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Abstract	The many examples in the previous chapters should leave no doubt that hidden attractors are common in nonlinear dynamical systems. Remarkably, hidden attractors can have basins that fill the entire space with every initial condition on the attractor. Two such examples are shown here.

Globally Attracting Hidden Attractors

Julien Clinton Sprott

Abstract The many examples in the previous chapters should leave no doubt that hidden attractors are common in nonlinear dynamical systems. Remarkably, hidden attractors can have basins that fill the entire space with every initial condition on the attractor. Two such examples are shown here.

1 Introduction

The many examples in the previous chapters should leave no doubt that hidden attractors are common in nonlinear dynamical systems. Previous authors have echoed the claim that they are hard to find because there is no systematic method to identify initial conditions in their basin of attraction.

Thus it is fitting to temper those claims with some examples of hidden attractors that are *globally* attracting. Not only is every initial condition in their basin of attraction, but every initial condition lies on the attractor, and thus they could hardly be less hidden. Furthermore, such attractors have been known and studied long before the recent hoopla about hidden attractors, and they have other remarkable properties to be recounted here.

2 Conservative Nosé–Hoover System

The interest in chaotic systems whose orbit visits the entire state space (called *ergodic*) arose long ago from a quest among molecular dynamicists to find a simple

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dynamical system that would model the behavior of a harmonic oscillator in thermal equilibrium with a heat bath at a constant temperature. Prior to the modern chaos era, it had been assumed that any such model would need many variables. The first breakthrough came in 1984 when Shuichi Nosé found a Hamiltonian system with four variables consistent with the necessary condition that the probability distribution functions of position and momentum should be Gaussian [1] as expected for Gibbs' canonical ensemble [2].

The following year, Bill Hoover showed that Nosé system could be reduced to a three-dimensional form with the same properties, now known as the Nosé–Hoover system [3]:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - zy \\ \dot{z} = y^2 - 1. \end{cases} \tag{1}$$

This system was independently discovered in a search for three-dimensional chaotic flows with five terms and two quadratic nonlinearities, and thus it is also known as the Sprott A system [4, 5] and has been widely studied. It is the simplest in a large class of systems with similar properties [6].

Absent the zy term, this system is a simple harmonic oscillator with x playing the role of position, and y is its canonically conjugate momentum. The zy term represents a nonlinear damping (for positive z) or anti-damping (for negative z) with z controlled by the $\dot{z}$ equation such that the damping averages to zero when the mean square momentum $\langle y^2 \rangle$ is unity. Thus z acts as a thermostat, controlling the average energy of the chaotic oscillator, but allowing it to fluctuate as desired to model an oscillator in equilibrium with a heat bath [7].

System (1) is unusual because it has no equilibrium points, but neither does it have an attractor because it is derived from a Hamiltonian [8] in which the fourth variable is a slave of the other three and thus does not influence the dynamics. Hence the system is conservative with the missing energy in the hidden variable, and the oscillator is isothermal rather than isoenergetic. Such systems are called nonuniformly conservative [9], and they share many of the properties of conventional conservative systems.

The third variable allows the system to oscillate chaotically with a chaotic sea whose Lyapunov exponents are (0.0139, 0, –0.0139) and that stretches to infinity in all three dimensions, but that encloses an intricate set of nested and intertwined invariant tori on which the orbits are quasiperiodic with Lyapunov exponents of (0, 0, 0). All orbits, both in the chaotic sea and on the tori, repeatedly cross the $z = 0$ plane, which allows the dynamics to be completely characterized by examining a cross section of the orbit in that plane. In particular, quasiperiodic orbits embedded anywhere in the chaotic sea will appear as 'holes' in that cross section of the flow. The system is time-reversal invariant under the transformation $(x; y; z; t) \rightarrow (x; -y; -z; -t)$ as expected for a conservative system.

3 Dissipative Nosé–Hoover System

There are many ways to add dissipation to a thermostatted oscillator without introducing equilibrium points and thus producing a hidden attractor. For example, the constant term in the $\dot{z}$ equation of system (1) can be replaced by a function $f(x)$ that is everywhere positive:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - zy \\ \dot{z} = y^2 - f(x). \end{cases} \tag{2}$$

Physically, this corresponds to a harmonic oscillator in a heat bath with a one-dimensional temperature gradient given by $\frac{df}{dx}$.

