Quantifying the robustness of a chaotic system

J. C. Sprott[a)]
Physics Department, University of Wisconsin-Madison, 1150 University Avenue, Madison, Wisconsin 53706, USA

(Dated: 11 January 2022)

As a way to quantify the robustness of a chaotic system, a scheme is proposed to determine the extent to which the parameters of the system can be altered before the probability of destroying the chaos exceeds 50%. The calculation uses a Monte–Carlo method and is applied to several common dissipative chaotic maps and flows with varying numbers of parameters.

In recent decades, hundreds of examples of iterated maps and systems of ordinary differential equations with chaotic solutions have been reported and studied. Some of these systems are intended as models of natural phenomena, but most are mathematical illustrations of particular dynamical behaviors. In either case, it is useful to know how much the parameters can vary from their nominal values before the chaos is destroyed since that will indicate how realistic the model is and how difficult it might be to employ the system in some practical application. For example, when constructing an electrical circuit designed to produce a chaotic signal, it is useful to know how carefully the component values must be chosen and controlled.[1] However, such information is rarely provided in the published literature. Thus it is useful to propose a quantitative measure that others can use and to give values for some common chaotic systems as a baseline for comparison.

I. INTRODUCTION

One of the defining characteristics of chaos is the sensitive dependence on initial conditions, usually quantified by calculation of the Lyapunov exponent(s).[2] Generally, a change in initial conditions will greatly alter the subsequent trajectory but will not affect the attractor for a dissipative dynamical system. However, a sufficiently large change in initial conditions can put the orbit in the basin of a different attractor or can make it unbounded and approach infinity. Thus it is useful to know the shape and size of the basin of attraction, and a method for doing so has been suggested.[3]

Similarly, a small change in the *parameters* of a chaotic dynamical system will greatly alter the trajectory but will only slightly distort the attractor unless the chosen parameters happen to be close to a bifurcation point, in which case the attractor can be destroyed or can undergo a qualitative change such as becoming a periodic limit cycle. Hence it is customary to choose the parameters of a chaotic system to be well away from any such bifurcations. Note that parameters can usually be converted to initial conditions by adding variables to a dynamical system.[4]

Mathematically, a robust dynamical system can be defined as one in which *all* small perturbations of the parameters away from their nominal values leave the system qualitatively unchanged.[5] Said differently, quantities that characterize the dynamics and topology such as Lyapunov exponents and attractor dimension change continuously as a function of the size of the perturbation for a robust system. A system that is not robust is said to be 'fragile.' It is usually also required that a robust system has no coexisting attractors. For some applications such as secure communications,[6] it is critically important to have a rigorously robust chaotic system. There is a large literature devoted to the design and proof of robust systems.[7]

However, for many purposes, a less rigid definition of robust is useful, and it is informative to assign a numerical value to quantify the robustness of the system, rather than to have a simple binary classification. Such a quantity should be a dimensionless number in the range of zero to one, or 0 to 100% to facilitate comparisons among diverse dynamical systems. There are many ways such a number could be constructed, and what follows is only one reasonable suggestion.

II. EXAMPLE: HÉNON MAP

To illustrate the idea, it is useful to consider in detail the simple two-dimensional iterated map introduced by Hénon[8] and given by

$$\begin{aligned} X_{n+1} &= 1 - a{X_n}^2 + bY_n \\ Y_{n+1} &= X_n \end{aligned} \qquad (1)$$

with chaotic solutions for the parameters $a = 1.4$ and $b = 0.3$ and initial conditions $X_0 = Y_0 = 0$.

The choice of initial conditions will generally alter the results unless the attractor is globally attracting, which the Hénon map is not. However, initial conditions at or near the origin are usually appropriate since chaos is born in nearly all mechanical and electronic systems through a route that begins when the equilibrium at the origin loses its stability.

a) Electronic mail: sprott@physics.wisc.edu.

