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Effects of Amplitude, Maximal Lyapunov Exponent, and Kaplan–Yorke Dimension of Dynamical Oscillators on Master Stability Function

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Obtaining the master stability function is a well-known approach to study the synchronization in networks of chaotic oscillators. This method considers a normalized coupling parameter which allows for a separation of network topology and local dynamics of the nodes. The present study aims to understand how the dynamics of oscillators affect the master stability function. In order to examine the effect of various characteristics of oscillators, a flexible oscillator with adjustable amplitude, Lyapunov exponent, and Kaplan–Yorke dimension is used. Not surprisingly, it is demonstrated that the amplitude of the oscillations has no substantial effect on the master stability function, i.e. the coupling strength needed for the complete synchronization is not changed. However, the flexible oscillators with larger maximal Lyapunov exponent synchronize with larger coupling strength. Interestingly, it is shown that there is no linear connection between the Kaplan–Yorke dimension and coupling strength needed for complete synchronization.

Keywords: Synchronization; master stability function; chaos.

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1. Introduction

Synchronization is an important collective behavior in networks of dynamical systems. This phenomenon has been observed in a wide range of complex networks, including networks of connected biological [Schöll, 2020], technological [Chowdhury & Khalil, 2021], and physical systems [Zhang & Strogatz, 2021]. For instance, in the brain as a complex network, synchronization is an integral feature of its dynamics at different scales of the nervous system, from neural activity to the entire brain [Traub *et al.*, 2004; De Stefano *et al.*, 2019; Wang *et al.*, 2020]. In the literature, several studies have been done on synchronization [Dai *et al.*, 2020; Chowdhury *et al.*, 2019; Wang *et al.*, 2019]. Researchers have examined the effect of different factors on synchronization, including coupling strength, time delays, and noise [Santos *et al.*, 2019; Shi & Lu, 2009; Sun *et al.*, 2015].

The synchronizability of a given network is first dependent on the network structure that describes the interaction between oscillators. Second, it is dependent on the local dynamics of oscillators [Lodi *et al.*, 2020; Xi *et al.*, 2020]. Master Stability Function (MSF) can determine the necessary and sufficient conditions for a network to reach a synchronized state [Pecora & Carroll, 1998]. MSF analyzes the synchronization independent of network topology, no matter how complex it is. In particular, the MSF is a maximal Lyapunov exponent of a set of variational equations. In variational equations, the dynamics of oscillators, type of couplings, and a normalized coupling parameter K are considered. Normalizing coupling parameter makes the analysis independent of the topology. Several works have studied the synchronization of single and multilayer networks with regular, small-world, and random typologies [Della Rossa & DeLellis, 2020; Barahona & Pecora, 2002; Della Rossa *et al.*, 2020]. In a network of coupled oscillators, the necessary condition for synchronization is negative MSF for an interval of normalizing control parameter K [Pecora & Carroll, 1998]. Authors in [Huang *et al.*, 2009] have introduced a general scheme to classify the behavior of MSFs in terms of crossing zero value into $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$. In this classification, Γ_m indicates that the MSF possesses m finite cross points. There are some research works that have focussed on MSF. To study a network with stochastic inner dynamics, an upgraded MSF was suggested [Della Rossa & DeLellis, 2020]. This method concluded that if noise diffuses evenly in the network, it could benefit synchronization. Zero-lag and cluster synchrony of delay-coupled dynamical oscillators using the master stability approach was discussed in [Ladenbauer *et al.*, 2013]. In [Berner *et al.*, 2021], the MSF was used for various adaptive networks. MSF can also be developed for hypergraphs [Mulas *et al.*, 2020]. MSF for various delay distributions and network structures was discussed in [Kyrychko *et al.*, 2014]. In [Sun *et al.*, 2009], a generalized MSF for a network of oscillators with small but arbitrary parametric changes was studied.

