an accuracy of better than 1% up to $\theta$=1.9. From $J_o$, the first order Bessel function can be generated from

$$J_1(x) = - dJ_o(x)/dx$$

which in terms of the power series above is

$$J_1(2\theta) = \theta - \theta^3/2 + \theta^5/12 - \theta^7/144 + \theta^9/2880 - \cdots$$

From these expressions, one can write the fields:

$$B_t(r)/B_{to} = 1 - (\theta r/a)^2 + (\theta r/a)^4/4 - (\theta r/a)^6/36 + (\theta r/a)^8/576 - (\theta r/a)^{10}/14400 + \cdots$$

$$B_p(r)/B_{to} = \theta r/a - (\theta r/a)^3/2 + (\theta r/a)^5/12 - (\theta r/a)^7/144 + (\theta r/a)^9/2880 - \cdots$$

The corresponding toroidal and poloidal current densities can be calculated from

$$\underline{J} = 2\theta \underline{B} / \mu_o a$$

Since the toroidal field on axis $B_{to}$ is not an easily measured quantity, it is useful to relate it to the average field using

$$\langle B_t \rangle = \frac{2}{a^2} \int_o^a r B_t dr$$

for which the power series expansion gives

$$\langle B_t \rangle / B_{to} = 1 - \theta^2/2 + \theta^4/12 - \theta^6/144 + \theta^8/2880 - \theta^{10}/86400 + \cdots$$

In order to calculate $B_{to}$ from the measured quantity $\langle B_t \rangle$, the above expression can be inverted to give

$$B_{to}/\langle B_t \rangle = 1 + \theta^2/2 + \theta^4/6 + 7\theta^6/144 + 13\theta^8/960 + 107\theta^{10}/28800 + \cdots$$

The function $F(\theta)$ can also be expanded as a polynomial with the result

$$F = 1 - \theta^2/2 - \theta^4/12 - \theta^6/48 - \theta^8/180 - 13\theta^{10}/8640 - 11\theta^{12}/26880 - \cdots$$

## III. Modified Bessel Function Model

The difficulty with the Bessel function model is that a finite current density is required at the wall ($r=a$). Since the plasma temperature approaches zero at the wall, the resistivity there is high, and a given electric field cannot drive a significant current. The modified Bessel function model is a modification of the Bessel function model in which the toroidal and poloidal current densities are made to go to zero at the wall. Experiments[2-4] show it to be a better representation of reality. The usual modified Bessel function model[5] assumes $\lambda$ constant out to some radius (typically 0.7a) beyond which it falls linearly to zero at $r=a$. We here adopt a slightly different strategy and express the fields as truncated power series of radius in which the coefficients of the terms in the series depend upon F and are adjusted to satisfy the requirements of zero current at the wall and nearly constant $\lambda$ at small radii. The coefficients are written in terms of F rather than $\theta$ because it is a better expansion parameter since the typical RFP state has $F \sim -0.2$ and $\theta \sim 1.6$.

As a starting point we express the toroidal field, normalized to the average toroidal field, as a fourth order polynomial in radial position with the odd powers absent as required for cylindrical symmetry:

$$B_t(r)/\langle B_t\rangle = f_o(F) + f_1(F)(r/a)^2 + f_2(F)(r/a)^4$$

From the definition of F we obtain

$$f_o + f_1 + f_2 = F$$

From the definition of $\langle B_t\rangle$ we obtain

$$f_o + f_1/2 + f_2/3 = 1$$

The condition of no poloidal current density at the wall requires $dB_t(a)/dr = 0$ or

$$f_1 + 2f_2 = 0$$

Combining these three equations gives

$$f_o = 3 - 2F$$

$$f_1 = -6(1 - F)$$

$$f_2 = 3(1 - F)$$

or

$$B_t(r)/\langle B_t\rangle = 3 - 2F - 6(1-F)(r/a)^2 + 3(1-F)(r/a)^4$$

The toroidal field profile calculated above is compared with the result from the Bessel function model for various values of F in figure 1. The values of F are chosen to represent a non-reversed discharge (F=0.5), a just reversed discharge (F=0), and a highly reversed discharge (F=-0.5). From $J_p(r) = -dB_t/dr/\mu_o$, the normalized poloidal current density is

$$\mu_o a J_p(r)/\langle B_t\rangle = -2f_1(F)(r/a) - 4f_2(F)(r/a)^3$$

or

$$\mu_o a J_p(r)/\langle B_t\rangle = 12(1-F)[r/a - (r/a)^3]$$

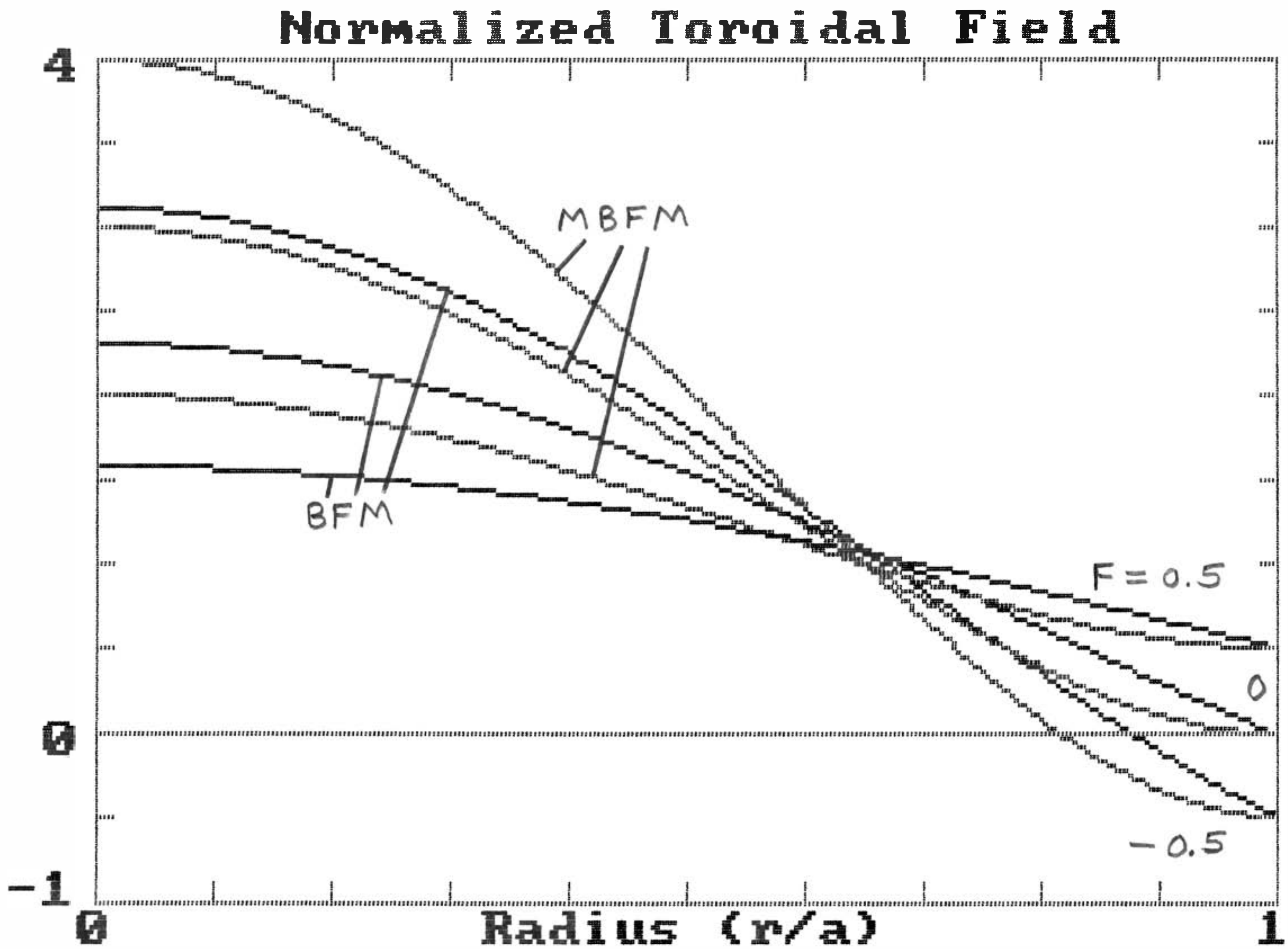
The poloidal current density profile calculated above is compared with the result from the Bessel function model for various values of F in figure 2.