This system with $f(x) = 1 + \varepsilon \tanh(x)$ (corresponding to a temperature that varies from $1 - \varepsilon$ at $x = -\infty$ to $1 + \varepsilon$ at $x = \infty$ with a maximum gradient of ε at $x = 0$) was originally proposed and studied by Posch and Hoover in 1997 [10]. For $\varepsilon = 0.38$ it has a hidden chaotic attractor that extends to infinity in all three dimensions and encloses a region in the vicinity of the origin with conservative tori and quasiperiodic orbits as shown in Fig. 1 [11]. The chaotic attractor fills the entirety of its basin of attraction, but with a highly nonuniform measure.

The attractor has Lyapunov exponents of (0.0019, 0, –0.0020) and a Kaplan–Yorke dimension of 2.945. Thus it differs markedly from essentially all the other chaotic attractors in this book for three-dimensional autonomous flows whose Kaplan–Yorke dimensions are only slightly greater than 2.0. In fact, the attractor is multifractal with a capacity dimension of exactly 3.0, and it stretches to infinity in all three dimensions but with a rapidly decreasing measure. Furthermore, as ε is decreased, the Kaplan–Yorke dimension further increases until it reaches a value of 3.0 for $\varepsilon = 0$, where the standard Nosé–Hoover system (1) with a chaotic sea is recovered.

As a consequence, its basin of attraction fills the entire space except for a finite region in the vicinity of the origin wherein tori with quasiperiodic orbits reside. This is an example of a *Class 1b* basin of attraction [12]. Although it is not a global attractor, a randomly chosen initial condition not too close to the origin is overwhelmingly likely to lie in the basin, and it will lie on the attractor, although usually in a region rarely visited by the orbit.

Remarkably, this dissipative system is time-reversal invariant, just like its conservative counterpart. The system has a repellor that overlaps the attractor, and that becomes an attractor when time is reversed. The attractor-repellor pair as shown by a portion of the orbits in Fig. 2 have only an imperceptible shift in the z-direction of $\langle z \rangle \approx \pm 1.2 \times 10^{-4}$ with $\langle x \rangle \approx -0.6855$ and $\langle y \rangle = 0$.

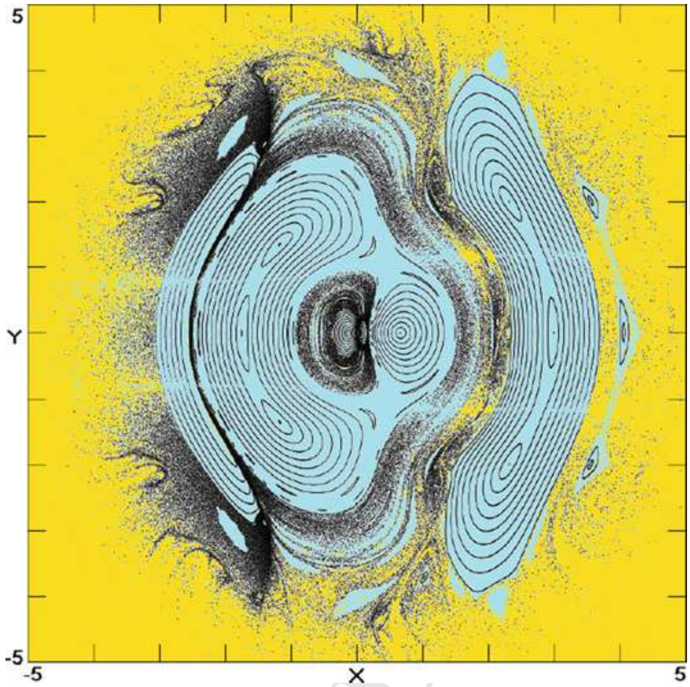

Fig. 1 Cross-section of the orbits in the $z = 0$ plane for system (2) with $f(x) = 1 + 0.38\tanh(x)$. Blue indicates the regions with conservative tori and quasiperiodic orbits, and yellow indicates the infinite basin of attraction of the dissipative hidden chaotic attractor shown in black [11]

4 Buncha System

As if the previous case were not remarkable enough, there are variants of the Nosé–Hoover system that are dissipative and ergodic with a hidden attractor that is the entirety of the three-dimensional state space. Probably the simplest and most elegant example is a reduced form of a general class of system proposed and studied by Buncha Munmuangsaen and collaborators [13] and given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - azy \\ \dot{z} = |y| - 1. \end{cases} \tag{3}$$

Like the Nosé–Hoover case, this system has a single bifurcation parameter a that can be put in any of the five terms, and that completely characterizes the system through a one-dimensional bifurcation diagram as shown in Fig. 3. For $a = 0$ the system is a simple conservative harmonic oscillator with an amplitude that depends only on the initial conditions.

