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Review

Constructing conditional symmetry in symmetric chaotic systems

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ABSTRACT

Based on the analysis of polarity balance and exhaustive computer searching, a series of symmetric chaotic flows is found for hosting conditional symmetry. Symmetric structure shapes the elegant symmetric phase trajectory, and conditional symmetry permits the convenience of embedding an extra set of coexisting symmetric attractors. Bifurcation analysis proves the coexistence of two independent processes of dynamical behavior under conditional symmetry. Circuit simulation confirms the numerical and theoretical analysis.

1. Introduction

Chaotic systems have been exhaustively studied from various aspects and directions. By studying chaotic flows with various types of equilibria [1–13], the mechanism for producing chaos is well documented. Chaos may exist in a dynamical system without any equilibrium points [1–3] or even with stable equilibria [4–6]. More dramatically, chaos may exist with lines [7–9] or closed curves [10,11], planes [12,13], or solids [13] of equilibria. Strange attractors, where the basin of which do not intersect the neighbor of any unstable equilibria are called hidden attractors [14–16]. Hidden attractors can be found in various systems with different equilibrium points and topology structures as well. Symmetry is another common characteristic of dynamical systems. Symmetrical systems are prone to exhibit a unique property of multistability, where coexisting attractors are isolated within separated basins of attraction. All the above issues including hidden attractor, symmetry, and multistability can be combined in a single system releasing complicated dynamics.

The symmetric structure yields elegant phase trajectories in chaotic systems. Even when the symmetry is broken, it usually still gives rise to a symmetric pair of coexisting attractors. As examples are listed in Table 1, there are three basic regimes of symmetry, *i.e.*, reflection symmetry, rotational symmetry, and inversion symmetry [17] according to the polarity reverse from one to three dimensions. Some symmetric systems release coexisting symmetric coexisting pairs of attractors rather than hidden [18,19]. Meanwhile, some systems with hidden attractors [20–22] may still exhibit multistability [20,22]. Symmetric systems exist widely in memristive systems [23] and conservative flows [24].

Conditional symmetry is the terminology to describe a special symmetry, where the polarity balance is reconstructed depending on variable reversal and inverse function from offset boosting [25]. Pure variable reversal results in symmetry while mixed polarity reversal leads to conditional symmetry. Conditional symmetry bridges the gap between symmetry and asymmetry, where polarity balance [26] can be reconstructed by necessary offset boosting [25–27]. However, conditional symmetry does not preclude symmetric systems. It is found that even a symmetric system with offset boosting can give coexisting attractors with conditional symmetry [28] where any of the coexisting attractors shares an elegant symmetric structure. In this paper, a series of chaotic flows with a compound structure of symmetry and conditional symmetry is proposed. Typical cases are described along with basic analysis in Section 2. A case study is carried out in Section 3 for showing the unique dynamics, including bifurcation and multistability. Circuit verification is provided in Section 4, followed by some discussion.

2. Symmetric chaotic flows and corresponding versions with conditional symmetry

There are many possibilities for providing conditional symmetry. For a 3-D chaotic system, there are several types of symme-

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Table 1
Three regimes of symmetric systems.

Authors	Year	Symmetry	Hidden attractor	Multistability
Sprott JC	2014	Involutional symmetries	No	Yes
Li C, Hu W, Sprott JC, Wang X	2014	Reflection symmetry	No	Yes
Zhang X, Li C, Min F, IU HC, Gao H.	2020	Rotational symmetry	No	Yes
Bayani A, Rajagopal K, Khalaf AJM, Jafari S, Leutcho GD, Kengne J.	2019	Inversion symmetry	Yes	Yes
Zhou L, Wang C, Zhou L.	2018	Rotational symmetry	Yes	No
Munmuangsaen B, Srisuchinwong B	2018	Rotational symmetry	Yes	Yes

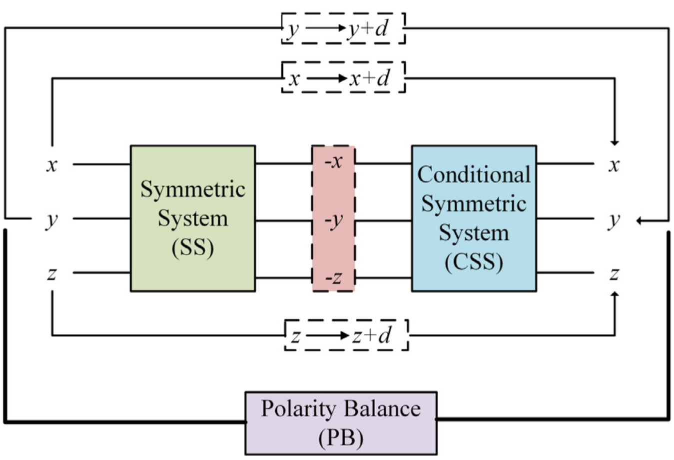

Fig. 1. Polarity balance for unifying symmetry and conditional symmetry.

try transformation. A 3-D symmetric system has three regimes according to the variable reversal, namely reflection symmetry, rotational symmetry, and inversion symmetry. For conditional symmetry, since offset boosting is a mandatory operation at least in one dimension for polarity balance, there are two types of conditional symmetry: conditional reflection symmetry and rotational symmetry. Moreover, conditional symmetry may be induced by 1-D or 2-D offset boosting. Thus polarity balance may be reconstructed from one dimension to another. Specifically, reflection symmetry in the x variable may become rotational symmetry for y and z. Furthermore, offset boosting may be associated with a symmetrically reversed variable or with a static variable. All these cases under the control of polarity balance are shown in Fig. 1. It is not easy to find such a specific symmetric structure for hosting conditional symmetry, even in the many symmetric nonlinear systems [29–33] that exhibit chaos. Whether a dynamical system can be transformed to be conditionally symmetric constitutes one of the criteria for system classification.

