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Infinite lattice of hyperchaotic strange attractors

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ABSTRACT

By introducing trigonometric functions in a 4-D hyperchaotic snap system, infinite 1-D, 2-D, and 3-D lattices of hyperchaotic strange attractors were produced. Furthermore a general approach was developed for constructing self-reproducing systems, in which infinitely many attractors share the same Lyapunov exponents. In this case, cumbersome constants are necessary to obtain offset boosting; correspondingly additional periodic functions are needed for attractor hatching. As an example, a hyperchaotic system with a hidden attractor was transformed for reproducing 1-D, 2-D infinite lattices of hyperchaotic attractors and a 4-D lattice of chaotic attractors.

1. Introduction

In the real physical world multistability is common, like in the gas laser [1], delayed systems [2], biological systems [3], atoms [4], lactose utilization networks [5], fiber lasers [6], phosphorylation systems [7], electroencephalograms [8], neural networks [9], ice sheets [10], and even in semiconductor superlattices [11]. Multistability can be of a potential threat or even a possible asset for engineering applications. The mechanisms hidden in multistability are different. When the symmetry of a dynamical system is broken, a symmetric pair of attractors may appear instead of a single symmetric one [12–21]. Asymmetric systems can also give coexisting attractors due to the specific damping or equilibria [22–25]. Variable-boostable chaotic systems can also return the symmetry by a boosted offset and generate coexisting conditionally symmetric attractors or infinitely many attractors [26–30]. Offset boosting is an operation by which the average level of the variable can be controlled. From the view of electronic signal offset boosting means revising its DC component. In fact, offset boosting is an effective means to realize attractor hatching so long as it is discretized and periodic. In this paper, infinite lattices of hyperchaotic strange attractors are constructed by transforming a variable-boostable system into a self-reproducing system or by introducing periodic functions into normal hyperchaotic systems. In Section 2, the method of attractor hatching for reproducing infinitely many attractors is discussed theoretically. In Section 3, an infinite 3-D lattice of hyperchaotic attractors is derived from a hyperchaotic snap system. In Section 4, the method is generalized to common dynamical systems for constructing infinite multidimensional lattices of attractors. Conclusion and discussions are given in the last section.

2. Attractor hatching from offset boosting

A self-reproducing system can be generated from an offset boostable dynamical system, where the offset boosting provides a direct entrance for attractor hatching. The mechanism for constructing infinitely many attractors is shown in Fig. 1. To cause a dynamical system to be self-reproducing, it is necessary to consider two inherent operations. The offset boosting operation makes the attractor shift in any desired direction, while the periodization operation of the offset-boostable variable causes attractor hatching. By this means, a dynamical system becomes self-reproducing [30] where infinitely many attractors are hatched. It is convenient to perform the above two operations in a variable-boostable system. A jerk system provides offset boosting in cascade [29] by introducing new intermediate variables. Correspondingly, an n-D hyperjerk system can be revised to be a (n-1)-D self-reproducing system by the periodic offset boosting [30].

Theorem 2.1. *A hyperjerk flow* $\frac{d^4x}{dt^4} = (x, \dot{x}, \ddot{x}, \dddot{x})$ *can be transformed to a 3-D variable-boostable system [26–30] by introducing three other*

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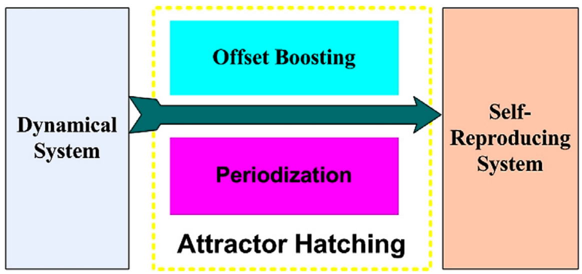

Fig. 1. Attractor hatching in a self-reproducing system.

variables y, z and u according to

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = u, \\ \dot{u} = f(\dot{x}, \dot{y}, \dot{z}, x) \end{cases} . \tag{1}$$

Offset boosting of the variables y, z and u can be obtained by introducing additional constants in the first three equations without revising the other dimensions.