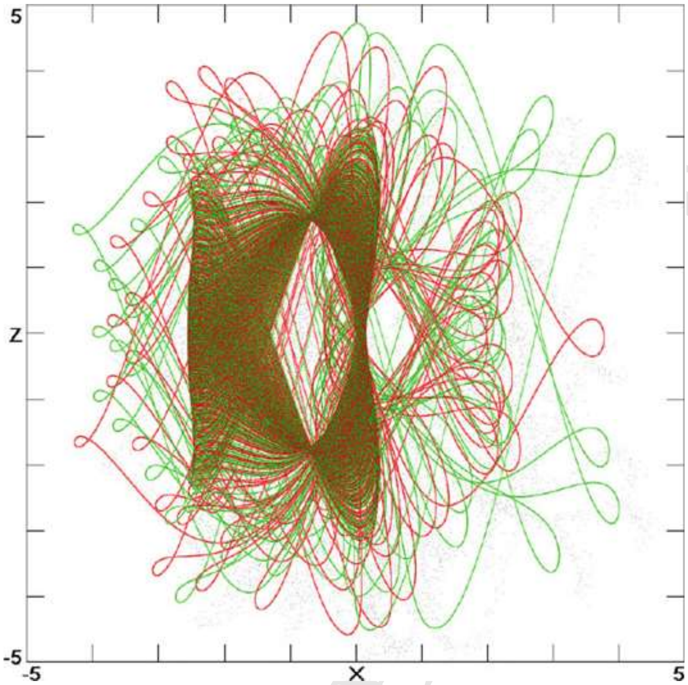

Fig. 2 An orbit on the hidden attractor (in red) and on the corresponding repellor (in green) for system (2) with $f(x) = 1 + 0.38\tanh(x)$

For $a > 0$, there are three distinct regions with bifurcations in the vicinity of $a = 0.9$ and $a = 2.1$. For a less than about 0.9, the dynamic is dominated by nested invariant tori with conservative quasiperiodic orbits but surrounded by a dissipative region with limit cycles and/or strange attractors. In the range of a between about 0.9 and 2.1, there is a conservative region containing nested tori that are linked by a symmetric pair of dissipative limit cycles and a long-duration chaotic transient whose orbit eventually collapses onto one of the limit cycles with a riddled basin of attraction. At $a \approx 2.0$, the limit cycles merge into one large limit cycle at $a \approx 2.07$ that gives birth to a strange attractor surrounding the tori. As a is increased further, the tori shrink and eventually vanish at $a \approx 3.07$, leaving only dissipative regions with a single strange attractor that fills all of space.

The Kaplan–Yorke dimension of the attractor continues to increase with increasing a, reaching a maximum of about 2.9924 at $a = 7$ before slowly decreasing, except for narrow periodic windows in the vicinity of $a = 2.28, 3.78, 4.00$, and 6.00, as well as other values that are unresolved in Fig. 3. In these periodic windows, there is a long-duration chaotic transient.

Although system (3) exhibits a variety of unusual behaviors, our interest here is in the regime where there is a single ergodic strange attractor that fills all of space and thus is technically "hidden" because the system has no equilibrium points. For that