For clarifying the process from symmetry to conditional symmetry, here we focus on the type of symmetry and the dimension with offset boosting.

Case I: Symmetry preserving after offset boosting

Conditional symmetry is constructed based on offset boosting with any polarity reversal-free variable. Here both 1-D and 2-D offset boosting can restore the symmetry with original polarity reversal. For example, the original system has reflection symmetry according to the x variable (denoted as: S:X – symmetry according to x), the attached offset boosting for y or z may return the reflection symmetry in x (marked like: CS: X-1DY – conditional symmetry according to x based on the offset boosting in the dimension of y). There are other issues for this case like CS: X-1DZ, CS: X-2DYZ, and CS: XY-1DZ.

Case II: Symmetry transferring after offset boosting

Conditional symmetry may be different for polarity reversal compared with the original system. Any change in the variable with polarity reversal belongs to this category. For a symmetric system S: X, the corresponding conditional symmetric version maybe CS: Y-1DZ, CS: Z-1DY, CS: XY-1DZ, CS: XZ-1DY.

Case III: Symmetry violation for offset boosting

The variable with polarity reversal in a symmetric system may become involved in offset boosting for conditional symmetry. For a chaotic system with reflection symmetry such as S: X, the corresponding versions could be: CS: Y-1DX, CS: Y-2DXZ, CS: Z-1DX, CS: Z-2DXY, and CS: YZ-1DX. For a chaotic system with rotational symmetry such as S: XY, the corresponding versions could be: CS: Z-1DX, Z-1DY, X-1DY, Y-1DX, and Z-2DXY. Note that even for a chaotic system with inversion symmetry such as S: XYZ, the corresponding versions could be: CS: X-1DY, X-1DZ, Y-1DX, Y-1DZ, Z-1DX, Z-1DY, XY-1DZ, XZ-1DY, YZ-1DX, X-2DYZ, Y-2DXZ, and Z-2DXY.

All three cases were exhaustively searched for elegant equations. For example, if conditional symmetry is to be derived in a chaotic system with rotational symmetry S: XZ, the structure designed for the case of symmetry violation for 2-dimensional offset boosting as CS: Z-2DXY,

$$\begin{cases}\dot{x} = a_1xy + a_2yz,\\ \dot{y} = a_3x^2 + a_4y^2 + a_5z^2 + a_6xz + a_7,\\ \dot{z} = a_8x + a_9z.\end{cases} \tag{1}$$

If conditional symmetry is applied to a chaotic system of reflection symmetry S: X, the structure designed for the case of symmetry transferring after 2-dimensional offset boosting as CS: Z-2DXY,

$$\begin{cases}\dot{x} = a_1xy,\\ \dot{y} = a_2x^2 + a_3y^2 + a_4z^2 + a_5yz + a_6,\\ \dot{z} = a_7z + a_8z|x| + a_9z|y|.\end{cases} \tag{2}$$

Therefore, eight typical cases are listed in Table 2, with corresponding symmetric strange attractors shown in Fig. 2. The system SCS1 is the diffusionless Lorenz system, which is also Sprott B

Table 2
Symmetric chaotic systems (SCS) and their corresponding conditional symmetric systems (CSS).