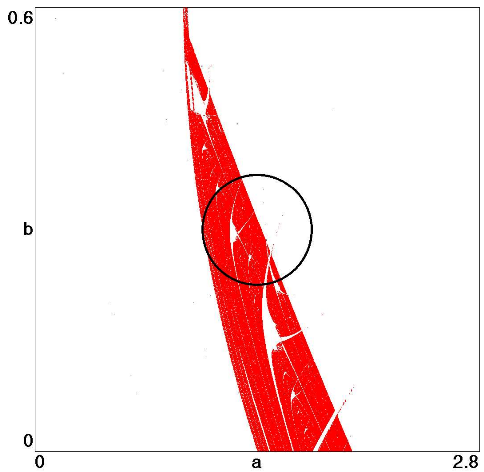

FIG. 1. Regions of parameter space (in red oline) for which the Hénon map has chaotic solutions with $X_0 = Y_0 = 0$. The black circle is centered on the nominal parameters ($a = 1.4$ and $b = 0.3$) and has a radius such that half of the enclosed parameters give chaotic solutions.

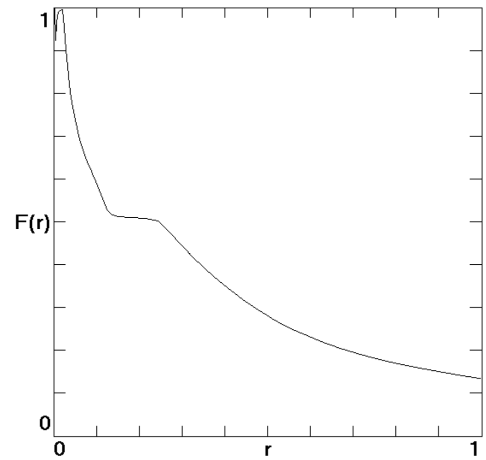

FIG. 2. Fraction of chaotic solutions within a distance r of the nominal normalized parameter values in parameter space for the Hénon map with $X_0 = Y_0 = 0$.

Figure 1 shows a 1000×1000 grid of parameters in the range of $0 < a \leq 2.8$ and $0 < b \leq 0.6$ with the chaotic regions colored in red (online). The nominal parameter values are at the center of the plot, and the dimensionless parameters ($a/1.4$ and $b/0.3$) range from 0 to 2 (a $\pm 100\%$ variation in each parameter).

In this case, chaos is identified by eliminating solutions that are unbounded ($|x| > 1000$) or periodic (with periods up to 1000) and assuming those that remain are chaotic. Using a positive Lyapunov as a criterion gives a similar result. Exactly 107786 of the million points (approximately 11.8%) are chaotic, a value of some interest in its own right, and one that could serve as a measure of robustness.

A careful examination of the figure suggests that the parameter space is dense in periodic windows[9] as it typical of low-dimensional dynamical systems with a smooth nonlinearity.[10] Thus the system does not satisfy the mathematical definition of robust. However, small perturbations of the parameters are much more likely than not to preserve the chaos.

This notion can be quantified by constructing circles of radii $r = \sqrt{(a/1.4 - 1)^2 + (b/0.3 - 1)^2}$ centered on the nominal values and calculating the fraction $F(r)$ of points within each circle that are chaotic. Figure 2 shows the result of such a calculation for the Hénon map.

Generally, but certainly not always, $F(r)$ is a monotonically decreasing function of r, that for this case first falls below $F = 0.5$ at $r = 0.246$, hereafter denoted as r_0. Hence we conclude that the Hénon map is about 25% robust in the sense that a 25% variation in parameters is more likely than not to destroy the chaos. Figure 1 shows a circle of radius r_0, the interior of which contains an equal number of parameter values that give chaotic and nonchaotic solutions.