Proof 2.1. Let $x = x'$, $y = y' + m$, $z = z' + n$, $u = u' + p$, where x', y', z', u' are new state space variables, while m, n, and p represent the new introduced constants,

$$\begin{cases} \dot{x}' = y' + m, \\ \dot{y}' = z' + n, \\ \dot{z}' = u' + p, \\ \dot{u}' = f\left(\dot{x}', \dot{y}', \dot{z}', x'\right). \end{cases} \tag{2}$$

Since $f(\dot{x}', \dot{y}', \dot{z}', x')$ depends on the time derivative of x', y', z' rather than y', z', u', which is not altered by the constants m, n, p, Eq. (2) in its hyperjerk representation is identical to the hyperjerk form of Eq. (1) and thus has the same dynamics while providing offset boosting of the variables y, z, and u. Normally, a 3-D jerk system can be revised to be more than a 4-D hyperjerk one since the attractor in 3-D space can be put in a space of higher dimension and consequently can realize offset boosting with greater dimensions.

Offset boosting provides attractor shifting by introducing independent constants in separate dimensions as shown in Eq. (2). Reasonably, if the offset-boostable variable is a periodic function, infinitely many attractors can be hatched since the offset boosting is contained in the period. To give a clear demonstration, we introduce three periodic functions in a 3-D offset-boostable hyperjerk system.

Theorem 2.2. *Periodic functions can be introduced into an offset-boostable system for constructing a self-reproducing system like,*

$$\begin{cases} \dot{x} = F(y), \\ \dot{y} = G(z), \\ \dot{z} = H(u), \\ \dot{u} = f(\dot{x}, \dot{y}, \dot{z}, x). \end{cases} \tag{3}$$

and therefore infinitely many identical attractors can be hatched in the corresponding 3-D space if system (3) has a bounded solution (an attractor) for one period. Here the functions F(y), G(z), and H(u) are periodic.

Proof 2.2. Since the functions $F(y)$, $G(z)$, and $H(u)$ are periodic, suppose p_1, p_2, and p_3 are their period, *i.e.*, $F(y) = F(y + kp_1)$, $G(z) = G(z + lp_2)$ and $H(u) = H(u + mp_3)$. For $x = x'$, $y = y' + kp_1$, $z = z' + lp_2$, $u = u' + mp_3$ (k, l, $m \in Z$), system (3) becomes

$$\begin{cases} \dot{x}' = F\left(y'\right), \\ \dot{y}' = G\left(z'\right), \\ \dot{z}' = H\left(u'\right), \\ \dot{u}' = f\left(\dot{x}', \dot{y}', \dot{z}', x'\right). \end{cases} \tag{4}$$

System (4) is identical to system (3), showing that introducing the constants kp_1, lp_2 and mp_3 does not change the dynamics of system (3) but gives a corresponding offset boost in the dimensions y, z, and u, which consequently gives birth to infinitely many attractors on a lattice in the y-z-u space. The specific new introduced constants (associated with the periods) in the offset-boostable variables can be easily removed in the left hand of Eqs. (3) and (4) for the time derivatives and also removed in the right hand side for the property of periodic functions, which contributes the property required for attractor hatching.

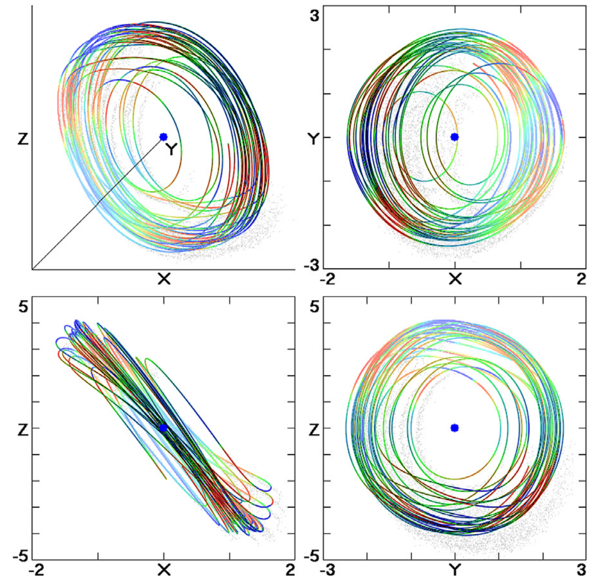

Fig. 2. Hyperchaotic attractor from system (5) with $a = 3.6$, $b = 1$ and initial conditions (0, 0, −2, 0), the colors indicate the value of the local largest Lyapunov exponent with positive values in red and negative values in blue. Equilibrium point is shown as blue dots. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