The present study seeks to address the question of "How amplitude, Maximal Lyapunov Exponent (MLE), and Kaplan–Yorke dimension (D_{KY}) of chaotic oscillators impact the synchronization?" With this aim, the synchronization of two coupled flexible chaotic oscillators is studied. The term "flexible" means that the amplitude, MLE, and D_{KY} of the oscillator are controllable by proper control parameters [Chen *et al.*, 2018; Munmuangsaen *et al.*, 2015]. While it is predictable that the amplitude of oscillations does not affect their synchronisability and the MLE has a linear relation with the MSF, we are aware of no work in the literature discussing the effect of D_{KY} on the MSF. The paper is organized into four sections. In Sec. 2, we briefly provide the mathematics of the flexible chaotic system and MSF formulation. Section 3 examines how the amplitude, MLE, and D_{KY} of oscillators affect the MSF. Finally, in Sec. 4, the conclusion gives a summary and critique of the findings.

2. MSF for a Flexible Chaotic System Under $x-x$ Coupling Scheme

In the following, the formula of the flexible chaotic system and its corresponding MSF under $x-x$ coupling scheme are presented.

2.1. *System dynamics*

The flexible system is defined as follows [Chen *et al.*, 2018],

$$\begin{aligned}
\dot{x} &= k_2 y, \\
\dot{y} &= k_2\left(-x + \frac{yz}{k_1}\right), \\
\dot{z} &= k_2(ak_1 - |y| - bz).
\end{aligned} \tag{1}$$

This system was based on a previously analyzed system with an adjustable attractor dimension [Munmuangsaen *et al.*, 2015; Sprott, 2020] which is known as Buncha system. System (1) was proposed to have adjustable amplitude, MLE, and D_{KY} [Chen *et al.*, 2018]. The oscillator shows chaotic dynamics in $a = 5$, $b = 0.5$, $k_1 = 1$ and $k_2 = 1$ and initial conditions $[6, 10, -2]$. In the equations of the system, k_1, k_2, and b are parameters that control the amplitude of oscillations, MLE, and D_{KY}, respectively [Chen *et al.*, 2018].

2.2. *Master stability function*

For a system described by $\frac{dX_i}{dt} = F(X_i)$, $i = 1, 2, \ldots, N$, where X_i is a m-dimensional vector of variables for the ith oscillator and $F(X_i)$ defines the velocity field, a general form of the identical network is described by

$$\frac{dX_i}{dt} = F(X_i) - \varepsilon \sum_{j=1}^{N} G_{ij} H(X_j), \tag{2}$$

where N is the number of coupled oscillators, and ε is the strength of the global coupling. Here, $G : R^N \to R^N$ is the coupling matrix satisfying $\sum_{j=1}^{N} G_{ij} = 0$, and $H : R^m \to R^m$ is the coupling function that determines which variables are coupled. Considering the synchronous state of the network $\frac{ds}{dt} = F(s)$ with $s = X_i$ for $i = 1, 2, \ldots, N$, the block diagonally decoupled form of variational equation is as follows,

$$\frac{d\xi_k}{dt} = [DF(s) - KDH(s)]\xi_k. \tag{3}$$

DF and DH are the Jacobian functions of $F(s) : R^m \to R^m$, and $H(s) : R^m \to R^m$. K is a normalized coupling parameter. $K_k = \varepsilon\mu_k$ $(k \geq 2)$ is a specific set of K with μ_k being a nonzero eigenvalue of the matrix $G : R^N \to R^N$, and $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_N$ [Pecora & Carroll, 1998; Huang *et al.*, 2009]. Consequently, the MSF is defined as the MLE of Eq. (3). When oscillators are coupled via x variable, the Jacobian function of $H(s)$ becomes,

$$DH = E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

So the simplified form of Eq. (3) for the given system [Eq. (1)] is

$$\frac{d\xi_k}{dt} = \left[\begin{pmatrix} 0 & k_2 & 0 \\ -k_1 & \frac{k_2}{k_1} z & \frac{k_2}{k_1} y \\ 0 & -k_2 \operatorname{sign}(y) & -bk_2 \end{pmatrix} - K \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \xi_k. \tag{5}$$

It should be noted that for y values near zero, $-0.005 < y < 0.005$, the approximation $\operatorname{sign}(y) \approx \tanh(\omega y)$ with $\omega = 500$ is considered.