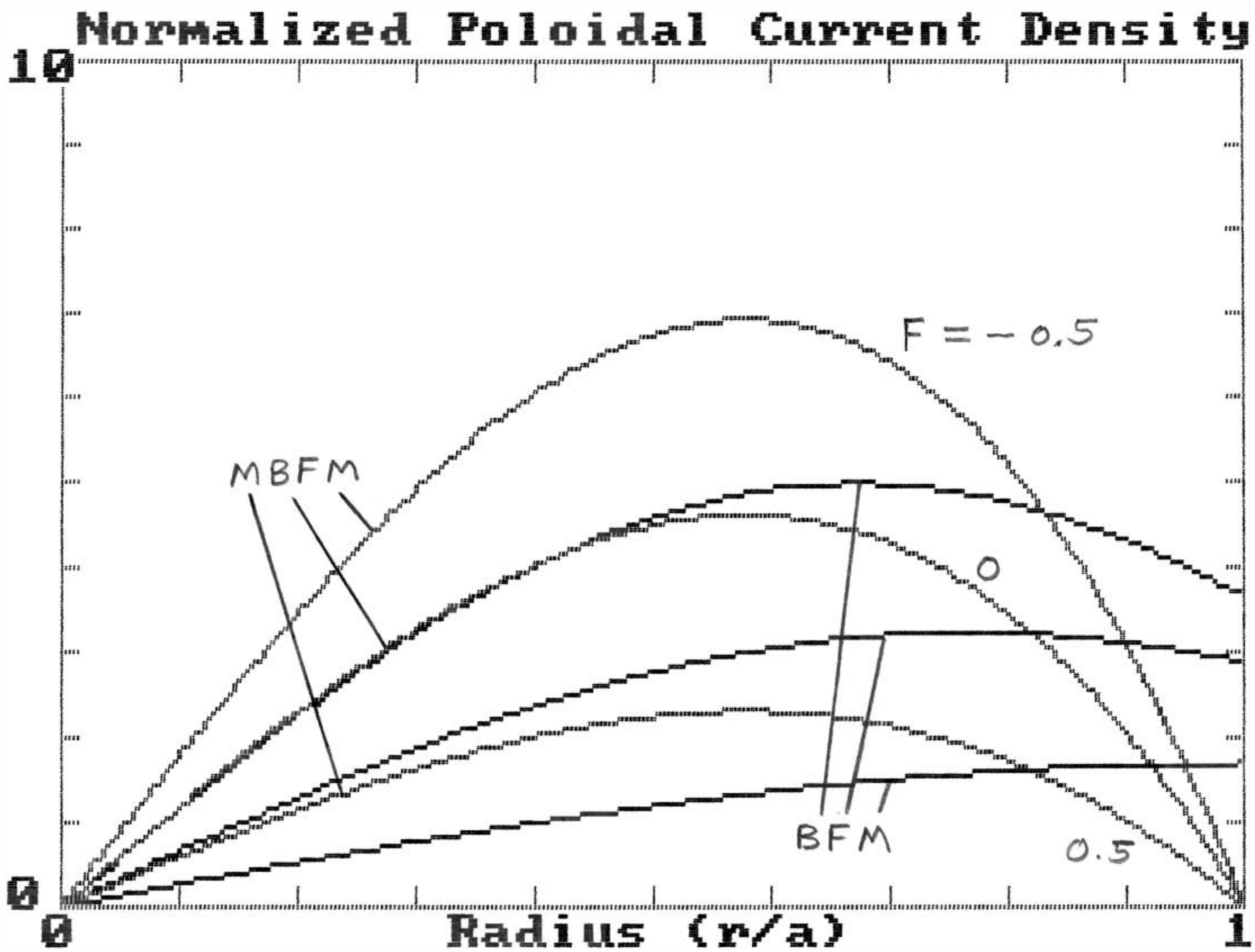
A similar procedure can be applied by writing the normalized toroidal current density as a power series,

$$\mu_o a J_t(r)/\langle B_t\rangle = g_o(F) + g_1(F)(r/a)^2 + g_2(F)(r/a)^4$$

From $J_t(r) = d(rB_p)/dr/\mu_o r$, the normalized poloidal magnetic field is

$$B_p(r)/\langle B_t\rangle = g_o(F)(r/a)/2 + g_1(F)(r/a)^3/4 + g_2(F)(r/a)^5/6$$


Normalized Toroidal Field
4
0
-1
0
Radius (r/a)
1
MBFM
BFM
F = 0.5
0
- 0.5



Normalized Poloidal Current Density
10
0
0
Radius (r/a)
1
F = -0.5
MBFM
0
BFM
0.5

On axis (r=0), $\lambda$ has the value $\lambda_o = \mu_o J_t(0)/B_t(0) = g_o/f_o a$. To have $\lambda$ as constant as possible, we require $\mu_o J_p'(0)/B_p'(0) = -4f_1/g_o a = \lambda_o$, or

$$g_o^2 = -4f_o f_1$$

where the ' denotes a derivative with respect to r. To further ensure the constancy of $\lambda$ near r=0 we require $\mu_o J_t''(0)/B_t''(0) = g_1/f_1 a = \lambda_o$, or

$$g_1 = g_o f_1/f_o$$

where the '' denotes a second derivative with respect to r. Finally, from the requirement that $J_t(a) = 0$, we obtain

$$g_2 = -g_1 - g_o$$

Substituting the previously derived values of $f_o$, $f_1$, and $f_2$ gives

$$g_o = [72 - 120F + 48F^2]^{1/2}$$

$$g_1 = -6g_o(1-F) / (3-2F)$$

$$g_2 = g_o(3-4F) / (3-2F)$$

With these coefficients, the normalized poloidal field can be written

$$B_p(r)/\langle B_t\rangle = [18-30F+12F^2]^{1/2}[r/a - 3(1-F)(r/a)^3/(3-2F) + (3-4F)(r/a)^5/3(3-2F)]$$

and the normalized toroidal current density can be written

$$\mu_o a J_t(r)/\langle B_t\rangle = [72-120F+48F^2]^{1/2}[1 - 6(1-F)(r/a)^2/(3-2F) + (3-4F)(r/a)^4/(3-2F)]$$

The field and current density profiles calculated above are compared with the results from the Bessel function model for various values of F in figures 3 and 4, respectively.