Cases	Systems	Equations	Parameters	Equilibria	Eigenvalues	LEs	D_{KY}
S: XY	SCS1 (Diffusionless Lorenz system)	$\begin{cases}\dot{x}=y-x,\\ \dot{y}=-xz,\\ \dot{z}=xy-a.\end{cases}$	$a=1$	$(\pm1,\pm1,0)$	$(-1.3532,$ $0.1766\pm1.2028i)$	0.21020 −1.2102	2.1737
CS: X-2DYZ	CSS1	$\begin{cases}\dot{x}=F(y)-x,\\ \dot{y}=-xG(z),\\ \dot{z}=xF(y)-a,\end{cases}$ $F(y)=\lvert y\rvert-5,$ $G(z)=\lvert z\rvert-5.$	$a=1$	$(1,\pm6,\pm5)$ $(-1,\pm4,\pm5)$ $(1,\pm6,\mp5)$ $(-1,\mp4\pm5)$	$(-1.3532,$ $0.1766\pm1.2028i)$ $(-1.1028\pm0.6655i,\ 1.2056)$	0.21020 −1.2102	2.1737
S: XY	SCS2 (Sprott B)	$\begin{cases}\dot{x}=ayz,\\ \dot{y}=x-y,\\ \dot{z}=1-xy.\end{cases}$	$a=1$	$(\pm1,\pm1,0)$	$(-1.3532$ $0.1766\pm1.2028\mathrm{i})$	0.2100 −1.210	2.1736
CS: Y-2DXZ	CSS2	$\begin{cases}\dot{x}=ayG(z),\\ \dot{y}=F(x)-y,\\ \dot{z}=1-F(x)y,\end{cases}$ $F(x)=\lvert x\rvert-7,$ $G(z)=\lvert z\rvert-7.$	$a=1$	$(\pm8,1,\pm7)$ $(\pm6,-1,\pm7)$ $(\pm8,1,\mp7)$ $(\pm6,-1,\mp7)$	$(-1.3532,$ $0.1766\pm1.2028\mathrm{i})$ $(-1.1028\pm0.6655\mathrm{i},$ $1.2056)$	0.2100 −1.210	2.1736
S: XZ	SCS3 (Sprott C)	$\begin{cases}\dot{x}=-yz,\\ \dot{y}=x^2-1,\\ \dot{z}=x-az.\end{cases}$	$a=2$	$(1,\ -2,\ 1/2)$ $(-1,\ 2,\ -1/2)$	$(-2.6590,$ $0.3295\pm0.8023\mathrm{i})$ $(-0.5\pm1.3229\mathrm{i},$ $-1)$	0.05690 −2.0569	2.0276
CS: Z-2DXY	CSS3	$\dot{x}=-G(y)z,$ $\{\dot{y}=(F(x))^2-1,$ $\dot{z}=F(x)-az,$ $F(x)=\lvert x\rvert-4,$ $G(y)=\lvert y\rvert-5.$	$a=2$	$(\pm5,\pm5,\frac{1}{2})$ $(\pm3,\pm5,-\frac{1}{2})$ $(\pm5,\mp5,\frac{1}{2})$ $(\pm3,\mp5,-\frac{1}{2})$	$(-2,\pm1i)$ $(-2,\pm1)$	0.05680 −2.0568	2.0276
S: XZ	SCS4	$\begin{cases}\dot{x}=az,\\ \dot{y}=-x^2+1,\\ \dot{z}=bx-z-x\lvert y\rvert.\end{cases}$	$a=18,$ $b=1.93$	(± 1,1.93,0) (± 1,–1.93,0)	$(3,\ -2$ $\pm2.8284i)$ $(1.3356$ $\pm2.8324i,$ $-3.6711)$	0.120 −1.12	2.1071
CS: XZ-1DY	CSS4	$\begin{cases}\dot{x}=az,\\ \dot{y}=-x^2+1,\\ \dot{z}=bx-z-x\lvert F(y)\rvert.\end{cases}$ $F(y)=\lvert y\rvert-22$	$a=18,$ $b=1.93$	(± 1,23.93,0) (±1,–23.93,0)	$(3,\ -2$ $\pm2.8284i)$ $(1.3356\pm2.8324i,$ $-3.6711)$	0.120 −1.12	2.1071
S: X	SCS5	$\begin{cases}\dot{x}=-ax+bx\lvert y\rvert,\\ \dot{y}=-x^2+z^2+yz,\\ \dot{z}=-cy^2-dyz+1.\end{cases}$	$a=0.33,$ $b=0.75,$ $c=0.35,$ $d=0.9$	(±2.5647,0.44,2.3541) (± 2.5647, −0.44, −2.3541) (0, −1.6903, 0) (0, 1.6903, 0)	$-0.1791,$ $1.0686\pm4.5462i,$ $(-2.0575,$ $0.0497\pm1.3771i)$ $(0.7606\pm1.1922i,$ $0.9377)$ $(-0.7606$ $\pm1.1922i,$ $0.9377)$	0.03010 −2.0419	2.0148
CS: X-2DYZ	CSS5	$\begin{cases}\dot{x}=-ax+bx\lvert F(y)\rvert,\\ \dot{y}=-x^2+(G(z))^2+F(y)G(z),\\ \dot{z}=-c(F(y))^2-dF(y)G(z)+1.\end{cases}$ $F(y)=\lvert y\rvert-5$ $G(z)=\lvert z\rvert-12$	$a=0.33,$ $b=0.75,$ $c=0.35,$ $d=0.9$	(0, 3.3097, 12) (0, 6.6903, 12) (0, 3.3097, −12) (0, 6.6903, −12) (± 2.5647, 5.44, 14.3541) (± 2.5647, 5.44, −14.3541) (± 2.5647, −5.44, 14.3541) (± 2.5647, −5.44, −14.3541)	$(0.9790\pm3.2561i,\ 0)$ $(-2.2926,\ 5.0427,\ 0)$ $(-0.1791,\ 1.0686\pm4.5462i)$ $(3.5404,\ -0.3951\pm0.9734i)$ $(-6.1376,\ -0.1785,\ 3.5660)$ $(-2.0575,\ 0.0497\pm1.3771i)$	0.03010 −2.0419	2.0148
S: X	SCS6	$\begin{cases}\dot{x}=-ax\lvert y\rvert+bx\lvert z\rvert,\\ \dot{y}=-x^2-yz+1,\\ \dot{z}=cy+z.\end{cases}$	$a=16.8,$ $b=2.8,$ $c=8.25$	(±1, 0, 0)	(1, 0, 0)	0.29850 −1.1319	2.2638
CS: XZ-1DY	CSS6	$\begin{cases}\dot{x}=-ax\lvert F(y)\rvert+bx\lvert z\rvert,\\ \dot{y}=-x^2-F(y)z+1,\\ \dot{z}=cF(y)+z.\end{cases}$ $F(y)=\lvert y\rvert-3$	$a=16.8,$ $b=2.8,$ $c=8.25$	(±1, 3, 0) (±1, −3, 0)	$(1,$ $\pm5.7966)$ $(1,$ $\pm5.7966i)$	0.29850 −1.1319	2.2638
S: XZ	SCS7	$\begin{cases}\dot{x}=-ayz+bx\lvert y\rvert-cx\lvert z\rvert,\\ \dot{y}=x^2-z^2+d,\\ \dot{z}=z+xy-z\lvert y\rvert.\end{cases}$	$a=7.3,$ $b=6.4,$ $c=118,$ $d=0.1$	(−0.0606, 0.8415, 0.3220)	$(-7.5250\pm10.0450i,$ $-0.0499)$	0.07080 −25.6071	2.0028

(*continued on next page*)

Table 2 (*continued*)