It is also useful to quantify the sensitivity of the chaos to the parameters a and b individually with the other held constant at its nominal value. Such a calculation is straightforward and leads to $r_0 = 11\%$ for a and $r_0 = 100\%$ for b. This result is consistent with the vertical elongation of the chaotic region in Fig. 1. The value of 100% means that most values for $a = 1.4$ and $0 < b < 0.3$ give chaos, while most values for $0.3 < b < 0.6$ do not.

To assess sensitivity of the robustness to initial conditions, the calculation was repeated with $X_0 = Y_0 = 0.9$, which is close to the boundary of the basin of attraction. The calculated robustness was 23.2%, which is close to the value of 24.6% for initial conditions at the origin. Whether this result is typical is an open question.

III. MONTE–CARLO ALGORITHM

The method just described is meant to illustrate the concept with a simple example, but it is unwieldy and computationally-intensive, especially for systems with more than two parameters where the circles become spheres or hyperspheres in parameter space and for systems of ordinary differential equations where identification of chaos requires calculation of the largest Lyapunov exponent or some equivalent quantity. Thus it is useful to describe a Monte–Carlo algorithm[11] that gives a good approximation to r_0 with orders of magnitude less computation and without the need to make plots and graphs.

The method begins by making an initial guess for the value of r_0 such as $r_0 = 0.5$ and randomly choosing a parameter point within a circle of radius $r = \sqrt{2}r_0$. The factor $\sqrt{2}$ is not critical and can be replaced by any value

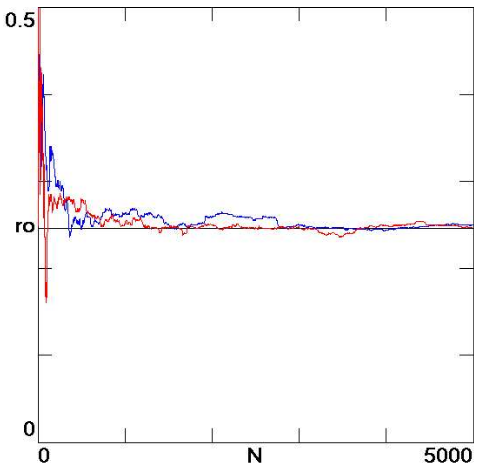

FIG. 3. Two Monte–Carlo calculations of the robustness of the Hénon map with $X_0 = Y_0 = 0$ as a function of the number of cases tested. The black horizontal line is the value of $r_0 = 0.246$ calculated from $F(r) = 0.5$ in Fig. 2.

```
ro = 0.5                        'Initial guess
FOR i = 1 TO 5000               'Iterate 5000 times
    DO                          'Choose random parameters
        a = 2.8*RND
        b = 0.6*RND
        rsq = (a/1.4 - 1)^2 + (b/0.3 - 1)^2
    LOOP UNTIL rsq < 2*ro*ro
    r2(i) = rsq                 'Save r squared values in array
    tot = 1
    FOR j = 1 TO i              'tot is the number of cases < ro
        IF ABS(r2(j)) < ro*ro THEN INCR tot
    NEXT j
    x(0) = 0                    'Set initial conditions
    y(0) = 0
    FOR n=1 TO 1e5              'Iterate the map 100,000 times
        x(n) = 1 - a*x(n-1)^2 + b*y(n-1)
        y(n) = x(n-1)
        IF ABS(x(n)) > 1000 THEN EXIT FOR       'Unbounded
        p = 0
        FOR j = n - 1 TO MAX(0, n - 1000) STEP -1
            IF x(j) = x(n) AND y(j) = y(n) THEN
                p = i           'Solution has period p
                EXIT FOR
            END IF
         NEXT j
        IF p > 0 THEN EXIT FOR
    NEXT n
    IF n > 1e5 THEN             'It's chaotic!
        r2(i) = -ABS(r2(i)) 'Negative r2 signifies chaos
        c = 1
        FOR j = 1 TO i          'c is the number of chaotic cases
            IF r2(j) < 0 AND r2(j) > -ro*ro THEN INCR c
        NEXT j
        Fr = c/tot              'Fraction of cases that are chaotic
        ro = 2*ro*Fr            'Update the estimate of ro
        PRINT Fr, ro
    END IF
NEXT i
```