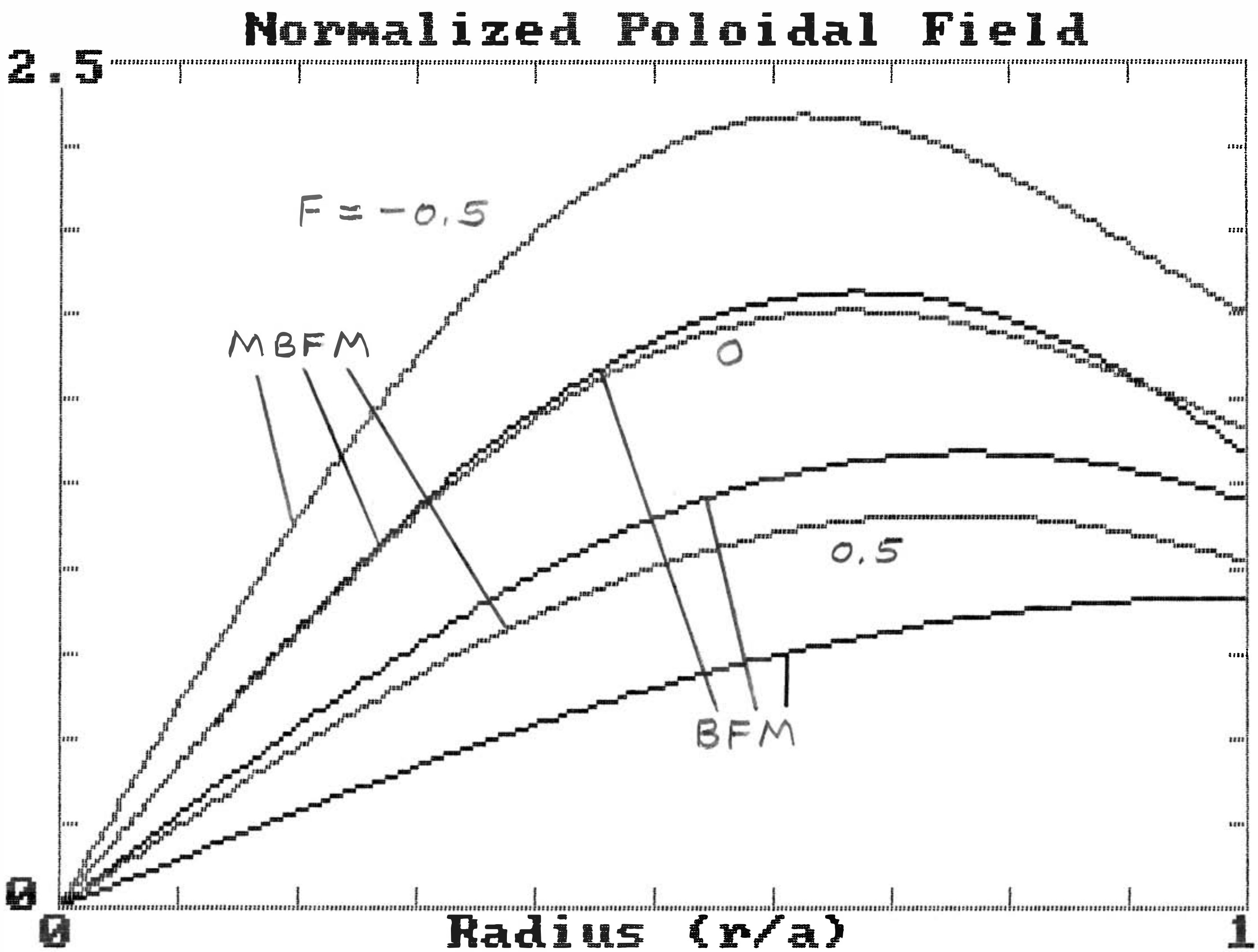
The q-profile given by

$$q(r) = rB_t/R_oB_p$$

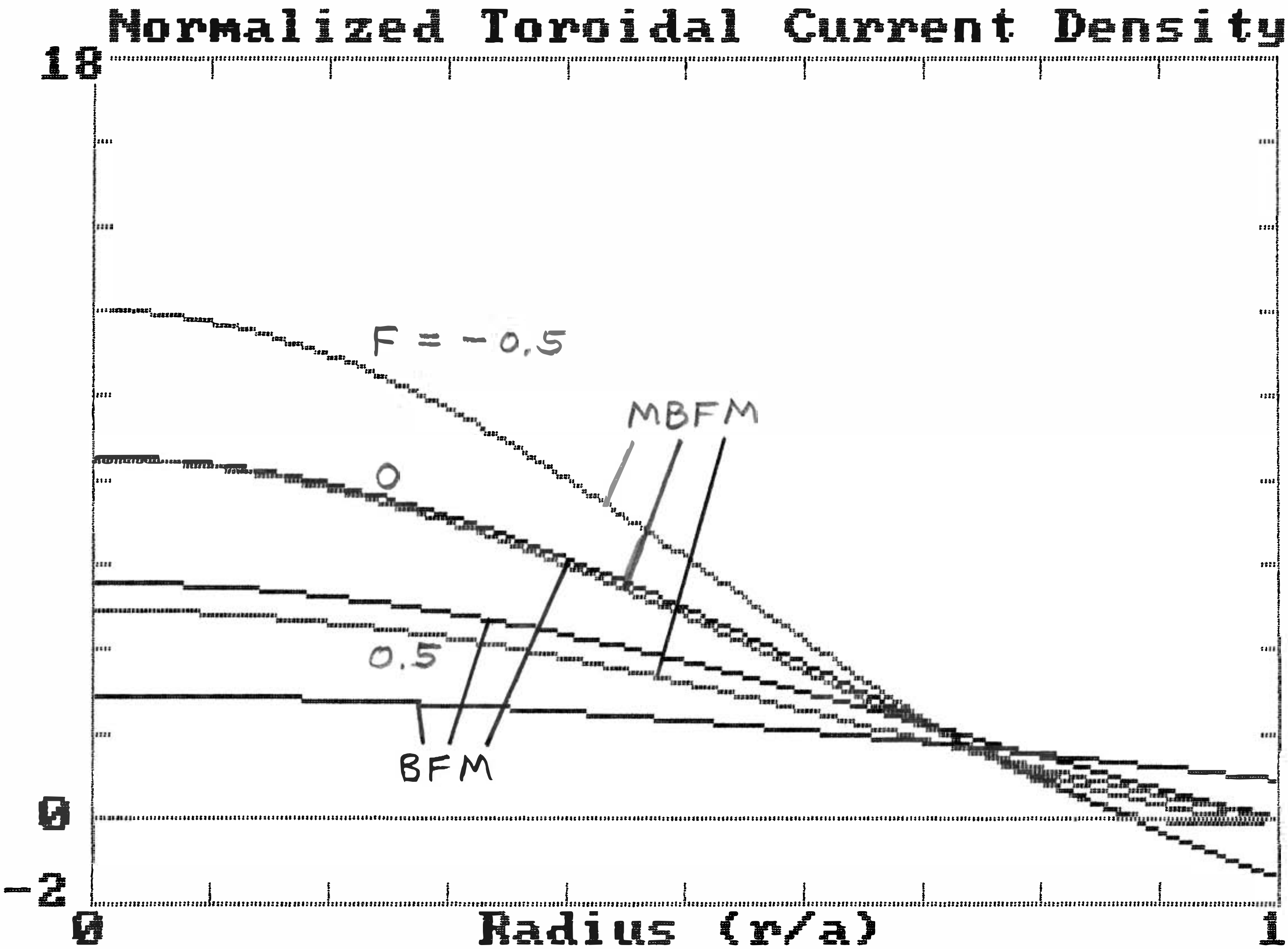
is plotted in figure 5 for the Bessel function model and the modified Bessel function model for various values of F and a typical aspect ratio of $R_o/a=3$. One curious feature is the near constancy of q on axis as a function of F (or $\theta$) in the RFP state,