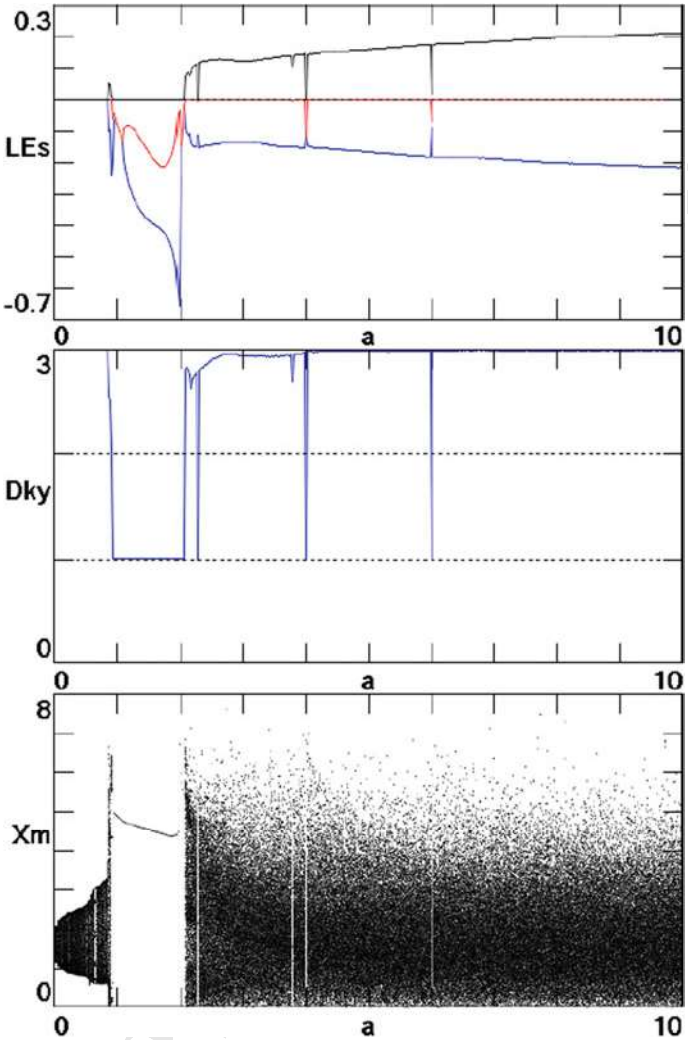

Fig. 3 Lyapunov exponents (LEs), Kaplan–Yorke dimension (Dky), and the local maxima of x (Xm) as a function of the bifurcation parameter a in (3) over the range $0 < a < 10$

purpose, we focus on the case $a = 5$ for which the Lyapunov exponents are (0.1610, 0, –0.1633), the Kaplan–Yorke dimension is 2.9858, and the orbit is as shown in Fig. 4. In this plot, the colors indicate the value of the local largest Lyapunov exponent with red positive and blue negative. While the attractor appears to be bounded, it has a fuzzy edge, and after a sufficiently long time the orbit will come arbitrarily close to every point in the three-dimensional state space.

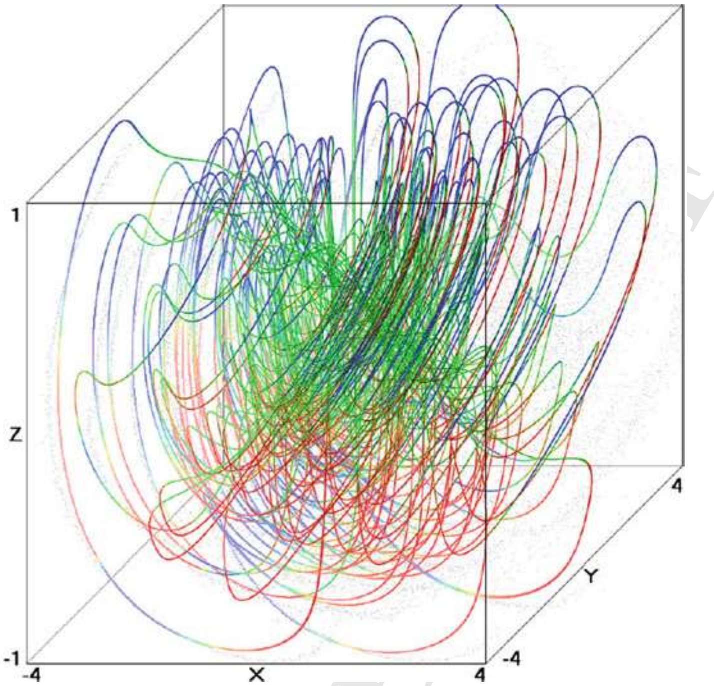

Fig. 4 An orbit on the ergodic strange attractor for system (3) with $a = 5$. The colors indicate the value of the local largest Lyapunov exponent with red positive and blue negative

To confirm that the system is ergodic with no embedded tori and quasiperiodic orbits, it suffices to examine a cross section of the flow at $z = 0$. Figure 5 shows such a plot. Other than the nullclines at $y = \pm 1$, where the orbit is tangent to the plane, there are no holes that would indicate a lack of ergodicity. A single orbit eventually visits every point in the plane, and every initial condition produces the same plot. Said differently, the attractor is globally attracting with a $Class\ 1a$ basin of attraction [12], and the attractor fills the whole of its basin.