FIG. 4. Pseudocode used to implement the Monte–Carlo calculation that produced Fig. 3.

somewhat greater than 1.0. For this first point, the fraction that is chaotic $F_0(r_0)$ will be either 1.0 if the solution is chaotic or zero if not. Continue the process if necessary N times until $F_N(r_0) = 1/N > 0$ (until there is one chaotic case with $r < r_0$). On average, this will occur for $N = 2^{p/2}$, where p is the dimension of the parameter space (the number of parameters).

Then replace r_0 by $2r_0F_N(r_0)$ and continue iterating $F_N(r_0)$ until it converges to a value of 0.5 and r_0 converges to a value that is no longer changing by a significant amount. This procedure is nothing more than Newton's method for finding the root $r = r_0$ of the equation $F(r) - 0.5 = 0$ assuming the local slope is $dF/dr = -1/2r$ at $r = r_0$, which is a crude approximation to the curve in Fig. 2. Newton's method converges rapidly if the slope is known accurately, which is unfortunately not the case here.

Nonetheless, applying the method to the Hénon map for two different instances (different sequences of randomly chosen parameters), gives the result in Fig. 3. Despite the fact that this is a slowly converging case because of the shoulder on the curve in Fig. 2 near $F(r) = 0.5$, the value of r_0 appears to converge to the expected value of $r_0 = 0.246$ (shown as a horizontal line in Fig. 3) to within a few percent after a few thousand iterations. Thus the method is several orders of magnitude faster than the million-point method in the previous section and simpler to implement.

Pseudocode, written in a dielect of BASIC that implements the Monte–Carlo calculation that produced Fig. 3 is shown in Fig. 4.

The computational time required to obtain a meaningful value for the robustness will depend on the complexity of the system (in particular, whether it is an iterated map or system of ordinary differential equations), the computer used, the efficiency of the compiler, the numerical method, and the desired accuracy. The result in Fig. 3 required about 20 minutes of computation using the code in Fig. 4, while the result from Fig. 1 required about six days of computation on a common desktop personal computer using the PowerBASIC Console Compiler.

Of course, there is some loss of accuracy with such a Monte–Carlo method, but it does not make sense to seek an overly accurate value because it will depend on the particular choice of nominal parameters and initial conditions. Choosing nominal values where the robustness is greatest may be a good strategy for some purposes. The method is best suited for comparing various systems and identifying ones that are highly robust and others that are very fragile. The next section will give some examples of each.

IV. ROBUSTNESS OF FAMILIAR SYSTEMS

Most of the familiar examples of chaos occur in systems of ordinary differential equations with simple polynomial and piecewise-linear nonlinearities, and it is instructive to calculate the robustness for some of those cases.

A. Lorenz system

Perhaps the most familiar and extensively studied case is the Lorenz system,[12]

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= -xz + \rho x - y \\ \dot{z} &= xy - \beta z, \end{aligned} \tag{2}$$

with chaotic solutions for the parameters $\sigma = 10, \rho = 28$, and $\beta = 8/3$. The resulting attractor has a global basin of attraction (all initial conditions go to the attractor except for a set of measure zero representing the three equilibrium points and the infinitely many unstable periodic orbits).[13] Since the origin is an equilibrium point for all values of the parameters, it is necessary to take different initial conditions such as $x_0 = y_0 = z_0 = 0.01$, but the results should be and are independent of the choice.