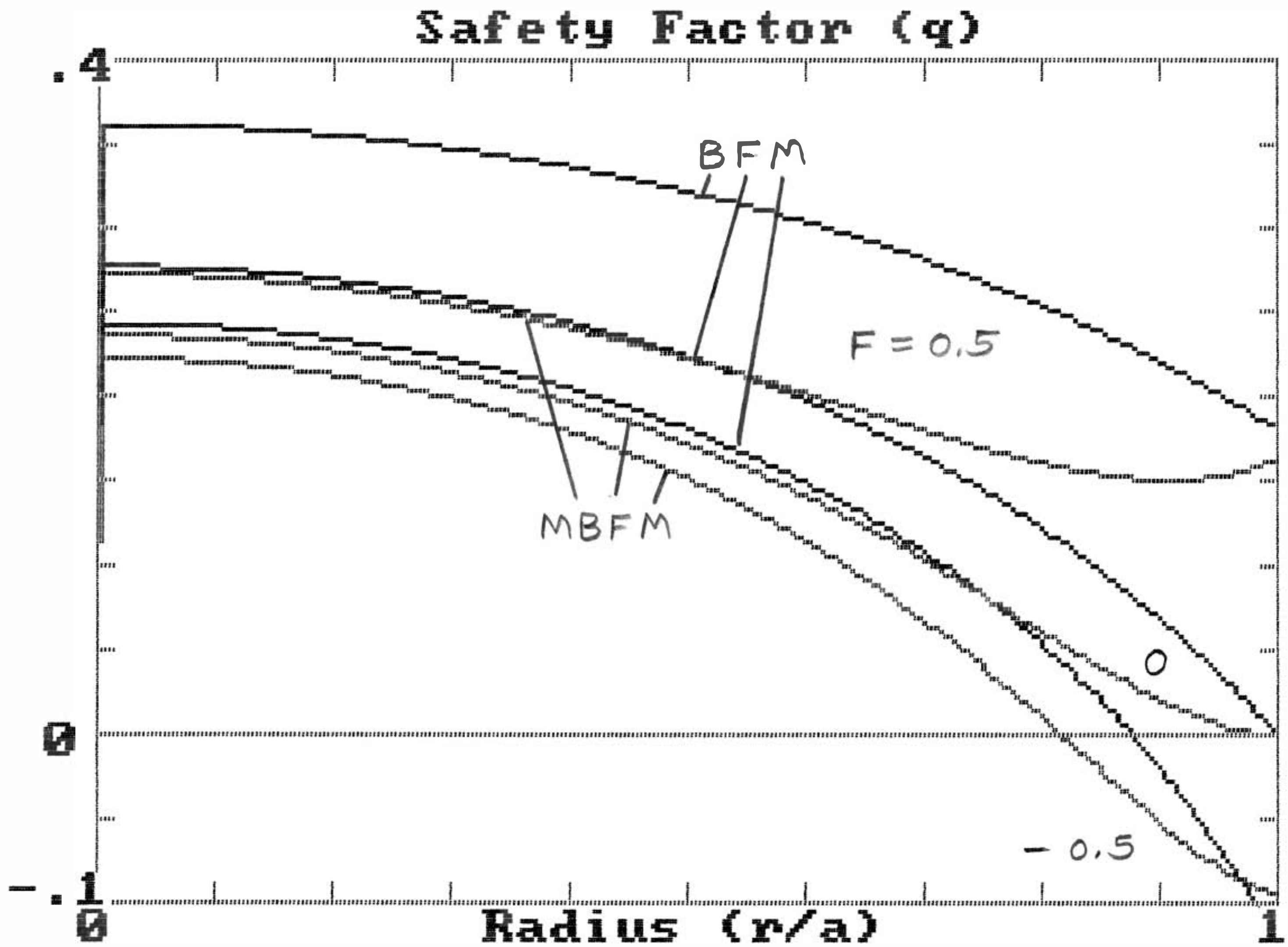
$$q(0) = a(3-2F) / R_o[18-30F+12F^2]^{1/2}$$

The safety factor on axis varies from $0.707a/R_o$ at F=0 to



Normalized Poloidal Field
2.5
0
0
1
Radius (r/a)
F = -0.5
MBFM
0
0.5
BFM


Normalized Toroidal Current Density
18
0
-2
0
1
Radius (r/a)
F = -0.5
MBFM
0
0.5
BFM


Safety Factor (q)
.4
0
-.1
0
Radius (r/a)
1
BFM
F = 0.5
MBFM
0
− 0.5

$0.645a/R_0$ at $F=-1$. The safety factor on axis is plotted as a function of $\theta$ in figure 6 for a typical aspect ratio of $R_0/a=3$ and compared with the Bessel function model result of $q(0)=a/R_0\theta$, which is much less constant.