The local largest Lyapunov exponent has a complicated structure as evidenced by the variations in color. Furthermore, the equations are time-reversible under the transformation $(x, y, z, t) \rightarrow (x, -y, -z, -t)$ just like the previous cases. When time is reversed, the attractor becomes a repellor that looks identical to the attractor. Thus there exists a symmetric strange attractor-repellor pair that is coincident, except for a tiny offset in the z direction of $\langle z \rangle = \pm 0.0023$, and they exchange roles when time is reversed.

The attractor and repellor are both multifractal with a capacity dimension of exactly 3.0 but with a highly nonuniform measure that is far from the Gaussian that characterizes the usual conservative ergodic harmonic oscillator. The probability distribution functions for the three variables, along with the first six even moments of the distribution, are shown in Fig. 6. None of the distributions have a sharp cutoff,

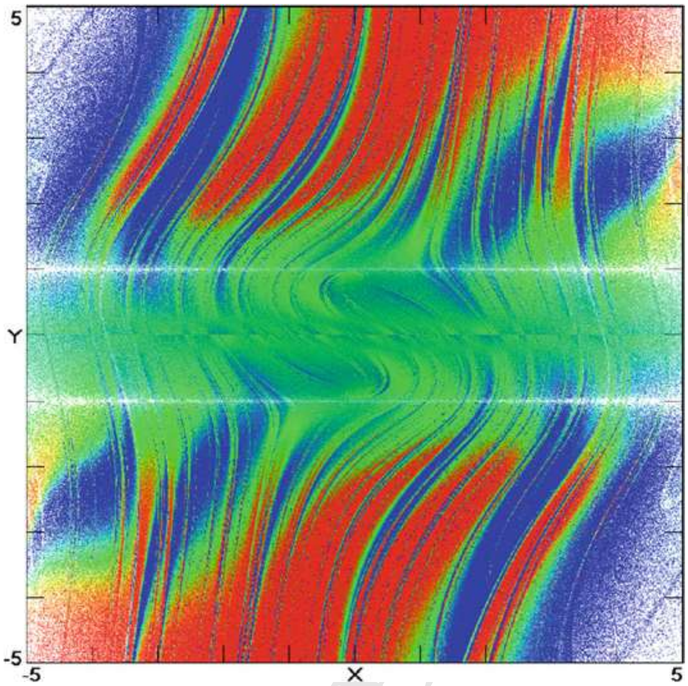

Fig. 5 Cross section of the ergodic strange attractor in the $z = 0$ plane for system (3) with $a = 5$. The colors indicate the value of the local largest Lyapunov exponent with red positive and blue negative

but rather they have long tails that extend to infinity in all directions. Hence the attractor and its basin fill the whole of the state space.

5 Signum Thermostat Dissipative System

Finally, we consider a dissipative chaotic system that is fully ergodic with a measure that more nearly approximates a Gaussian with a hidden global attractor and that is presented here for the first time. This system is a variant of the dissipative Nosé–Hoover system (2) but with the zy term replaced by $2\text{sgn}(z)y$ and is called the signum thermostat [14]. To preserve symmetry in the x probability distribution, $f(x)$ is taken as $f(x) = \exp(-\varepsilon x^2)$ to give

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - 2\text{sgn}(z)y \\ \dot{z} = y^2 - \exp(-\varepsilon x^2). \end{cases} \tag{4}$$