The parameter space is three-dimensional with spheres of radius $r = \sqrt{(\sigma/10-1)^2 + (\rho/28-1)^2 + (3\beta/8-1)^2}$ representing points equidistant from the nominal parameter values. Using the Monte–Carlo method described in the previous section gives a value of $r_0 \approx 66\%$. For this and the following cases, N is at least ten thousand, and the value of r_0 appears to have converged to the two quoted significant digits. The sensitivity to each parameter individually is 93% for σ, 64% for ρ and 79% for β. Thus the Lorenz system is relatively robust, at least compared with the Hénon map..

It is instructive to add coefficients to the remaining four terms in Eq. (2) with nominal values of 1.0 and calculate the robustness in the resulting seven-dimensional parameter space. The result is $r_0 \approx 87\%$, which is slightly greater than the three-dimensional case. Thus the robustness of a system appears not to depend strongly on the chosen parameters provided there are sufficiently many to completely characterize the dynamics. In general, this means that the number of parameters should be equal to the number of terms in the equations minus $D+1$, where D is the dimension of the system since D of the variables and time can be linearly rescaled without altering the dynamics.

B. Rössler system

Similar to the Lorenz system but with a single quadratic nonlinearity is the Rössler system,[14]

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c), \end{aligned} \tag{3}$$

with chaotic solutions for the parameters $a = b = 0.2$ and $c = 5.7$ and initial conditions $x_0 = y_0 = z_0 = 0$.

The parameter space is three-dimensional with spheres of radius $r = \sqrt{(5a-1)^2 + (5b-1)^2 + (c/5.7-1)^2}$ representing points equidistant from the nominal parameter values. Using the Monte–Carlo method described in the previous section gives a value of $r_0 \approx 51\%$. The sensitivity to each parameter individually is 53% for a, 82% for b and 57% for c. Thus the Rössler system is only slightly less robust than the Lorenz system.

C. Chua's Circuit

Probably the most famous and highly studied chaotic electrical circuit was developed by Chua[15] and can be modeled by the piecewise-linear equations

$$\begin{aligned} \dot{x} &= c[y - x + bx + \tfrac{1}{2}(a-b)(|x+1| - |x-1|)] \\ \dot{y} &= x - y + z \\ \dot{z} &= -dy, \end{aligned} \tag{4}$$

with chaotic solutions for the parameters $a = 8/7, b = 5/7, c = 9$, and $d = 100/7$ and initial conditions $x_0 = y_0 = z_0 = 0.01$.

The parameter space is four-dimensional with hyperspheres of radius $r = \sqrt{(7a/8-1)^2 + (7b/5-1)^2 + (c/9-1)^2 + (7d/100-1)^2}$ representing points equidistant from the nominal parameter values. Using the Monte–Carlo method described in the previous section gives a value of $r_0 \approx 17\%$. The sensitivity to each parameter individually is 48% for a, 63% for b, 22% for c, and 25% for d. Thus Chua's circuit is somewhat less robust than the Lorenz system despite having a similar double-lobe attractor. As a model of an electrical circuit, it might be more reasonable and instructive to choose the parameters to be values of the circuit components.

D. Jerk Circuit

Even simpler than Chua's circuit is the two-parameter piecewise-linear system given by[16]

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -az - by + |x| - 1, \end{aligned} \tag{5}$$

with chaotic solutions for the parameters $a = 0.6$ and $b = 1$ and initial conditions $x_0 = y_0 = z_0 = 0$. Note that the constant 1 in the $\dot{z}$ equation is an amplitude parameter[17] that only affects the size of the attractor and thus cannot be used as a bifurcation parameter.

Equation (5) is called a 'jerk system' because it can be written in compact form as $\dddot{x} = -a\ddot{x} - b\dot{x} + |x| - 1$, where $\dddot{x}$ is the time derivative of the acceleration $\ddot{x}$ in a mechanical system where x is the displacement.[18] The form of the nonlinearity $|x|$ makes it especially amenable to electronic implementation using diodes, and it is just one of a large family of similar systems with various nonlinearities.[19]

The parameter space is two-dimensional with circles of radius $r = \sqrt{(a/0.6-1)^2 + (b-1)^2}$ representing points equidistant from the nominal parameter values. Using the Monte–Carlo method described in the previous section gives a value of $r_0 \approx 5.4\%$. The sensitivity to each parameter individually is 9% for $a$and 7% for b. Thus the electronic circuit is less robust than Chua's circuit, but it is simple to construct and operates reliably provided one of the circuit components can be carefully adjusted.