From the toroidal current density profile and the definition of $\theta$, it is straightforward to show that

$$\theta = g_0/2 + g_1/4 + g_2/6 = g_0(3-F) \;/\; 6(3-2F)$$

or

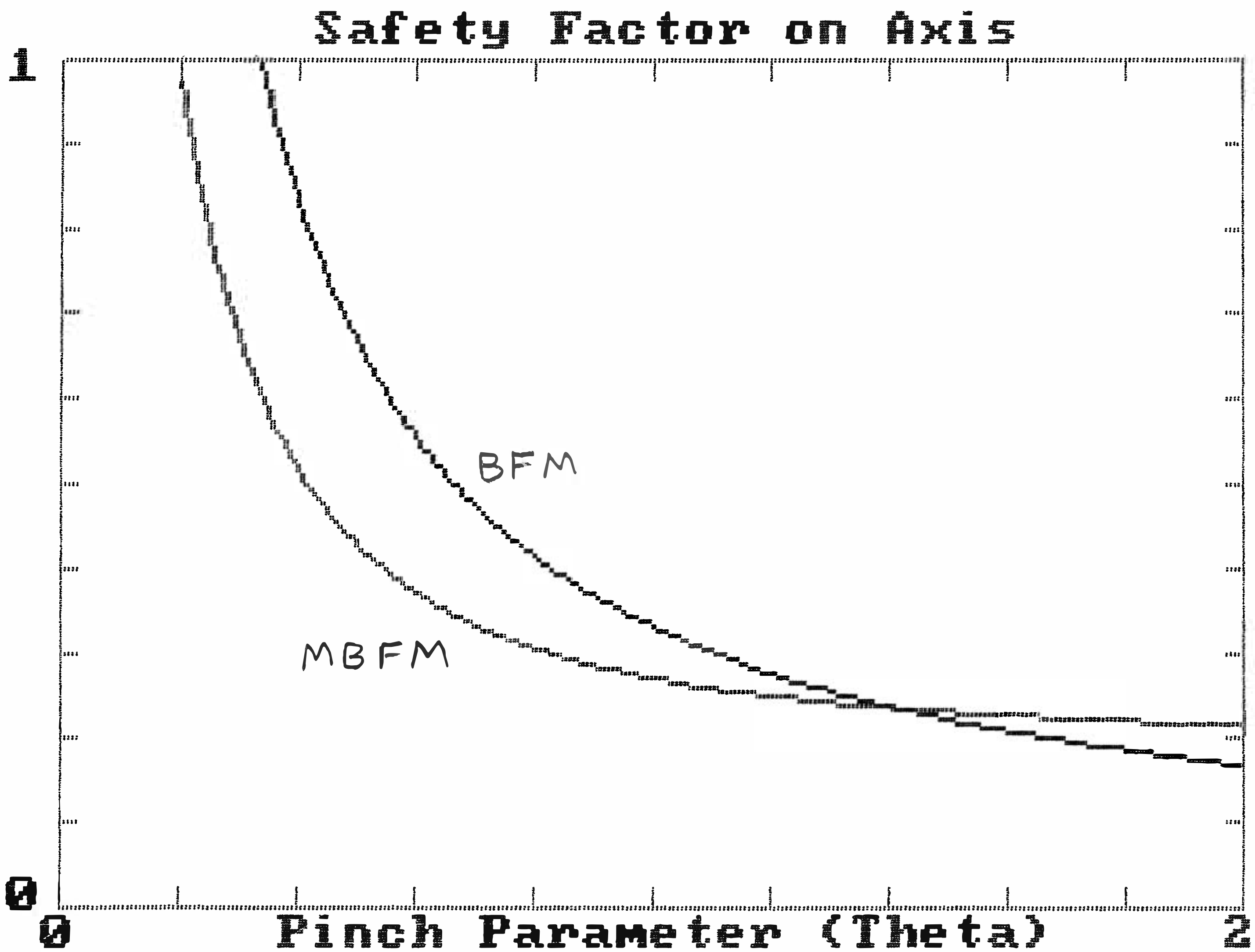
$$\theta = (3-F)[2-10F/3+4F^2/3]^{1/2} \;/\; (3-2F)$$

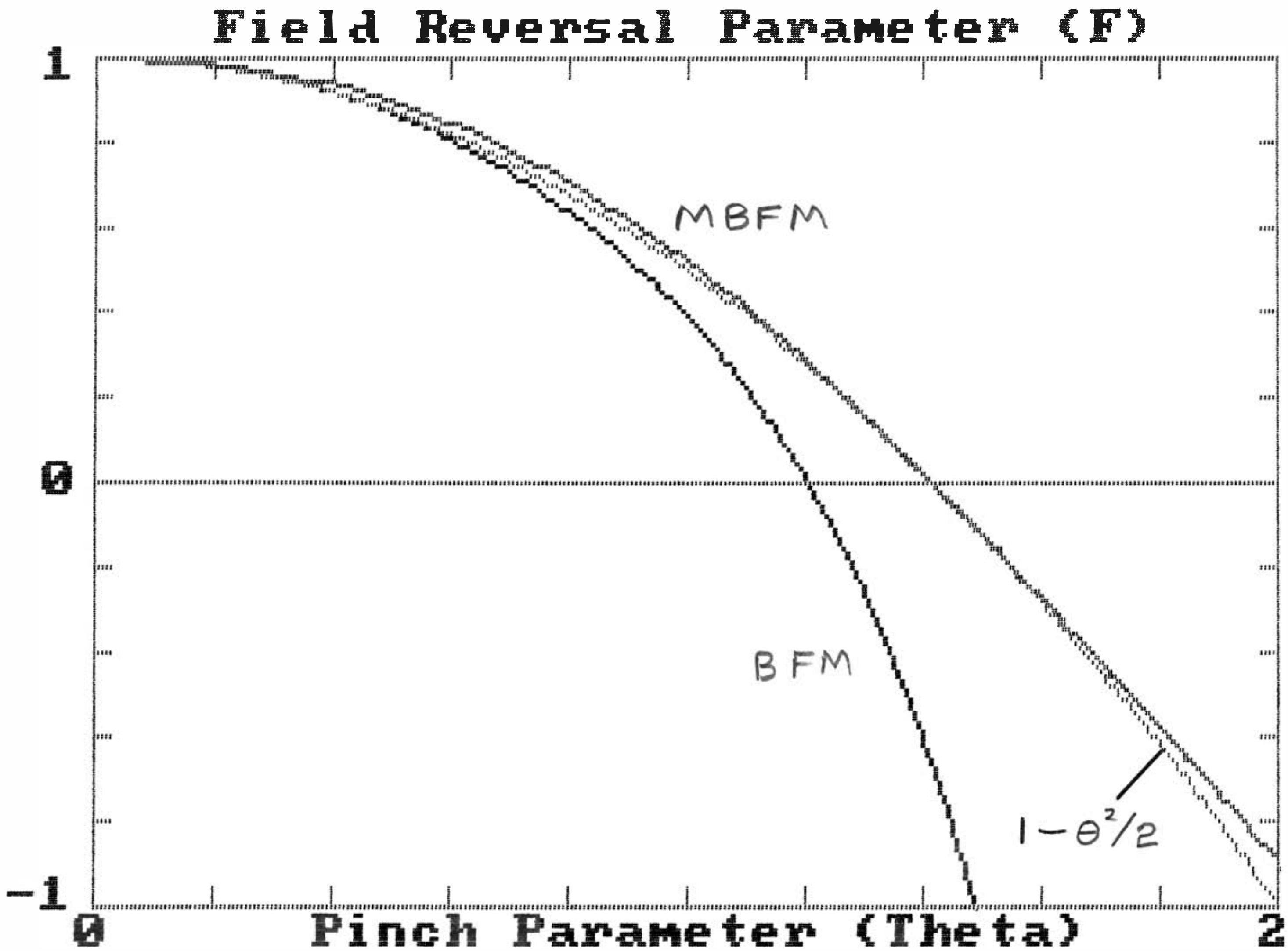
Note that an RFP state exists for $\theta > 2^{1/2}$. The F-$\theta$ curve calculated above is compared with the Bessel function model result in figure 7. For many purposes it is useful to have $\theta$ as a function of F. Inverting the above equation is difficult, but $F(\theta)$ can be reasonably well fit to a surprisingly simple function,

$$F \approx 1 - \theta^2/2$$

as shown in figure 7. This relation bears an amusing resemblance to $F(\theta)$ for the Bessel function model as derived earlier.