3. An infinite 3-D lattice of hyperchaotic attractors

In the following, we construct an infinite 3-D lattice of strange attractors based on a hyperchaotic snap system. Chlouverakis and Sprott [31] proposed what may be the algebraically simplest hyperchaotic snap system given by $\ddddot{x} + x^4\dddot{x} + a\ddot{x} + \dot{x} + x = 0$ with a single parameter $a = 3.6$ and Lyapunov exponents (0.1310, 0.0358, 0, −1.2550) and a Kaplan-Yorke dimension of 3.1329. The hyperchaotic snap system can be revised to be a 3-D offset-boostable one in the variables y, z, and u whose governing equations are,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = u, \\ \dot{u} = -x^4\dot{z} - a\dot{y} - b\dot{x} - x. \end{cases} \tag{5}$$

when $a = 3.6$ and $b = 1$, system (5) has a hyperchaotic strange attractor as shown in Fig. 2. System (5) has inversion symmetry and has an equilibrium point (0, 0, 0, 0), which is a saddle-focus whose eigenvalues are ($0.1604 \pm 1.8395i$, $-0.1604 \pm 0.5172i$).

System (5) can be revised to be a self-reproducing one providing a 3-D lattice of hyperchaotic strange attractors in a form with

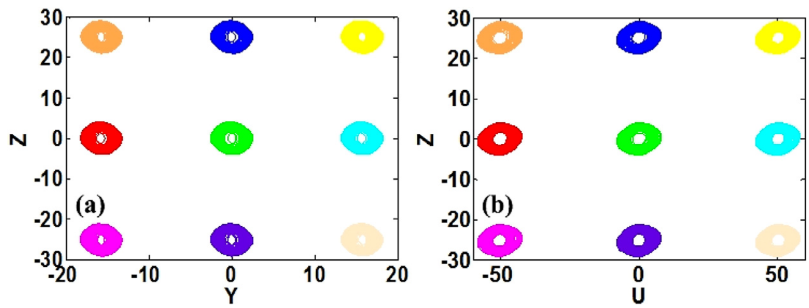

Fig. 3. 3-D lattice of coexisting hyperchaotic attractors from system (6) (a) y-z plane when the initial conditions are (0, $5k\pi$, $-2+8k\pi$, 0), (b) u-z plane when the initial conditions are (0, 0, $-2+8k\pi$, $16k\pi$) ($k \in \{-1, 0, 1\}$).

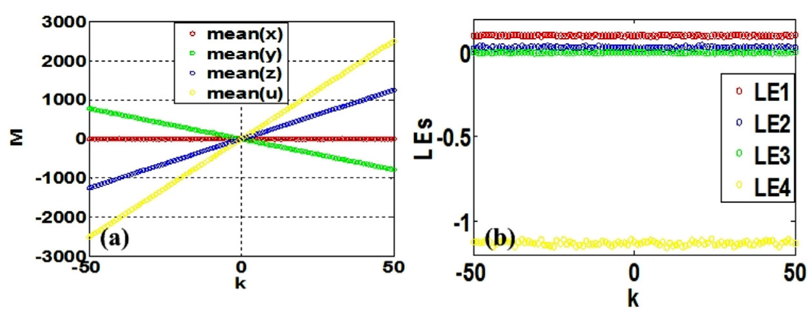

Fig. 4. Offset regulating of system (6) with invariable Lyapunov exponents (a) regulated offset when initial conditions are (0, $-5k\pi$, $-2+8k\pi$, $16k\pi$ ($-50 \le k \le 50$)), (b) invariable Lyapunov exponents.

Table 1
Equilibria and their eigenvalues for system (6).

Equilibria	Eigenvalues	Equilibria	Eigenvalues
(0, $5k\pi$, $8k\pi$, $16k\pi$)	$0.1604 \pm 1.8395i$ $0.1604 \pm 0.5172i$	(0, $2.5(2k+1)\pi$, $8k\pi$, $16k\pi$)	0.6395, -0.3996 $-0.1199 \pm 1.9745i$
(0, $5k\pi$, $8k\pi$, $8(2k+1)\pi$)	2.0769, -1.8309 $-0.1230 \pm 0.4979i$	(0, $2.5(2k+1)\pi$, $8k\pi$, $8(2k+1)\pi$)	-1.9621, 1.6104, 0.7652, -0.4136
(0, $5k\pi$, $4(2k+1)\pi$, $16k\pi$)	2.0769, -1.8309 $-0.1230 \pm 0.4979i$	(0, $2.5(2k+1)\pi$, $4(2k+1)\pi$, $16k\pi$)	-1.9621, 1.6104, 0.7652, -0.4136
(0, $5k\pi$, $4(2k+1)\pi$, $8(2k+1)\pi$)	$0.1604 \pm 1.8395i$ $0.1604 \pm 0.5172i$	(0, $2.5(2k+1)\pi$, $4(2k+1)\pi$, $8(2k+1)\pi$)	0.6395, -0.3996 $-0.1199 \pm 1.9745i$