3. Results

In a given network, synchronization can be affected by its topology (matrix G), and coupling variables (matrix H). Using the MSF method, the influence of network topology is considered as $K_k = \varepsilon\mu_k$ $(k \geq 2)$; that is a multiplication of global coupling strength with corresponding eigenvalues of G. Once MSF is calculated based on the normalized parameter K, it can be applied to any network topology. To determine the synchronizable regions of coupling strength, all K_ks must be located in the region for which $MSF < 0$. The MSF for different coupling schemes is shown in Fig. 1. In the following, all simulations are done using 5000 cycles of the system with a time-step of 0.01. The ODEs are approximated using the fourth-order Runge–Kutta method. The network's elements are coupled only through one variable, i.e. only one element of matrix DH is equal to one. So, each panel belongs to a different type of coupling. For instance, the notation $x \to y$ determines that couplings are from the x variable of one oscillator to the y variable of its connected neighbors. Seeking intervals with negative MSF shows us that only through the $x \to x$, $y \to y$, and $z \to z$ coupling schemes, the network can be synchronized; otherwise, there is no possibility to reach the synchronized state. Besides network topology and coupling type, individual oscillators play a crucial role in synchronizing the network. The relevance between the synchronization of the

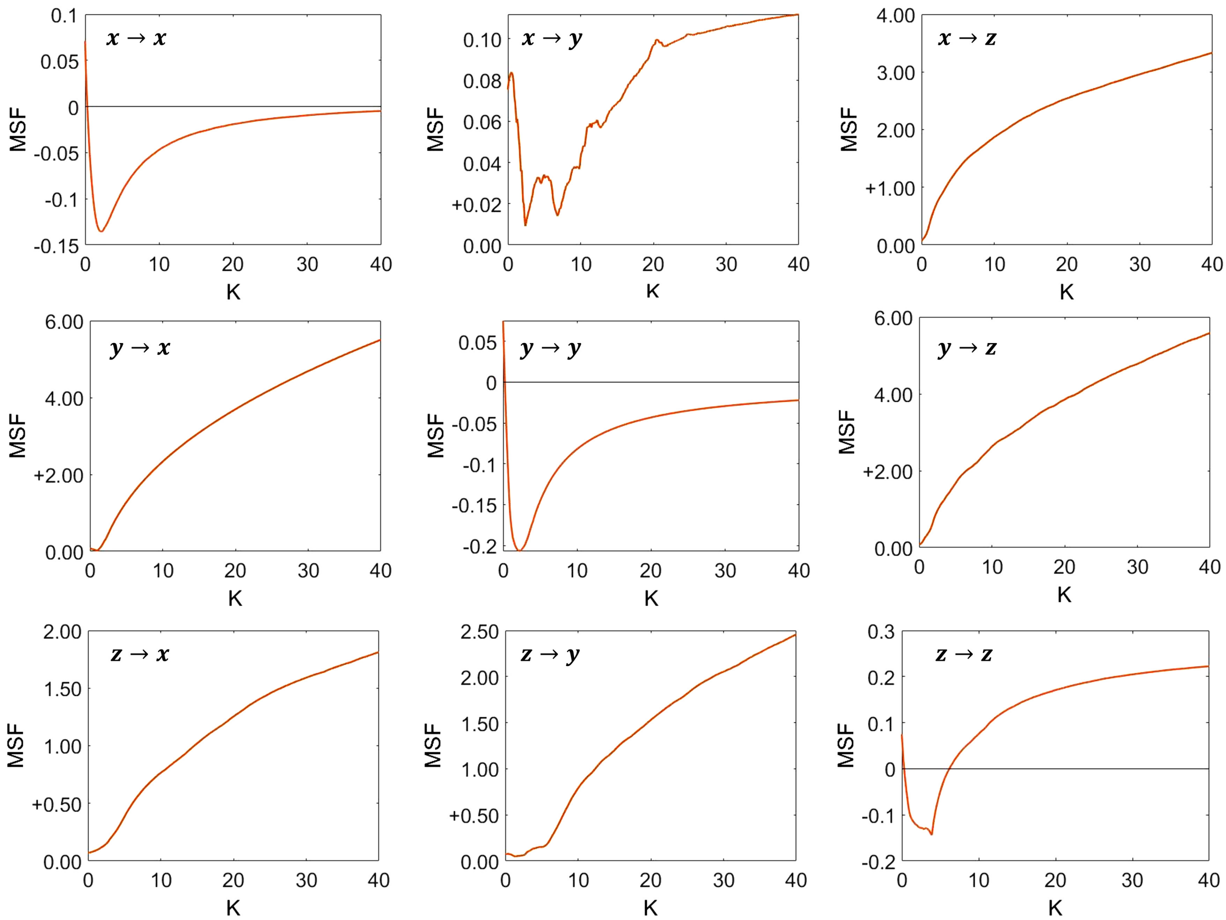


Fig. 1. MSFs of System (1) based on normalized parameter K for different coupling schemes. In each panel, the coupling types are specified by the name of involving variables in the coupling, e.g. the notation $x \to y$ determines that couplings are from the x variable of one oscillator to the y variable of its connected neighbor. The system variables are set to $a = 5$, $b = 0.5$, $k_1 = 1$, and $k_2 = 1$.

