The Jerk Dynamics of Lorenz Model

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The Lorenz model is widely considered as the first dynamical system exhibiting a chaotic attractor the shape of which is the famous butterfly. This similarity led Lorenz to name the *sensitivity to initial conditions* inherent to such chaotic systems, the *butterfly effect* making its model a paradigm of chaos. Nearly thirty years ago, Stefan J. Linz presented in a very interesting paper an "exact transformation" enabling to obtain the jerk form of the Lorenz model and a nonlinear transformation "simplifying its jerky dynamics". Unfortunately, the third order nonlinear differential equation he finally obtained precluded any mathematical analysis and made difficult numerical investigations since it contained exponential functions. In this work, we provide in the simplest way the jerk form of the Lorenz model. Then, a stability analysis of the jerk dynamics of Lorenz model prove that fixed points and their stability, eigenvalues, Lyapunov Characteristics Exponents and of course attractor shape are the exactly the same as those of Lorenz original model.

I. INTRODUCTION

At the very beginning of the sixties, Edward Norton Lorenz (1917-2008), a meteorologist from the famous M.I.T. (Massachusetts Institute of Technology) succeeded in establishing a model for atmospheric convection comprising only three variables (1). The solution of this weather forecasting model that Lorenz [10] plotted in a three-dimensional phase space is compelled to evolve on a chaotic attractor which resembles the wings of a butterfly. It is probably this form that prompted Lorenz to call the "sensitivity to initial conditions" (described by the French mathematician Henri Poincaré as early as 1908 in his philosophical writings *Science and Method* [12]) the "butterfly effect".

During these last two decades, the seminal works of Gottlieb [4] and Sprott [15–21] have triggered out an increasing interest in the study of chaotic oscillators based on jerk equations, that is, oscillators which can be completely described by third-order ordinary differential equations of the form $\dddot{x} = f(\ddot{x}, \dot{x}, x)$. In 1997, Stephan J. Linz [9] proposed in a very interesting paper an "exact transformation" enabling to obtain the jerk form of the Lorenz model and a nonlinear transformation "simplifying its jerky dynamics". Unfortunately, the third order nonlinear differential equation he finally obtained precluded any mathematical analysis and made difficult numerical investigations since it contained exponential functions. Let's notice that the jerk form in x of the Lorenz model that we will provide below is exactly the same as those obtained by Linz but presented in a different way. In 2014, Buscarino *et al.* [2] used *linear combinations* of the three nonlinear ordinary differential equations modeling the Chua's circuit to deduce its jerk forms in x and z. Recently, Xu and Cao [22] proposed to use the so-called *controllable canonical form* to provide all the jerk forms dynamics of Chua's circuit.

In this paper, following the method of *linear combinations* proposed by Buscarino *et al.* [2], we provide the jerk form in x of Lorenz model. Thus, by making a comparison of fixed points and their stability, eigenvalues, Lyapunov Characteristic Exponents and attractor shapes between the original three-order Lorenz model and its first jerk form in x we demonstrate the topological equivalence of both systems.

The paper is organized as follows. In the next section, we briefly recall some very well-known dynamics features of the Lorenz model. Then, in Sec. 3 the jerk equations of Lorenz model is derived. Then, mathematical and numerical results concerning the stability analysis are reported in Sec. 4.

II. LORENZ MODEL

The purpose of the model established by Edward Lorenz [10] was in the beginning to analyze the unpredictable behavior of weather. After having developed nonlinear partial derivative equations starting from the thermal equation and Navier-Stokes equations, Lorenz truncated them to retain only three modes. The most widespread form of the Lorenz model is as follows:

$$\frac{dx}{dt} = \sigma (y - x) ,$$
$$\frac{dy}{dt} = -xz + rx - y, \tag{1}$$
$$\frac{dz}{dt} = xy - \beta z,$$

where parameters represent respectively the Prandtl number (σ), the Rayleigh number (r), and the aspect ratio of the convection cylinders (β). In this study we will use the following parameters values (σ, r, β) = (10, 28, 8/3). With this parameter set, numerical integration of Lorenz model (1) has led to the famous strange attractor in the shape of a butterfly (see Fig. 1a & 1c). Dynamics features of the Lorenz model have been completely analyzed for many years in many works the most famous of which being that of Sparrow [13]. Lorenz model (1) has three *fixed points*:

$$O\left(0,0,0\right), \quad I\left(-\sqrt{\beta(r-1)}, -\sqrt{\beta(r-1)}, r-1\right), \quad J\left(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1\right), \tag{2}$$