Cases	Systems	Equations	Parameters	Equilibria	Eigenvalues	LEs	D_{KY}
CS: XZ-1DY	CSS7	$\begin{cases}\dot{x} = -aF(y)z + bx\lvert F(y)\rvert - cx\lvert z\rvert, \\ \dot{y} = x^2 - z^2 + d, \\ \dot{z} = z + xF(y) - z\lvert F(y)\rvert.\end{cases}$ $F(y) = \lvert y\rvert - 3$	$a = 7.3$, $b = 6.4$, $c = 118$, $d = 0.1$	(−0.0606, 0.8415, 0.3220) (−0.0606, −0.8415, 0.3220)	(−32.6450, 0.6449, −0.4518) (−32.6275, 0.0878 ± 0.5328i)	0.07080 −25.6071	2.0028
S: XYZ	SCS8	$\begin{cases}\dot{x} = y - y\lvert z\rvert, \\ \dot{y} = -y + y\lvert x\rvert, \\ \dot{z} = ax\lvert y\rvert - bz\lvert x\rvert.\end{cases}$	$a = 1$, $b = 20$	(0, 0, z) (x, 0, 0) (1, 20, 1) 1, −20, 1) (−1, 20, −1) (−1, −20, −1)	(−1, 0, 0) (0, −20|x|, |x| - 1) (0.4572 ± 4.3493i, −20.9145) (4.0760, −5.1985, −18.8775 (−39.8322, 20.3262, −0.4940) (−9.7469 ± 26.3653i, −0.5062)	0.4543, 0 −20.4543	2.0222
CS: Z-1DX	CSS8	$\begin{cases}\dot{x} = y - y\lvert z\rvert, \\ \dot{y} = -y + y\lvert F(x)\rvert, \\ \dot{z} = aF(x)\lvert y\rvert - bz\lvert F(x)\rvert.\end{cases}$ $F(x) = \lvert x\rvert - 4$	$a = 1$, $b = 20$	(± 4, 0, z) (x, 0, 0) ±5, ±20, 1 ±5, ∓20, 1	(−1, 0, 0) (0, 8–20|x|, |x| - 5) (0.4572 ± 4.3493i, −20.9145) (4.0760, −5.1985, −18.8775)	0.4543, 0 −20.4543	2.0222

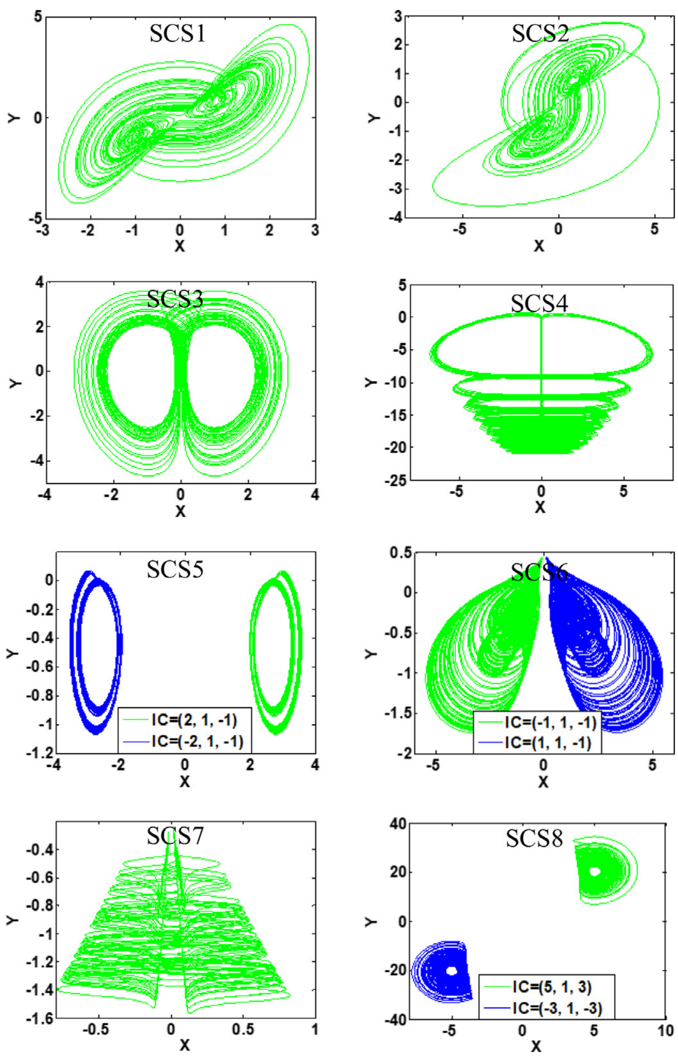

Fig. 2. Strange attractors of symmetric chaotic flows.

when a transformation $x \leftrightarrow y,\ z \to -z$ is applied. The system SCS3 is the Sprott C system after a transformation $y \leftrightarrow z,\ y \to -y$. Although the diffusionless Lorenz system is the Sprott B system with the algebraic form of rotational symmetry according to x and y, their corresponding versions of conditional symmetry are different; the former is CS: X-2DYZ, and the latter is Y-2DXZ. Furthermore, even some chaotic systems with inversion symmetry have been proposed, but none of them host conditional symmetry.

Based on offset boosting, phase trajectories of the derived chaotic systems with conditional symmetry are plotted in Fig. 3. Four cases obtain their conditional symmetry based on 1-D offset boosting, and the other four cases use 2-D offset boosting. This shows that the structure of the basin of attraction is related to the shape of an attractor. A symmetric system typically has attractors with symmetric basins of attraction [22] while conditional symmetry revises the structure of the original system such that the coexisting attractors have asymmetric basins of attraction, as shown in Fig. 4. Symmetric pairs of coexisting attractors in a symmetric system all are doubled as in other cases with a single symmetric attractor. The system SCS4 appears to be a tiny island of conservative chaos adjacent to a homoclinic orbit. The system SCS8 for $a = 1$ and $b = 8$ has a chaotic transient that attracts to an equilibrium point. There are two lines of equilibria at (x, 0, 0) and (0, 0, z) as well as some isolated equilibrium points, as listed in Table 1. Increasing b increases the duration of the transient. For $b = 20$, the chaos lasts at least for a time of $t = 1e7$ where the LEs are (0.4543, 0, −20.4543) with $D_{KY} = 2.0222$. The system SCS7 has one index-0 spiral node, which has one real negative eigenvalue, and a complex conjugate pair with a negative real part. Correspondingly, system SCS7 is multistable since the strange attractor coexists with a stable equilibrium point, and the strange attractor is hidden since it cannot be found by using initial conditions in the vicinity of the equilibrium [4].