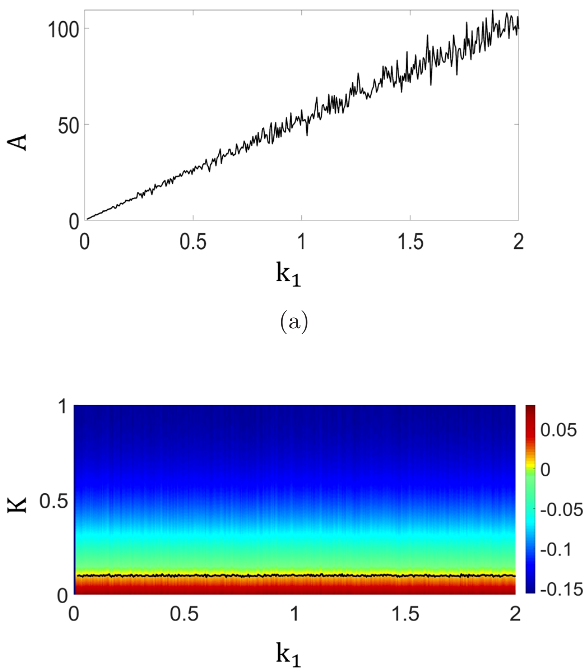
network and the amplitude, MLE, and D_{KY} of the oscillator is discussed in the following subsections.

3.1. *Amplitude*

It is important to understand how the amplitude of oscillations impacts synchronization. In the following, it is considered that oscillators in the network are coupled through x variable. The flexible chaotic system, Eq. (1), has an adjustable amplitude. Changing k_1 results in change to the amplitude of the dynamics. Figure 2(a) shows how the amplitude of oscillations changes with parameter k_1. In this figure, the horizontal axis shows the amplitude control parameter, and the vertical axis shows the amplitude of the attractors. The amplitude is measured by

$$A = \sqrt{(X_{\max} - X_{\min})^2 + (Y_{\max} - Y_{\min})^2 + (Z_{\max} - Z_{\min})^2}. \tag{6}$$

$X_{\max}$, $Y_{\max}$ and $Z_{\max}$ are the maximum, and $X_{\min}$, $Y_{\min}$ and $Z_{\min}$ are the minimum of oscillations in the settled attractor for the corresponding variables. As k_1 increases, the attractor of the system expands. Figure 2(b) represents how MSF is changing by altering the parameters k_1 and K. In this figure, the vertical axis is the normalized coupling parameter K in Eq. (3), and the horizontal axis is k_1. The color bar represents the value of MSF at each point (k_1, K). It changes from positive to negative values. Positive values of MSF correspond to regions of (k_1, K) where the network cannot be synchronized. However, the negative values of MSF correspond to



(a)

(b)