With this parameter set, eigenvalues corresponding to each of these fixed points are the following:

$$(-22.8277, -8/3, 11.8277) ,$$
$$(-13.8546, 0.09395 - 10.1945, 0.09395 + 10.1945i) , \tag{3}$$
$$(-13.8546, 0.09395 - 10.1945, 0.09395 + 10.1945i) .$$

Thus, the origin O is a *saddle-node* while I and J are *saddle-foci.* Then, according to Sparrow [13] a Hopf bifurcation [1, 5, 8, 11] occurs when the parameter r reaches the value:

$$r_H = \sigma \frac{\sigma + \beta + 3}{\sigma - \beta - 1} \tag{4}$$

With the original parameter set, i.e., with $\sigma = 10$ and $\beta = 8/3$, Sparrow [13] finds: $r_H = 470/19 \approx 24.74$. In order to complete the analysis of the effects of the control parameter, value r changes on the topology of the attractor of the Lorenz model (1), the bifurcation diagram has been built and plotted in Fig. 2a for $r \in [20, 80]$. Then, by using the Lyapunov Exponents Toolbox (LET) developed by Steve Siu for MatLab® and involving the two algorithms proposed by Wolf *et al.* [23] and Eckmann and Ruelle [3] (see https://fr.mathworks.com/matlabcentral/fileexchange/233-let) we have obtained for this parameter set the following Lyapunov Characteristic Exponents (LCEs) for the Lorenz model:

$$(+0.906, 0, -14.572) \tag{5}$$

The Kaplan-Yorke conjecture [6] enabling to estimate the *fractal dimension* of a strange attractor is then equal to $d_{KY} \approx 2.062$. Thus, according to the classification of Klein & Baier [7] for (autonomous) continuous-time attractors of dynamical system, such LCEs (5) confirm the chaotic feature of the so-called Lorenz butterfly.

III. LORENZ JERK SYSTEM

Starting from the first equation of Lorenz model (1), we obtain:

$$y = \frac{\dot{x}}{\sigma} + x. \tag{6}$$

It follows that:

$$\dot{y} = \frac{\ddot{x}}{\sigma} + \dot{x}. \tag{7}$$

From the second equation of (1), we deduce that:

$$z = r - \frac{y + \dot{y}}{x}. \tag{8}$$

By replacing in this Eq. (8), y and $\dot{y}$ by their above expressions Eqs. (6,7), we have:

$$z = r - 1 - \frac{1}{\sigma x}\left[\ddot{x} + (\sigma + 1)\,\dot{x}\right]. \tag{9}$$

By taking the time derivative of Eq. (9), we find:

$$\dot{z} = \frac{\dot{x}}{\sigma x^2}\left[\ddot{x} + (\sigma + 1)\,\dot{x}\right] - \frac{1}{\sigma x}\left[\dddot{x} + (\sigma + 1)\,\ddot{x}\right]. \tag{10}$$

Then, by replacing equation (10) in the third equation of (1, we obtain finally:

$$\dddot{x} = -(\sigma + 1)\,\ddot{x} + \left[\ddot{x} + (\sigma + 1)\,\dot{x}\right]\frac{\dot{x}}{x} + \beta\,(r - 1)\,\sigma x - \beta\left[\ddot{x} + (\sigma + 1)\,\dot{x}\right] - x^2\left(\dot{x} + \sigma x\right). \tag{11}$$

Then, the jerk form in x is obtained by posing:

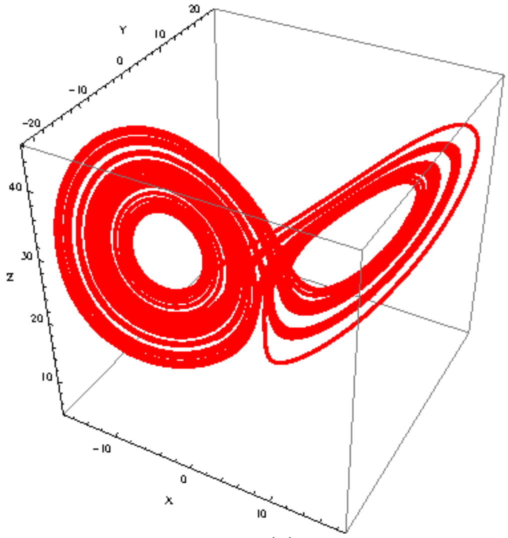
$$\dot{x} = y, \quad \ddot{x} = z, \quad \dddot{x} = f\left(x, \dot{x}, \ddot{x}\right). \tag{12}$$