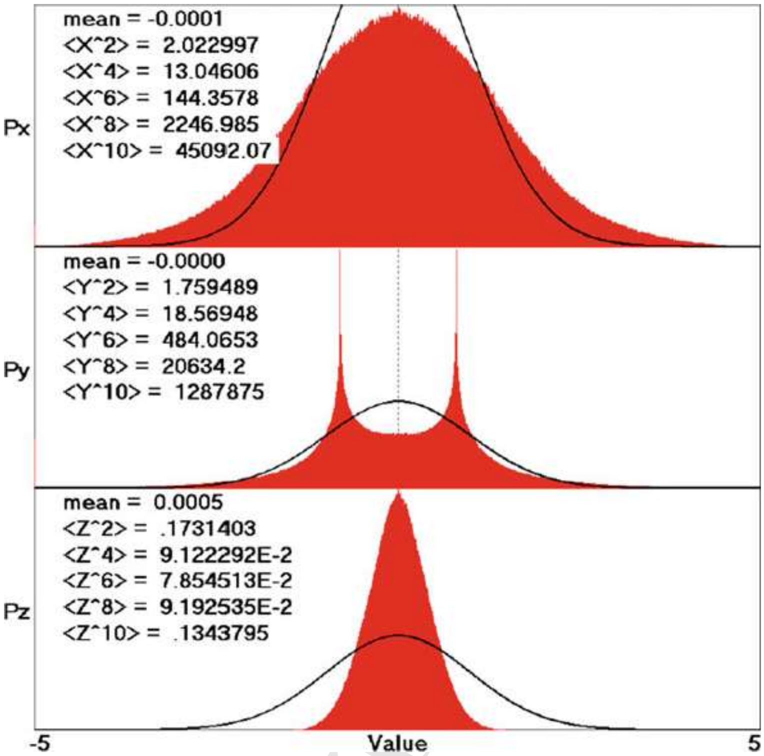

Fig. 6 Probability distribution functions of the ergodic strange attractor for system (3) with $a = 5$. The black curves show a Gaussian distribution with a variance (second moment) of 1.0

Physically, this corresponds to a harmonic oscillator in a heat bath with its highest temperature at $x = 0$ and that approaches absolute zero at $x = \pm\infty$. Since $f(x) > 0$ for all x, there is no equilibrium point, and so any attractor for the system is hidden by definition.

For $\varepsilon = 0$ (constant temperature), $\exp(-\varepsilon x^2) = 1$, and the system is nonuniformly conservative and ergodic with a chaotic sea whose probability distribution is given exactly by $P(x, y, z) = \exp(-x^2/2 - y^2/2 - 2|z|)/2\pi$. For ε small and positive, the system is dissipative and ergodic with a strange attractor whose probability distribution departs only slightly from the case with $\varepsilon = 0$.

For example, $\varepsilon = 0.1$ gives the cross section plot at $z = 0$ shown in Fig. 7. Aside from the nullclines at $y = \pm\exp(-x^2/20)$, there is no indication of quasiperiodic holes in the plot. The colors show that the local largest Lyapunov exponent has a considerable structure as is typical of these systems.

The evidence that the system is dissipative with a chaotic attractor comes from the Lyapunov exponents whose values are (0.03544, 0, –0.3636), the Kaplan–Yorke dimension whose value is 2.9746, and the time-averaged dissipation of $\langle 2\text{sgn}(z)\rangle \approx 9.2 \times 10^{-3}$. The attractor is multifractal with a capacity dimension of 3.0.

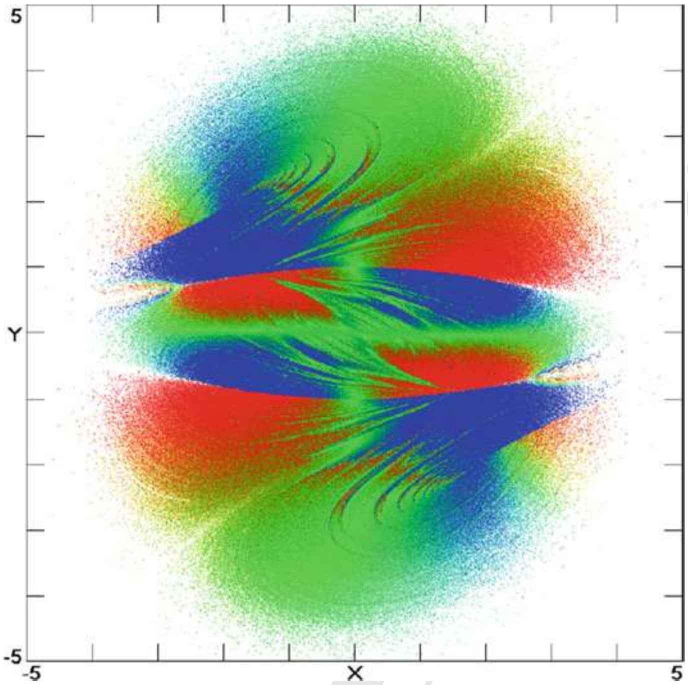

Fig. 7 Cross section of the ergodic strange attractor in the $z = 0$ plane for system (4) with $\varepsilon = 0.1$. The colors indicate the value of the local largest Lyapunov exponent with red positive and blue negative

Like the previous systems, this case is time-reversal invariant under the transformation $(x, y, z, t) \rightarrow (x, -y, -z, -t)$ with an attractor-repellor pair that fully overlap with only a tiny offset in the z-direction. The attractor has a global $Class\ 1a$ basin of attraction [12], and the attractor fills the entire basin.