Fig. 2. The association between MSF and k_1, the parameter that changes the amplitude of the oscillations, for $a = 5$, $b = 0.5$, and $k_2 = 1$. (a) The amplitude of oscillations by changing parameter k_1. (b) MSF values indicated by color bar for different values of k_1, and normalized coupling parameter K. The result shows that the zero-cross point of MSF, the black curve, does not change by changing parameter k_1, i.e. MSF is independent of the amplitude of oscillations.

the regions of (k_1, K) that the network potentially can be synchronized. The black curve is the first zero point of MSF in the direction of K. Results show that for the different amplitudes of oscillation, the black curve is approximately constant. It means that the zero-cross point of MSF, which defines the required coupling strength for synchronization, remains the same. Subsequently, changes in amplitude make the synchronization neither difficult nor easy.

3.2. *Maximal Lyapunov exponent*

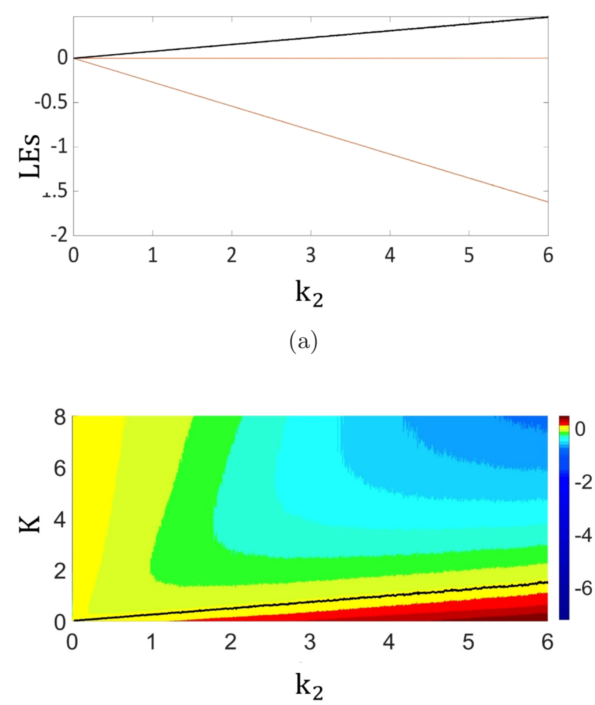
Maximal Lyapunov exponent is an important feature of dynamical oscillators. A definition of deterministic chaos is relevant to positive MLE in the Lyapunov spectrum. For Eq. (1), the MLE is adjustable by parameter k_2. Figure 3(a) shows the dependence of the Lyapunov exponents (indicated by three different colors) on the parameter k_2. Figure 3(b) shows the relation between MLE and MSF. The value of MSF related to each point (k_2, K) is depicted by a color bar. The black curve is the first zero point of MSF in the direction of K. This curve shows a boundary range of k_2 and K that the network potentially can or cannot be fully synchronized. The results show two conclusions. First, the zero-cross point of MSF increases as the MLE increases. It means that as the MLE of the oscillators increases, more coupling strength is needed to synchronize the network. Second, the shape of MSF contours is not changing by varying k_2. The zero-cross point of MSF is linearly dependent on k_2.

3.3. *Kaplan–Yorke dimension*

Besides MLE, other Lyapunov exponents are also informative for a dynamic system. Specifically for a chaotic system, D_{KY} is obtained from its Lyapunov spectrum as,

$$D_{\mathrm{KY}} = d + \sum_{i=1}^{d} \frac{\lambda_i}{\lambda_{d+1}}, \tag{7}$$

where d is the largest number of descending Lyapunov exponents whose summation is non-negative.



(a)

(b)

Fig. 3. The association between MSF and k_2, the parameter that changes MLE of the oscillator, for $a = 5$, $b = 0.5$, and $k_1 = 1$. (a) The LEs versus parameter k_2 and (b) MSF values indicated by colors based on MLE control parameter k_2, and normalized coupling parameter K. The result shows that the zero-cross point, the black curve, increases as the MLE increases. So, the more the oscillators are chaotic, more coupling strength is needed to synchronize their network.