3. Coexisting bifurcations and symmetry evolution

The transformation for conditional symmetry is independent of the fundamental dynamics when polarity balance is well protected. Take SCS3 for example,

$$\begin{cases}\dot{x} = -yz, \\ \dot{y} = x^2 - 1, \\ \dot{z} = x - az.\end{cases} \tag{3}$$

$$\begin{cases}\dot{x} = -G(y)z, \\ \dot{y} = (F(x))^2 - 1, \\ \dot{z} = F(x) - az.\end{cases} \tag{4}$$

Its derived version of conditional symmetry is written as Eq. (4), where $F(x) = |x| - 4,\ G(y) = |y| - 5$. SCS3 is of rotational symmetry, which is robust against polarity reversal of $x \to -x,\ z \to$

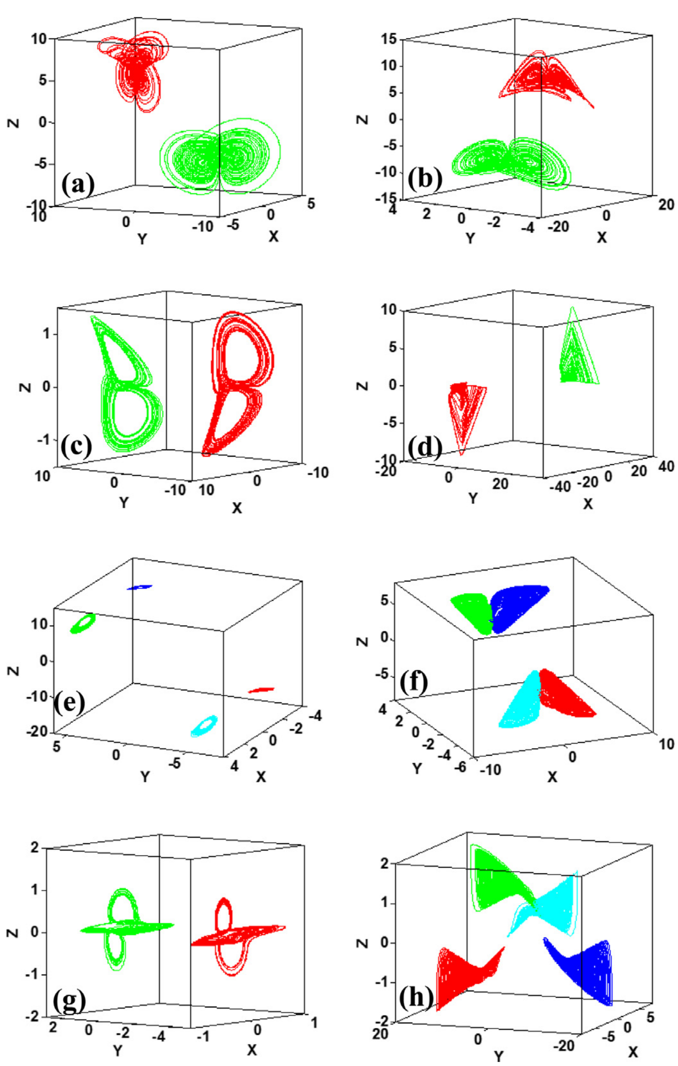

Fig. 3. Coexisting attractors with conditional symmetry: (a) IC1 = (1, 5, 6), IC2 = (1, −5, −4); (b) IC1 = (8, 1, 6), IC2 = (−6, 1, −8); (c) IC1 = (5, 6, −1), IC2 = (−3, −4, −1); (d) IC1 = (1, 23, −1), IC2 = (−1, −21, 1); (e) IC1 = (2, 3, 2), IC2 = (−2, −1, −4); (f) IC1 = (−1, 4, −1), IC2 = (1, −2, 1), IC3 = (1, 4, −1), IC4 = (−1, −2, 1); (g) IC1 = (1, 4, 1), IC2 = (−1, −2, 1); (h) IC1 = (5, 1, −3), IC2 = (−3, 1, 3), IC3 = (3, −1, 3), IC4 = (−5, 1, −3). (IC1 is red, IC2 is green, IC3 is blue and IC4 is cyan).

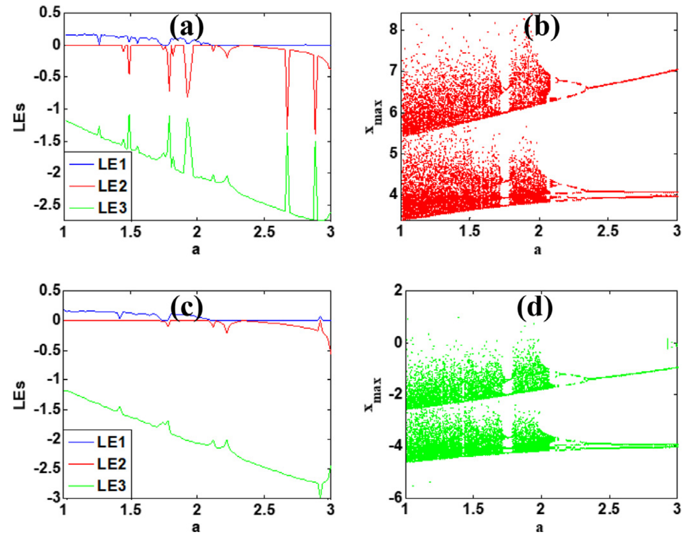

Fig. 5. Bifurcations and Lyapunov exponents in CSS3 when a varies in [1,]: (a) (b) IC = (5, 6, −1), (c) (d) IC = (−3, −4, −1).