A limitation of the modified Bessel function model above is


Safety Factor on Axis
1
0
0
2
Pinch Parameter (Theta)
BFM
MBFM


Field Reversal Parameter (F)
1
0
-1
0
2
Pinch Parameter (Theta)
MBFM
BFM
1 − θ²/2

that the current density $\underline{J}$ is not strictly parallel to $\underline{B}$ everywhere. Thus it is necessary to generalize the definition of $\lambda$ to include both parallel and perpendicular components:

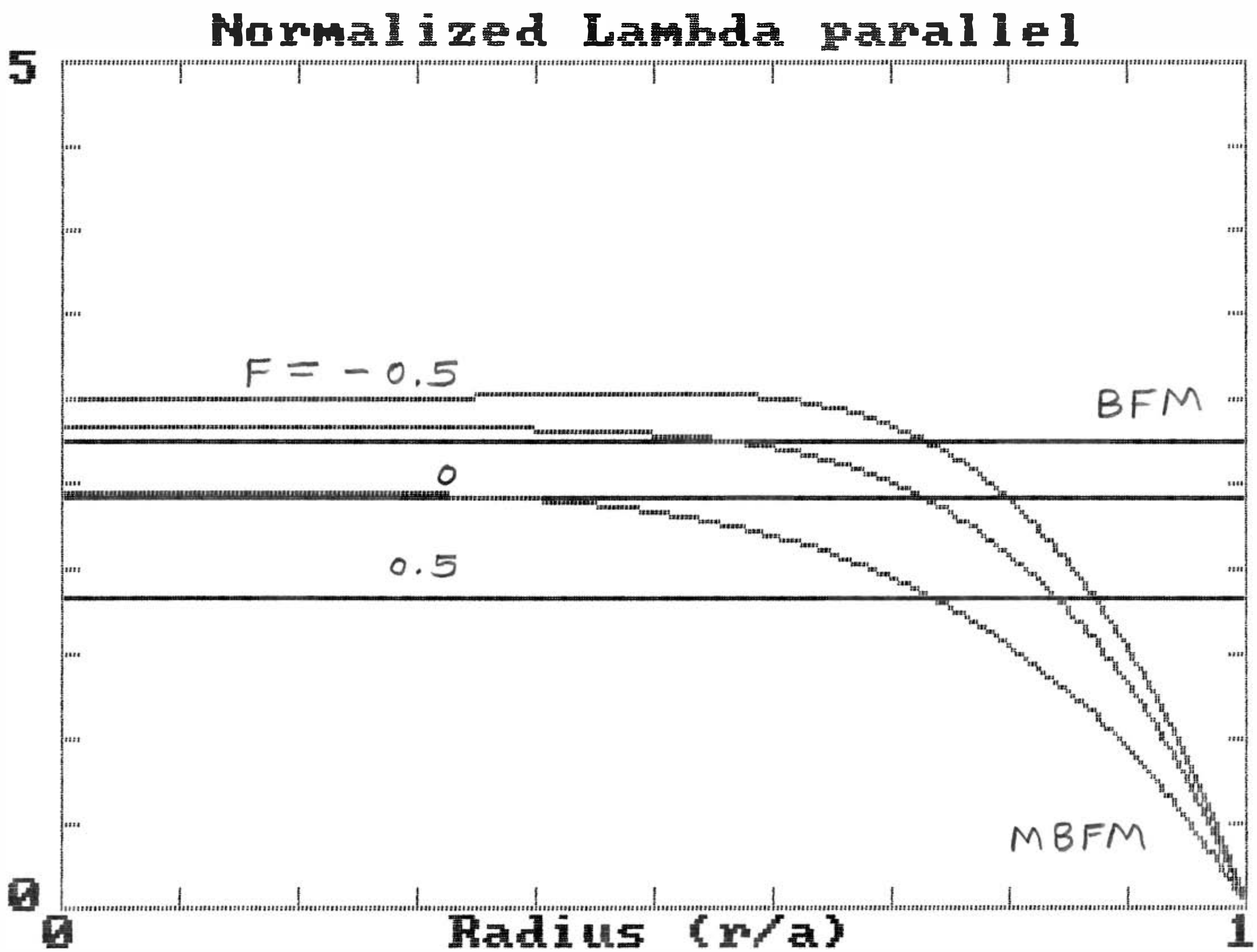
$$\lambda_{||} = \frac{\mu_o \underline{J}\cdot\underline{B}}{B^2} = \mu_o \frac{J_p B_p + J_t B_t}{B_p^{\,2} + B_t^{\,2}}$$