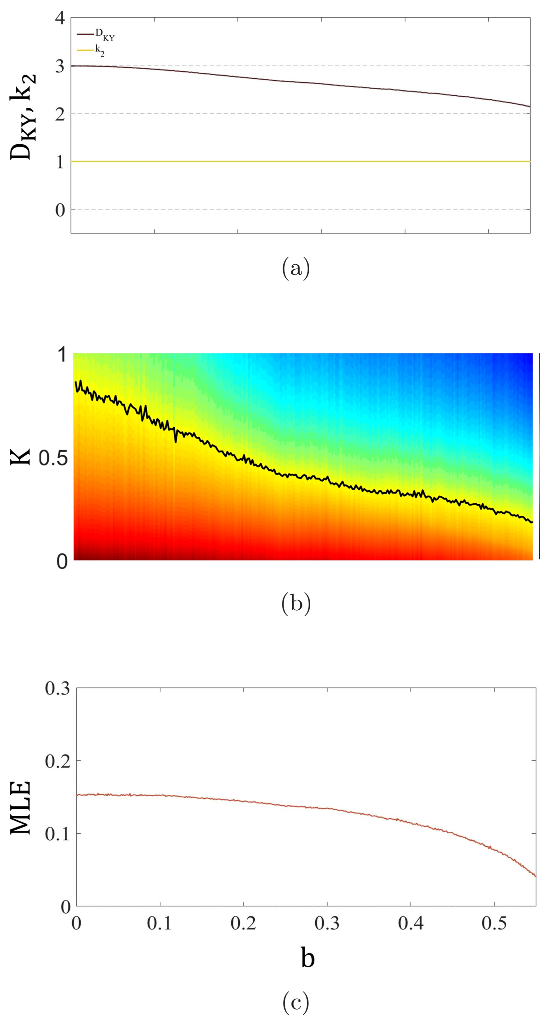


Fig. 4. The association between MSF and $D_{\rm KY}$ control parameter for a constant $k_2 = 1$. (a) The $D_{\rm KY}$, and k_2 values versus parameter b, (b) MSF values indicated by color bar based on $D_{\rm KY}$ control parameter b, and normalized coupling parameter K and (c) MLE values versus parameter b. Here, a and k_1 parameters are set to $a = 5$, and $k_1 = 1$.

In System (1) $D_{\rm KY}$ can be controlled using parameter b. In Fig. 4, the impact of $D_{\rm KY}$ on MSF is studied. In this case, as shown in Fig. 4(a), only the parameter b varies and $k_2 = 1$. It shows that as b increases, the $D_{\rm KY}$ changes from $D_{\rm KY} = 3$ to almost 2. So the dynamics change from conservative chaos to dissipative chaos. From Fig. 4(b), it can be discussed that synchronization is strongly associated with the dimension of the oscillations. In this figure, the horizontal axis belongs to control parameter b, and the vertical axis is normalized parameter K. Values of $D_{\rm KY}$ associated with (b, K) points are indicated by the color bar. The black curve that is the first zero point of MSF in the direction of K, shows the necessary condition to reach a synchronized state. Therefore, it could be concluded that the smaller $D_{\rm KY}$ results in a smaller zero-cross point, which corresponds to smaller coupling strength required for synchronization. However unlike the previous case, this does not happen in a linear way. The important point here is that as the $D_{\rm KY}$ changes, the MLE changes, too [Fig. 4(c)]. So it is necessary to find a way to examine the effect of $D_{\rm KY}$ independently. As seen in Sec. 3.2, the MLE has a linear relation with k_2. To keep the MLE constant on a value named MLE(desired), an adaptive parameter k_2 named k_2(desired) can be