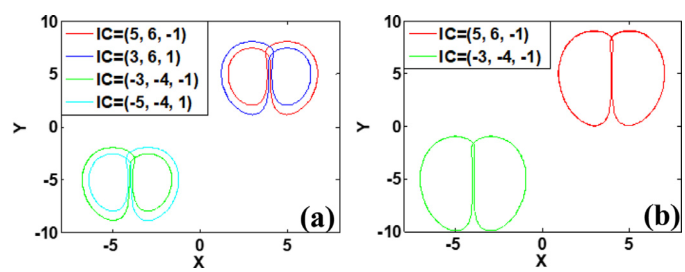

Fig. 6. Limit cycles of conditional symmetry in CSS3: (a) $a = 2.2$, (b) $a = 3$.

$-z$. Eq. (4) is the case of CS: Z-2DXY, which means that 2-D offset boosting in the x and y variables returns the polarity balance under the polarity reversal of $z \to -z$.

As shown in Fig. 5, two coexisting independent bifurcations of conditional symmetry occur when the parameter a varies in [1,3]. Typical inverse-period-doubling bifurcation takes place in two independent channels separated by conditional symmetry. Most of the dynamics including periodicity and chaos are preserved un-

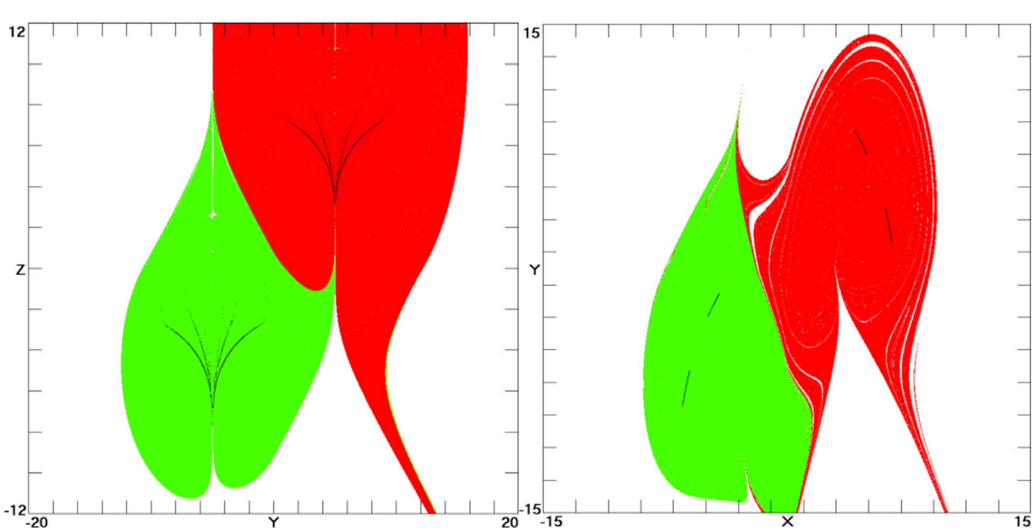

Fig. 4. Basins of attraction for the coexisting conditional symmetric attractors: (a) CSS1on the plane of $x = 0$, (b) CSS3 on the plane of $z = 0$.

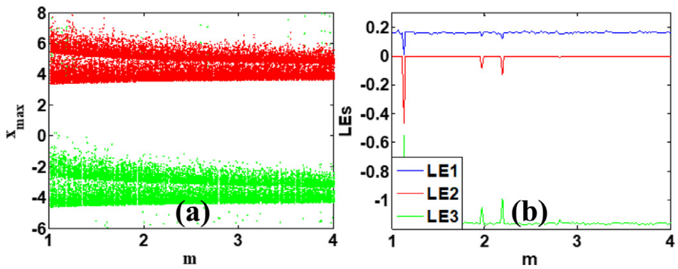

Fig. 7. Amplitude rescaling in CSS3 with $a = 1, F(x) = |x| - 4, G(y) = |y| - 5$: (a) bifurcation diagram, (b) Lyapunov exponents.

der conditional symmetry even when the symmetry is broken. As shown in Fig. 6, symmetric and asymmetric pairs of limit cycles are doubled when $a = 2.2$ and $a = 3$.

Furthermore, system SCS3 provides a single parameter m for amplitude control, which is the coefficient of x^2 in the second dimension of Eq. (3). A transformation $x \to mx, y \to y, z \to mz, t \to t(m > 0)$ leaves an independent parameter m in system SCS3. In system CSS3, the same parameter m can provide amplitude control of the variables x and z since the piecewise function F(x) is equivalent to a linear term x in degree.

$$\begin{cases} \dot{x} = -G(y)z, \\ \dot{y} = m(F(x))^2 - 1, \\ \dot{z} = F(x) - az. \end{cases} \tag{5}$$