Figure 8 shows the small departure of the probability distribution functions from the ones with $\varepsilon = 0$. The dissipative oscillator spends slightly less time in the vicinity of the origin where it is heated strongly as well as far from the origin where it is cooled, and relatively more time at intermediate values. Although the tails of the distributions are suppressed, they still extend to infinity in all directions so that every initial condition is on the attractor.

Smaller values of ε give distributions even closer to a Gaussian and Kaplan–Yorke dimensions that approach ever closer to the limit of 3.0. The system remains ergodic for ε up to about 3.4 except for periodic windows with long duration chaotic transients, whereupon the chaotic attractor is replaced by a globally attracting hidden limit cycle. This is not surprising since the temperature is an amplitude parameter that does not affect the dynamic in the conservative constant-temperature case with a signum thermostat. However, the probability distributions become increasingly peaked at intermediate values of x and y as ε increases.

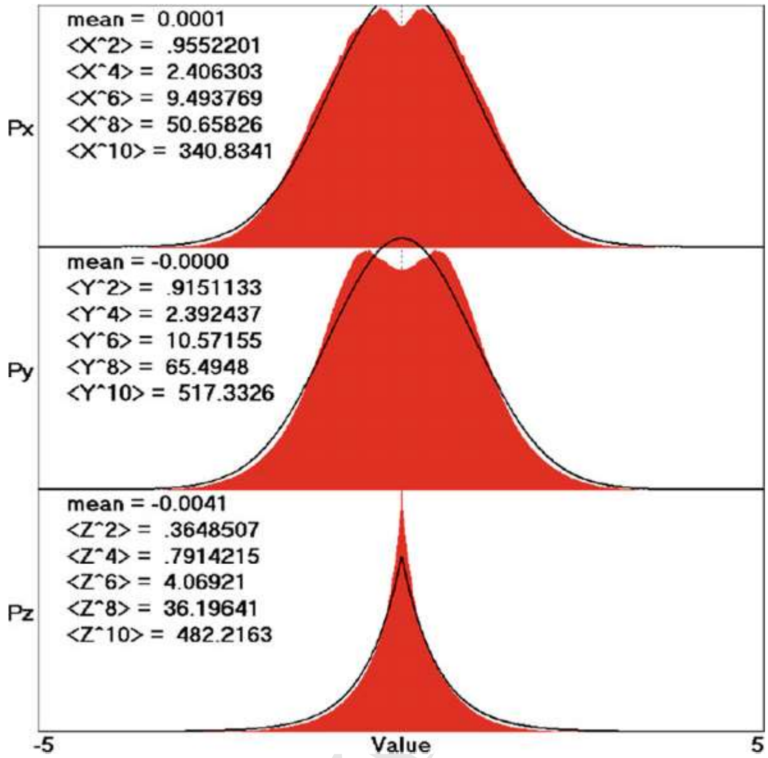

Fig. 8 Probability distribution functions of the ergodic strange attractor for system (4) with $\varepsilon = 0.1$. The black curves show the distributions for $\varepsilon = 0$

6 Summary and Conclusions

The Nosé–Hoover system is almost certainly the simplest example of a nonuniformly conservative chaotic flow without equilibria, but the chaotic sea coexists with regions of quasiperiodicity. It is the simplest example of a wide class of thermostatted oscillators, which are time-reversal invariant in accordance with Newton's laws and that exhibit aspects of thermodynamics and statistical mechanics such as a Gaussian probability distribution function. It is possible to eliminate the quasiperiodic regions and obtain systems that are fully ergodic with the orbit visiting every point in space as desired for a realistic physical model.

There are various ways to add dissipation to such systems and produce strange attractors that are hidden and that fill almost the entire state space, typically with a finite region that is occupied by tori with conservative quasiperiodic orbits. Most remarkably, it is also possible to modify the dissipative systems in such a way as to make them fully ergodic with a multifractal strange attractor that is globally attracting and fills the whole of space and yet satisfies the definition of being hidden. Two such examples were given here. These systems may be more realistic models of physical phenomena than are the purely mathematical models with hidden attractors that constitute most of the other examples in this book.

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