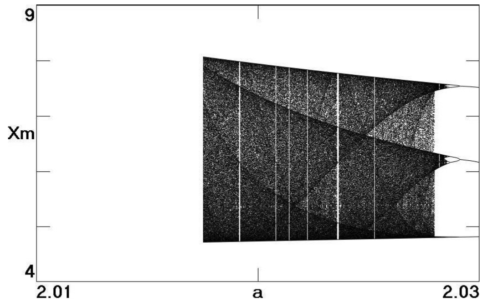

FIG. 5. Bifurcation diagram of the local maxima of x for the simplest chaotic system in Eq. (6) with $x_0 = y_0 = z_0 = 0.05$ showing seven of the infinitely many tiny periodic windows.

E. Simplest chaotic system

The final example is another jerk system but with a single parameter and a quadratic nonlinearity,[20]

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -az + y^2 - x, \end{aligned} \tag{6}$$

with chaotic solutions for the parameter $a = 2.02$ and initial conditions $x_0 = y_0 = z_0 = 0.05$. It can be written in compact form as $\dddot{x} = -a\ddot{x} + \dot{x}^2 - x$, and it has been rigorously proved that no simpler chaotic system with a single quadratic nonlinearity exists.[21] Variations of this system with different nonlinearities[22] have been implemented electronically.[23]

The parameter space is one-dimensional with $r = |a/2.02-1|$ representing points equidistant from the nominal parameter value of $a = 2.02$. Using the Monte–Carlo method described in the previous section gives a value of $r_0 \approx 1.7\%$. Thus this system is relatively fragile in part because of its small basin of attraction that does not include the origin and accounts for why it was not discovered much earlier.

With a single parameter, it is easy to visualize the behavior of the system in a conventional bifurcation diagram as shown in Fig. 5 where the local maximum of x is plotted. The system undergoes a common period-doubling route to chaos as the parameter a decreases. The plot shows the usual large period-3 window on the right with its period-doubling, but there are infinitely many tiny periodic windows in the vicinity of $a = 2.02$, mostly with very large periods, seven of which (with apparent periods of 4, 6, 8, 7, 5, 7, and 9 from left to right) are barely visible in the plot. These windows have transiently chaotic orbits, some of which require calculating for a time of $\sim 4 \times 10^4$ to resolve, which is rarely done in the literature and accounts for an overestimate of the robustness of some systems. Points on the left of the plot as well as those outside the basin of attraction have unbounded orbits.

TABLE I. Selected chaotic systems with their robustness r_0.

System	Eq.	Parameters	Init Cond	r_0
Hénon[8]	(1)	1.4, 0.3	0	25%
Lorenz[12]	(2)	10, 28, 8/3	0.01	66%
Rössler[14]	(3)	0.2, 0.2, 5.7	0	51%
Chua[15]	(4)	8/7, 5/7, 9, 100/7	0.01	17%
Jerk[16]	(5)	0.6, 1	0	5%
Simplest[20]	(6)	2.02	0.05	1.7%

V. CONCLUSION

This paper has described a simple method for quantifying the robustness of a chaotic system and given a number of examples. Table 1 summarizes the cases previously discussed, showing the wide range of their robustness. The method can be applied with equal ease to any dynamical system with any number of variables and parameters and any desired mode of behavior (stable equilibrium, periodic, quasiperiodic, chaotic, hyperchaotic). Such a calculation should probably be included as part of the complete description of any new chaotic system that is proposed or reported.

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