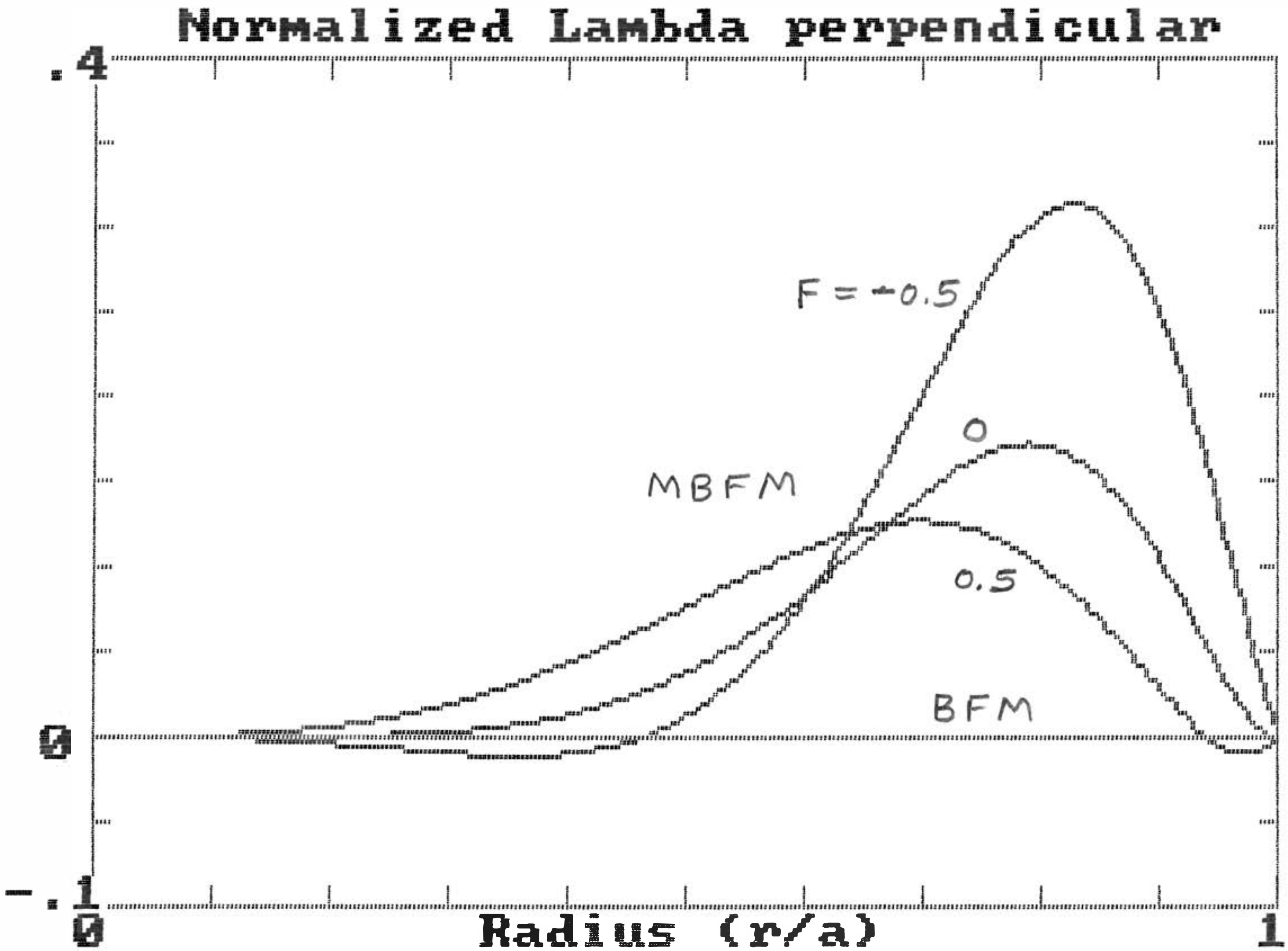
$$\lambda_{\perp} = \frac{\mu_o \, \underline{J}\times\underline{B}}{B^2} = \mu_o \frac{J_t B_p - J_p B_t}{B_p^{\,2} + B_t^{\,2}}$$

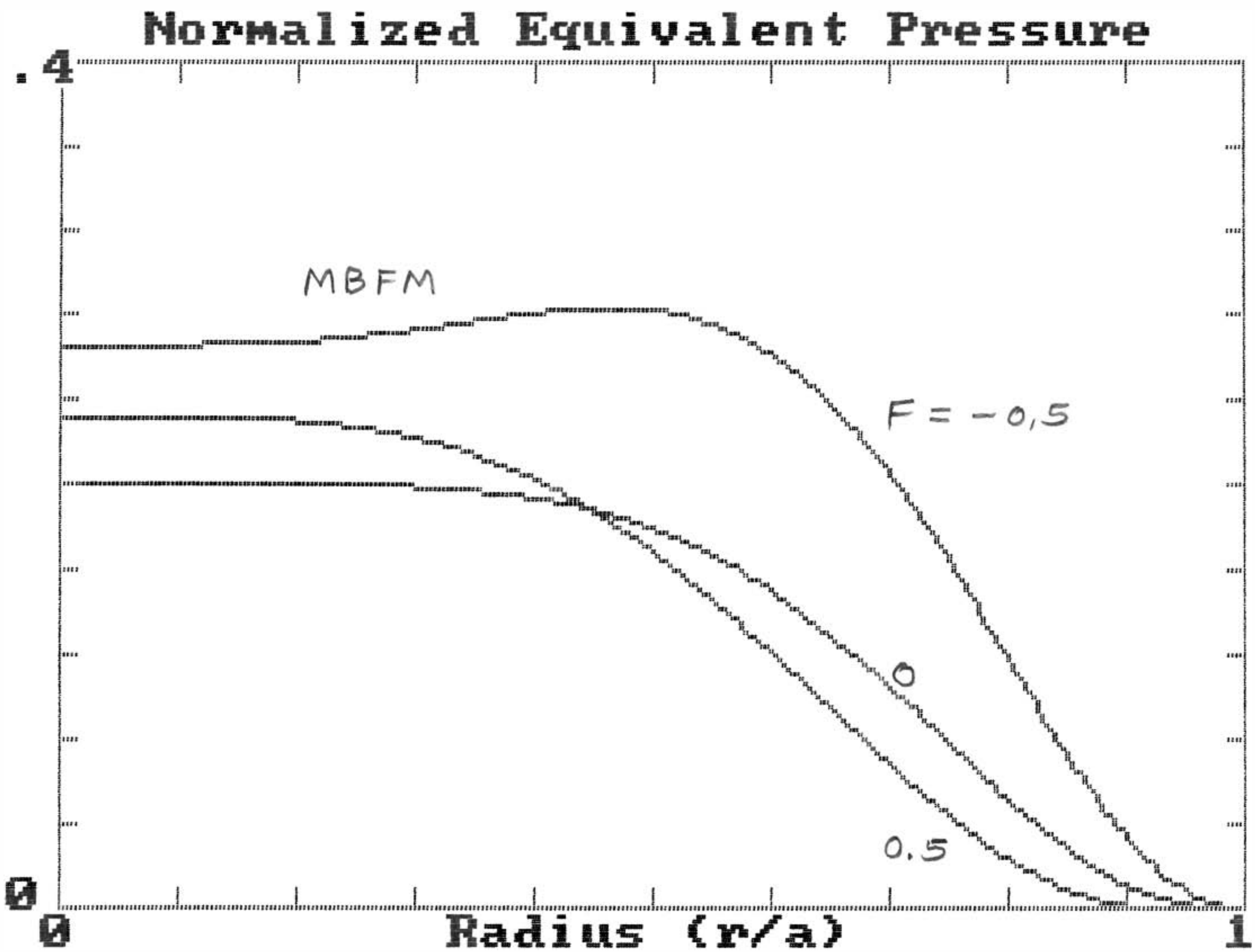
The evaluation of these expressions is long and tedious, and the algebraic results will not be displayed here, but figures 8 and 9 show graphs of $\lambda_{||}$ and $\lambda_{\perp}$ normalized to 1/a as a function of radius for various values of F. Also shown is the Bessel function model result in which $\lambda$ is independent of r and equal to $2\theta/a$ and $\lambda_{\perp}$ is identically zero. Note that $\lambda_{||}$ for the modified Bessel function model is nearly constant out to about r/a=0.7 and then falls to zero at r=a. The normalized value of $\lambda_{\perp}$ is much smaller than $\lambda_{||}$, and thus the approximation that $\underline{J}$ is parallel to $\underline{B}$ is reasonably well satisfied. Actually the existence of a small current perpendicular to $\underline{B}$ is equivalent to assuming that the plasma has a finite pressure. From $\underline{J}\times\underline{B}=\nabla p$, the equivalent pressure profile can be calculated from