six sinusoidal functions according to Theorem 2.2,

$$\begin{cases} \dot{x} = F(y), \\ \dot{y} = G(z), \\ \dot{z} = H(u), \\ \dot{u} = -x^4\dot{z} - a\dot{y} - b\dot{x} - x. \end{cases} \tag{6}$$

when $a = 3.6$, $b = 1$, $F(y) = 2.5\sin(0.4y)$, $G(z) = 4\sin(0.25z)$, $H(u) = 8\sin(0.125\,u)$, system (6) has an infinite 3-D lattice of hyperchaotic strange attractors with Lyapunov exponents (0.1013, 0.0306, 0, -1.1340) and a Kaplan-Yorke dimension of 3.1163. When the initial conditions vary according to the period of the sinusoidal function, system (6) produces infinitely many hyperchaotic attractors (shown in Fig. 3). As shown in Fig. 4, since the periods of $\sin(0.4y)$, $\sin(0.25z)$, and $\sin(0.125\,u)$ are $5k\pi$, $8k\pi$, $16k\pi$, when the initial conditions of the variables y, z, and u change according to k, the offset of y, z, and u will be modulated without changing the Lyapunov exponents.

System (6) retains the inversion symmetry with a rate of hypervolume contraction given by the Lie derivative, $\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{u}}{\partial u} = -\cos(0.125u)x^4$. System (6) has eight series of unstable equilibria P $= (0, 2.5k\pi, 4l\pi, 8m\pi)$, ($k, l, m \in Z$), whose eigenvalues are shown in Table 1.

If some of the offset-boostable variables retain their original form rather than being replaced with the sinusoidal function, system (6) will correspondingly generate a 1-D or 2-D lattice of hyperchaotic attractors, shown in Table 2. Other trigonometric functions can also be introduced in the offset-boostable variables, some of which can revise the intervals of the infinite lattice of attractors. ILHA and ILHD give a 1-D lattice of hyperchaos, ILHB, ILHC and ILHE give a 2-D lattice of hyperchaos while ILHF gives a 3-D lattice. As shown in Fig. 5, when the initial condition varies, system (6) visits different attractors according to the period of trigonometric functions. Different periods (5π for $\tan(0.2y)$ and 8π for $4\sin(0.25y)$) leading to different intervals between coexisting attractors.

4. Generalization for attractor hatching

The method for constructing a self-reproducing system can be generalized to other dynamical systems. Unlike those offset boostable systems, it is more complicated to transform a normal dynamical system for attractor hatching. To control the offset of some variables, it is necessary to revise more terms; and consequently for periodic offset boosting more introduced trigonometric functions may get coupled, which poses great influence on the dynamics. Generally all the variables in a dynamical system can be offset boosted by introducing separate constants into the original system, *i.e.*, offset boosting of the variables can be obtained if the number

Table 2
Lyapunov exponents and Kaplan-Yorke dimension for attractors from system (6) with $a = 3.6$, $b = 1$.

Cases	$(F(y), G(z), H(u))$	LEs	D_{KY}
ILHA	$(5\sin(0.2y), z, u)$	0.1149, 0.0355, 0, −1.1802	3.1274
ILHB	$(4\sin(0.25y), 4\sin(0.25z), u)$	0.1309,0.0446,0, −1.3661	3.1284
ILHC	$(4\sin(0.25y), 4\cos(0.25z), u)$	0.1309,0.0345,0, −1.3662	3.1284
ILHD	$(4\tan(0.2y), z, u)$	0.0961,0.0197,0,−0.8509	3.1359
ILHE	$(4.6\tan(0.2y), 4.6\tan(0.2z), u)$	0.1262,0.0322,0,−1.0543	3.1502
ILHF	$(2.5\sin(0.4y), 4\sin(0.25z), 8\sin(0.125u))$	0.1014, 0.0306, 0, −1.1332	3.1164