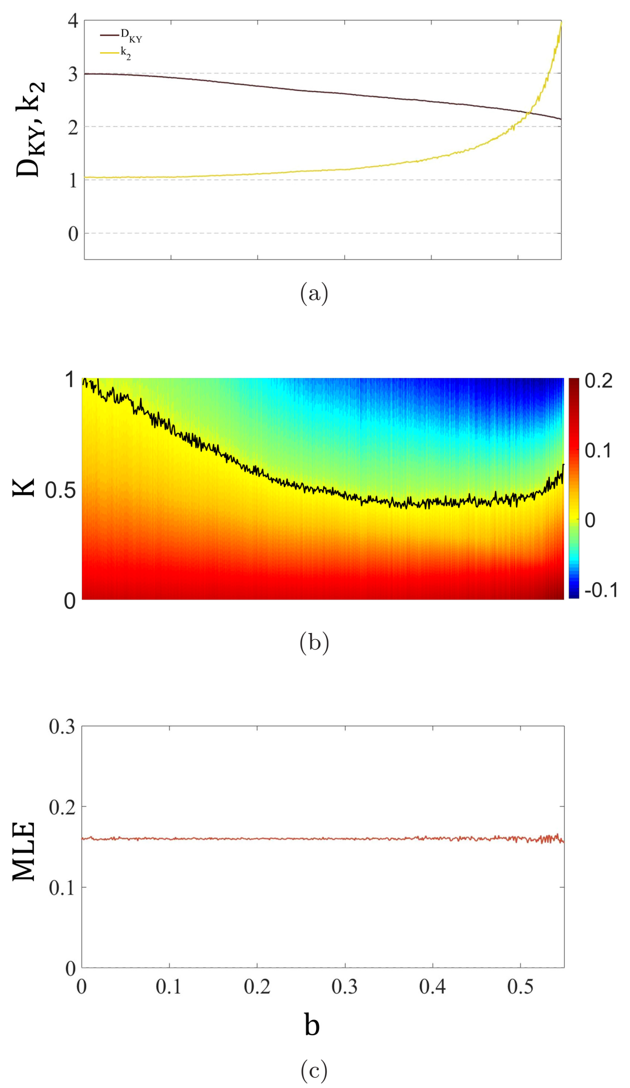


Fig. 5. The association between MSF and $D_{\rm KY}$ control parameter for updated k_2 values. (a) The $D_{\rm KY}$, and k_2 values versus parameter b, (b) MSF values indicated by color bar based on $D_{\rm KY}$ control parameter b, and normalized coupling parameter K with adaptive k_2 and (c) MLE values versus parameter b. Here, a and k_1 parameters are set to $a = 5$, and $k_1 = 1$.

used according to the following formula,

$$\frac{\mathrm{MLE(desired)}}{\mathrm{MLE}} = \frac{k_2(\mathrm{desired})}{k_2}. \tag{8}$$

Considering the result presented in Fig. 4 for $k_2 = 1$ and setting the desired value of MLE to MLE(desired) $= 0.16$, it is obtained that

$$k_2(\mathrm{desired}) = \frac{0.16}{\mathrm{MLE}}. \tag{9}$$

This calculation results in a vector of k_2 values by changing b. In the next step, at each point, both k_2 values and b values change. For each parameter b, a corresponding value of k_2 is determined. Then the MSF in each (k_2, b) point is calculated. Figure 5 shows that by selecting adaptive k_2 values, the MLE is kept approximately constant. So, in this condition, any change in the MSF is associated with D_{KY}, independent of MLE. It is found out that as D_{KY} decreases, the first zero point of MSF in the direction of K has a decreasing nonlinear trend. However, by decreasing D_{KY} more than a threshold, the first zero point of MSF in the direction of K increases.

4. Conclusion

This study set out to determine the impact of the dynamics of oscillators on synchronization. To this aim, the MSFs associated with a flexible chaotic system were examined. The results indicated that for $x \to x$, $y \to y$, and $z \to z$ coupling schemes, a network of flexible chaotic oscillators could be synchronized through type Γ_1. Otherwise, it could not be synchronized. Considering the $x \to x$ coupling type for the network, it was shown that the amplitude of oscillations does not influence synchronization. Then the effect of MLE on the MSF was investigated. It was shown that as the MLE of the chaotic oscillator increased by increasing parameter k_2, the zero-cross point moved to a larger value. So, more coupling strength was needed to reach the synchronized state. Further, as the Kaplan–Yorke dimension became smaller by increasing parameter b, the zero-cross point became smaller. However, by decreasing D_{KY} more than a threshold, the zero point of MSF increased.

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Conflict of Interest

The authors declare no conflict of interest.

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