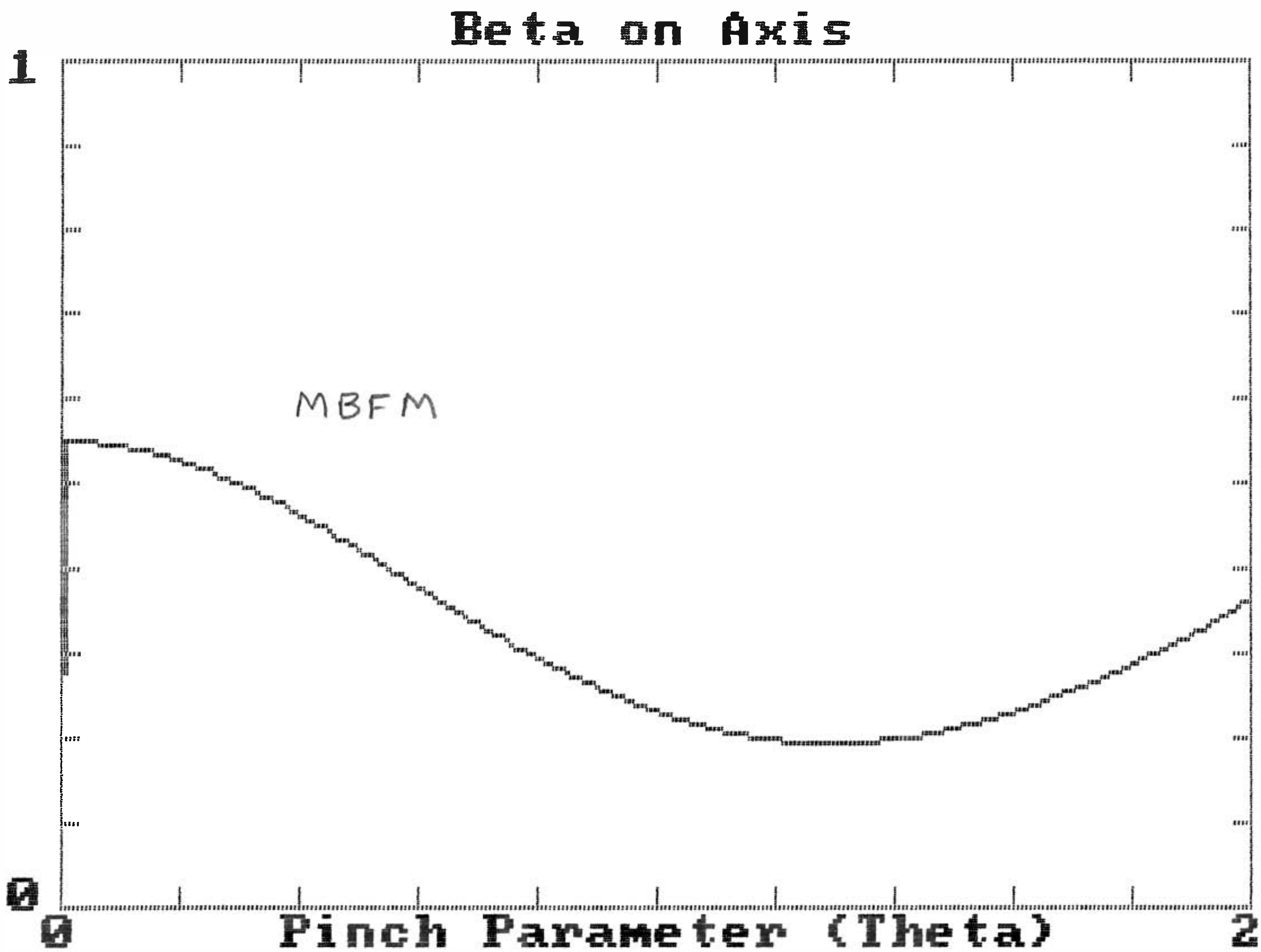
$$p(r) = \int_a^r (J_p B_t - J_t B_p)\,dr$$

The result of evaluating this expression for various F is shown in figure 10, normalized to $\langle B_p(a)\rangle^2/2\mu_o$ (beta). In figure 11, the beta on axis $[2\mu_o p(0)/\langle B_p(a)\rangle^2]$ is plotted versus $\theta$. The error arising from the assumptions of the modified Bessel function model for the RFP just equals and cancels the error due


Normalized Lambda parallel
5
F = -0.5
BFM
0
0.5
MBFM
0
0
Radius (r/a)
1


Normalized Lambda perpendicular
.4
0
-.1
0
Radius (r/a)
1
F = -0.5
0
MBFM
0.5
BFM


Normalized Equivalent Pressure
.4
0
MBFM
F = -0.5
0
0.5
0
Radius (r/a)
1


Beta on Axis
1
0
MBFM
0
Pinch Parameter (Theta)
2

to the neglect of plasma pressure at a beta of about 20-30%. Thus the modified Bessel function model described above should be quite a good representation of a real RFP.

## IV. Numerical Methods

Since the goal of deriving the analytic expressions in the previous sections was to facilitate repetitive numerical calculations, it is useful to determine the precise form of the expressions that yield the shortest calculation time. The results, of course, depend upon the compiler used as well as the computer hardware available. For the present purpose, tests were done on a standard IBM PC with 8087 math coprocessor using Turbo BASIC Version 1.0 from Borland, International. As an example, the value of F(θ) for the first three terms of the Bessel function model,

$$F \approx 1 - \theta^2/2 - \theta^4/12$$

was evaluated 10,000 times in the following equivalent formats with the resulting times (in seconds) indicated in parenthesis:

```
F = 1 - θ^2/2 - θ^4/12   (9.17)

F = 1 - θ*θ/2 - θ*θ*θ*θ/12   (3.56)

F = 1 - .5*θ*θ - .0833333*θ*θ*θ*θ   (3.17)

F = 1 - .5*θ*θ*(1+.1666666*θ*θ)   (3.00)
```

The conclusions are: 1) avoid exponentiation, 2) multiply rather than divide, and 3) use parentheses to avoid repetitive operations.

References

1. J. B. Taylor, Phys. Rev. Letters 32, 1139 (1974).

2. H. A. B. Bodin and A. A. Newton, Nucl. Fusion 20, 1255 (1980).

3. S. Ortolani, R. Paccagnella, and E. Zilla, University of Padova, Italy, Report I.G.I. 84/07 (1984).

4. R. J. LaHaye, T. N. Carlstron, R. R. Goforth, G. L. Jackson, M. J. Schaffer, T. Tamano, and P. L. Taylor, Phys. Fluids 27, 2576 (1984).

5. K. F. Schoenberg, R. F. Gribble, and J. A. Phillips, Nucl. Fusion 22, 1433 (1982).