Considering the Lorenz model (1), we obtain the dynamics of the jerk system:

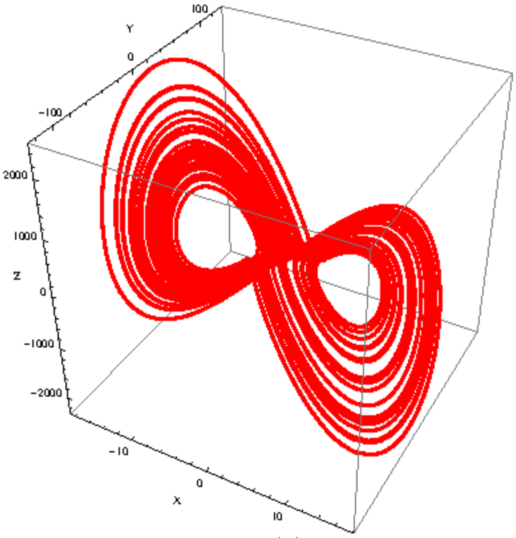
$$\begin{aligned}
\frac{dx}{dt} &= y,\\
\frac{dy}{dt} &= z,\\
\frac{dz}{dt} &= \beta\,(r - 1)\,\sigma x - \beta\,(\sigma + 1)\,y - (\beta + \sigma + 1)\,z + \left[z + (\sigma + 1)\,y\right]\frac{y}{x} - x^2\left(\sigma x + y\right),
\end{aligned} \tag{13}$$

Remark. Let's notice that Eq. (11) is identical to that obtained by Linz [9] (see his equation (18)), excepted the fact that he has posed for unknown reasons $\dot{x}/x = \ln x$ and used a "Cole-Hopf-like transformation" to express his jerk equation. Thus, his resulting third order nonlinear differential equation (20) contains exponential functions which preclude any mathematical analysis and made difficult numerical investigations.

Equations (13) represent in different state space representations Lorenz model (1) and thus maintain its same structural properties. The three-dimensional attractors for the previously defined parameters and for $(\sigma,\, r,\, \beta) = (10,\, 28,\, 8/3)$ are reported in the following figures. The original Lorenz model chaotic attractor is reported in Fig. 1a & 1c, the attractor of the equivalent jerk system represented by Eqs. (13) is reported in Fig. 1b & 1d.



(a)



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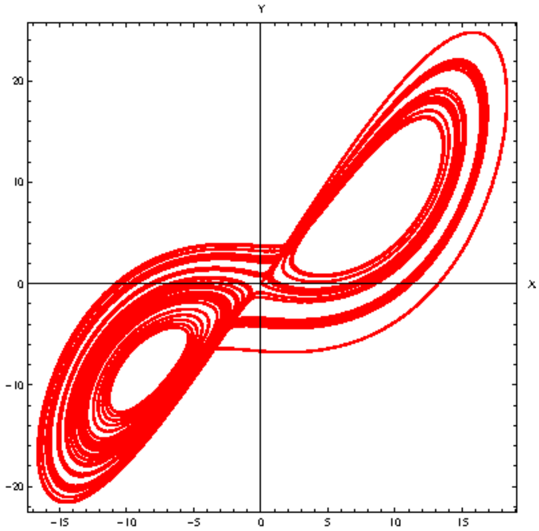

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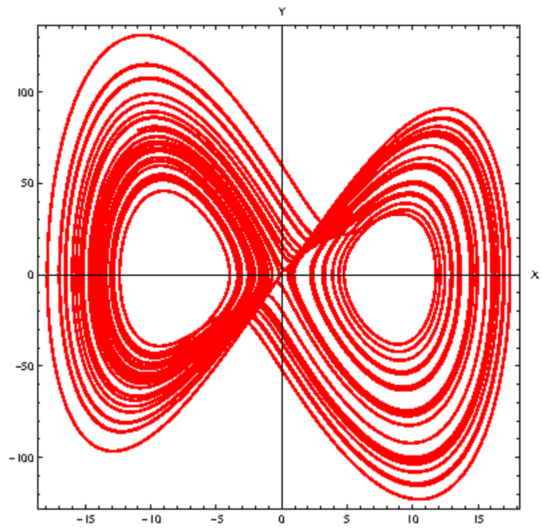