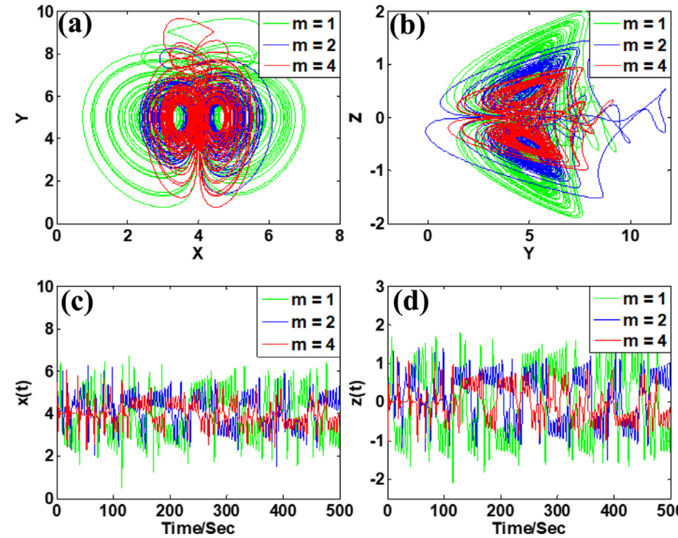

Fig. 8. Phase trajectories of system CSS3 with $a = 1$ under initial condition (5, 6, −1): (a) x - y plane, (b) $y - z$ plane, (c) signal x, (b) signal z.

Correspondingly, the parameter m in Eq. (5) modifies the amplitude of the coexisting attractors with conditional symmetry. In fact, this control should be limited in a certain region since the conditional symmetry depends on the attractor size when a specific function F(x) is pre-defined. To enlarge the space for ampli-

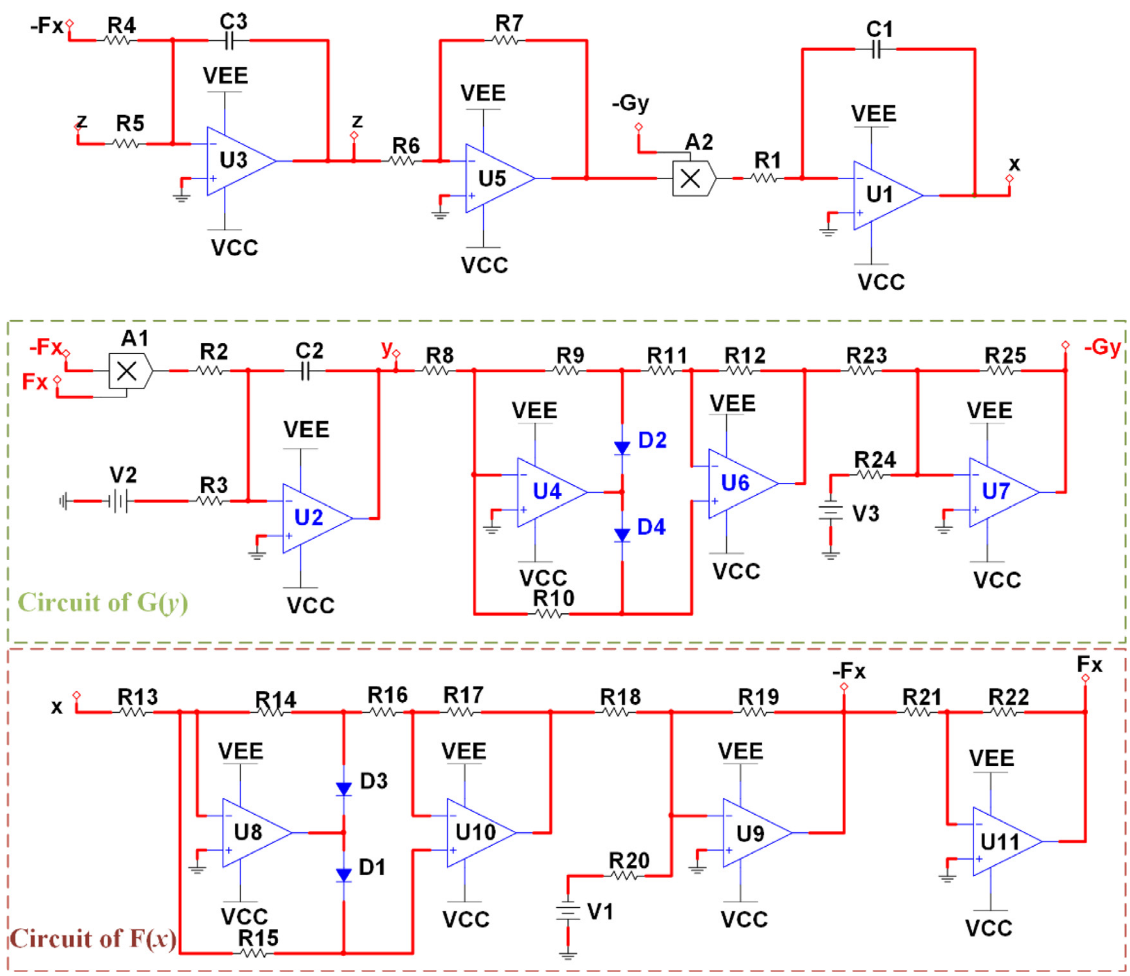

Fig. 9. Circuit schematic of system CSS3.