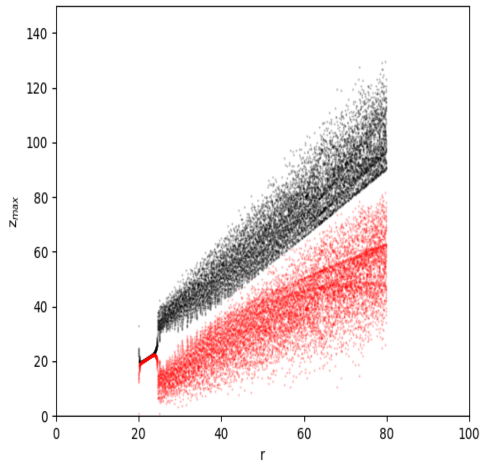
(d)

FIG. 1: Phase portraits of the Lorenz model (1) (left) and of its jerk form in x (13) (right).

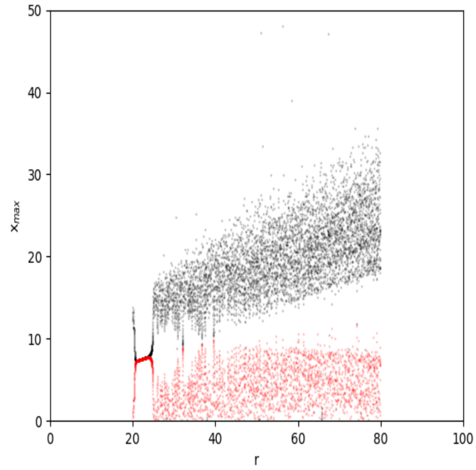
In order to state the topological equivalence between the original representation of the Lorenz model (1) and its jerk form in x (13), we have performed a stability analysis including the fixed points stability, the occurrence of Hopf bifurcation, the representation of the bifurcation diagram and the computation of the Lyapunov Characteristic Exponents for the jerk form in x (13) that we have compared to the stability analysis of the Lorenz model (1).

By using the classical nullclines method, it can be shown that the Lorenz jerk system (13) admits exactly the same fixed points (2) as the Lorenz model (1). With this parameter set, we have verified that both eigenvalues corresponding to the fixed points I and J are exactly the same as the Lorenz model (1) but are different for the origin O which is a saddle-focus $(5.99248, -9.32958 \pm 5.57737i)$ for the Lorenz jerk system (13). We have also verified that a Hopf bifurcation occurs for the same value of parameter r (4). Then, in the next two Figs. 2a & 2b, we have plotted both bifurcation diagrams of the Lorenz model (1) and its corresponding jerk form (13). Both figures 2a & 2b clearly demonstrate the equivalence of the two representations.

Finally, still using the Lyapunov Exponents Toolbox (LET) developed by Steve Siu for MatLab® we have obtained for this parameter set exactly the same Lyapunov Characteristic Exponents (LCEs) as for the Lorenz model (5) and, of course, the same Kaplan-Yorke *fractal dimension* for the strange attractor of the Lorenz jerk system (13).



(a)



(b)

FIG. 2: Bifurcation diagrams of the Lorenz model (1) (left) and of its jerk form in x (13) (right).

IV. DISCUSSION

In this paper, the jerk form in x of the Lorenz model have been derived following the method of *linear combinations* used by Buscarino *et al.* [2]. Then, a stability analysis of the jerk dynamics of Lorenz model prove that fixed points and their stability, eigenvalues, Lyapunov Characteristics Exponents and of course attractor shape are the exactly the same as those of Lorenz original model. Recently, Xu and Cao [22] proposed to use the so-called *controllable canonical form* to provide all the jerk forms dynamics of Chua's circuit. So, two perspectives could be given to this work. The first would be to verify if the jerk form in x of the Lorenz model can be also obtained by making use of the *controllable canonical form*. The second would be an electronic realization of the jerk dynamics of Lorenz model.

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