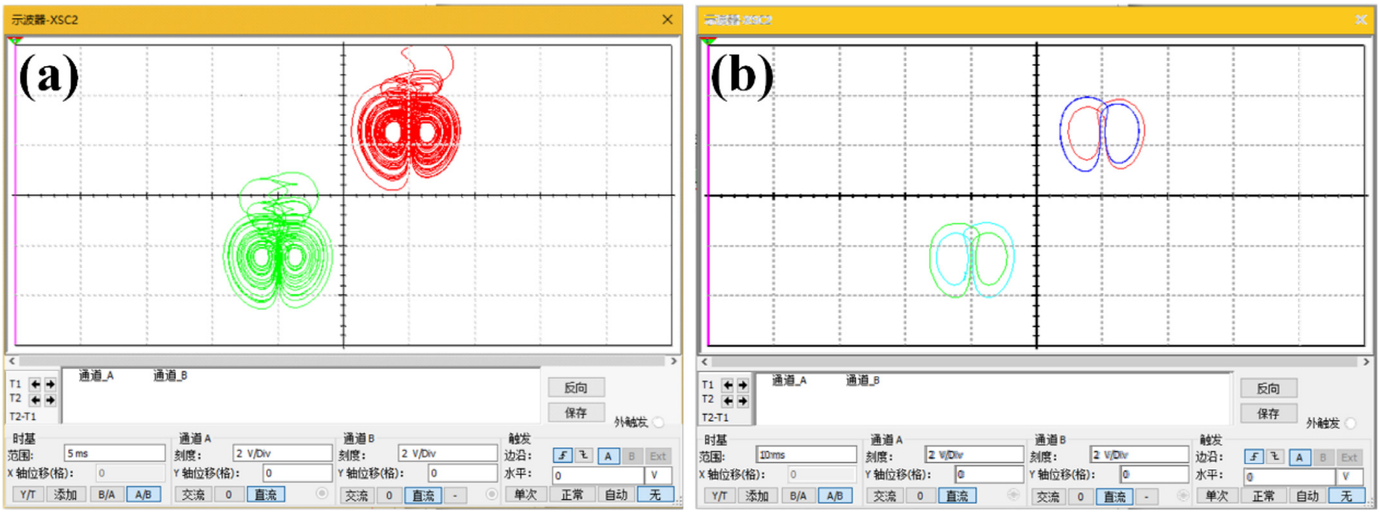

Fig. 10. Coexisting attractors in the oscilloscope obtained from Multisim.

tude control of coexisting attractors, the constants for offset boosting should be modified accordingly. When $a = 1, F(x) = |x| - 4$, $G(y) = |y| - 5$, the parameter m varies in [1,4], and system CSS3 exhibits coexisting chaos of conditional symmetry with rescaled amplitude in the variables x and z. Lyapunov exponent spectra and bifurcation diagrams are shown in Fig. 7. Fig. 8 shows the phase trajectories on the x-y and y-z planes and modified signals of x and z when the control parameter m varies. Conditional symmetry does not destroy the function of the coefficient of quadratic feedback in the y variable in CSS3.

4. Circuit verification of SCS3

In the following, based on Multisim software 14.0, circuit implementations are used to confirm the coexisting oscillations of conditional symmetry. By constructing a set of modules for integration, absolute value function operation, and phase reversal, system CSS3 is designed for reproducing the coexisting attractors, where the operational amplifier LM741, ideal multiplier, and diode D1N4148 are used for signal calculation.

To avoid saturation of the operational amplifier, the variables x and y in system CSS3 are reduced by a factor of two. Thus system CSS3 becomes to be,

$$\begin{cases} \dot{x} = -\dfrac{1}{2}(|2y| - 5)z, \\ \dot{y} = \dfrac{1}{2}(|2x| - 4)^2 - \dfrac{1}{2}, \\ \dot{z} = (|2x| - 5) - az. \end{cases} \tag{6}$$

Let the voltages across the capacitors V_{C1}, V_{C2} and V_{C3} represent the variables x, y, and z, and select a time scale of 1000 on the dimension-less equation, correspondingly state equations can be built based on Kirchhoff's law as,

$$\begin{cases} C_1 \dfrac{dx}{dt} = -\dfrac{G(y)}{R_1} z, \\ C_2 \dfrac{dy}{dt} = \dfrac{(F(x))^2}{R_2} - \dfrac{1}{R_3} V_2, \\ C_3 \dfrac{dz}{dt} = \dfrac{F(x)}{R_4} - \dfrac{1}{R_5} z. \end{cases} \tag{7}$$

where $F(x) = \frac{R_{22}}{R_{21}}\left(\frac{R_{19}}{R_{18}}|x| - \frac{R_{19}}{R_{20}}V_1\right)$ $G(y) = \left(\frac{R_{25}}{R_{23}}|y| - \frac{R_{25}}{R_{24}}V_3\right)$. Corresponding circuit parameters are set as, $C_1 = C_2 = C_3 = 10\text{nF}$, $R_1 = R_2 = R_3 = 20\text{k}\Omega$, $R_4 = R_5 = R_6 = R_7 = R_{19} = R_{21} = R_{22} = R_{25} = 10\text{k}\Omega$, $R_{18} = R_{23} = 5\text{k}\Omega$, $R_{24} = 2\text{k}\Omega$, $R_{20} = 2.5\text{k}\Omega$, $R_8 = R_9 = R_{10} = R_{11} = R_{12} = R_{13} = R_{14} = R_{15} = R_{16} = R_{17} = 100\text{k}\Omega$. The corresponding circuit for system CSS3 gives coexisting oscillations as shown in Fig. 9 and Fig. 10.

5. Conclusion

Conditional symmetry exists in asymmetric systems and in the symmetric system as well. Although polarity reversal with the variable and offset boosting provides a flexible combination of polarity balance, it is still difficult to construct conditional symmetry in symmetric systems. Several symmetric chaotic systems proposed in this work offer variable candidates for chaos-based engineering if the information device needs two pairs of coexisting signals. The newly found symmetric cases for constructing conditional symmetry further shows that polarity balance in a dynamical system has great diversity, from which more coexisting oscillations may be found.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Chunbiao Li: Conceptualization, Methodology, Writing – original draft, Project administration, Funding acquisition. **Julien Clinton Sprott:** Supervision, Investigation, Validation, Writing – review & editing. **Xin Zhang:** Methodology, Software, Visualization, Data curation. **Lin Chai:** Validation, Formal analysis. **Zuohua Liu:** Validation, Investigation, Resources.

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