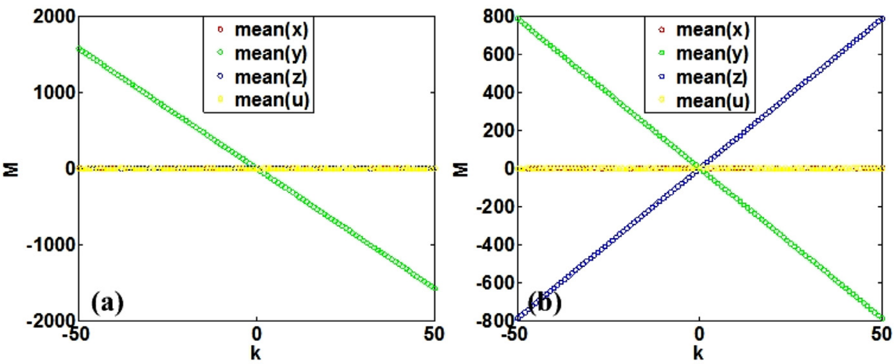

Fig. 5. Offset regulating of system (6) with $a = 3.6$, $b = 1$, $H(u) = u$ (a) $F(y) = 5\sin(0.2y)$, $G(z) = z$ when initial conditions revise as $(0,\ -10k\pi,\ -2, 0\ (-50 \le k \le 50))$, (b) $F(y) = 4.6\tan(0.2y)$, $G(z) = 4.6\tan(0.2z)$ when initial conditions revise as $(0,\ -5k\pi,\ -2 + 5k\pi, 0\ (-50 \le k \le 50))$.

of the introduced constants is not limited. For example, for system (7),

$$\begin{cases} \dot{x}_1 = F_1(x_1, x_2, \ldots, x_n), \\ \dot{x}_2 = F_2(x_1, x_2, \ldots, x_n), \\ \ldots\ldots, \\ \dot{x}_n = F_n(x_1, x_2, \ldots, x_n). \end{cases} \tag{7}$$

let $x_i = x_i' + c_i (i \in \{1, 2, \ldots, n\})$, where x_i' is a new state variable and c_i represents the new introduced constant,

$$\begin{cases} \dot{x}'_1 = F_1((x'_1 + c_1), (x'_2 + c_2), \ldots, (x'_n + c_n)), \\ \dot{x}'_2 = F_2((x'_1 + c_1), (x'_2 + c_2), \ldots, (x'_n + c_n)), \\ \ldots\ldots \\ \dot{x}'_n = F_n((x'_1 + c_1), (x'_2 + c_2), \ldots, (x'_n + c_n)). \end{cases} \tag{8}$$

Eq. (8) is identical to the form of Eq. (7) except for the revised offset, and thus it has the same dynamics while providing offset boosting for the variables x_i with a vector of new constants c_i.

Similarly, the offset boosting constants can be replaced by periodic functions, and the corresponding derived system is a self-reproducing system giving infinitely many attractors. Unlike the offset-boostable system, it is more complicated when the new periodic functions are introduced, and the coupling between various periodic functions can seriously influence the dynamics.

Theorem 4.1. *A dynamical system Eq. (7) can be revised to be a self-reproducing system,*

$$\begin{cases} \dot{x}'_1 = F_1(x'_1, x'_2, \ldots, x'_n), \\ \dot{x}'_2 = F_2(x'_1, x'_2, \ldots, x'_n), \\ \ldots\ldots \\ \dot{x}'_n = F_n(x'_1, x'_2, \ldots, x'_n). \end{cases} \tag{9}$$

when periodic functions are introduced into $F_i(x_1, x_2, \ldots, x_n)(i \in \{1, 2, ..., n\})$ *as:* $x_k = g_k(x_k')$, $(k \in \{j_1, j_2, \ldots, j_m\}, 1 \le j_1 < j_2 < \ldots < j_m \le n)$ *and* $x_k = x_k'$, $(k \in \{1, 2, \ldots, n\} \backslash \{j_1, j_2, \ldots, j_m\})$.

Proof 4.1. Since $g_l(x_l')$ is a periodic function, then there exists a constant P_l and an integer S_l subject to $g_l(x_l' + S_l P_l) = g_l(x_l')$, $l \in \{j_1, j_2, \ldots, j_m\}$. With a variable substitution $x_l'' = x_l' + S_l P_l$, $x_w'' = x_w'$, $w \in \{1, 2, .., n\} \backslash \{j_1, j_2, \ldots, j_m\}$, the following system is obtained:

$$\begin{cases} \dot{x}''_1 = F_1(x''_1, x''_2, \ldots, x''_n), \\ \dot{x}''_2 = F_2(x''_1, x''_2, \ldots, x''_n), \\ \ldots\ldots \\ \dot{x}''_n = F_n(x''_1, x''_2, \ldots, x''_n). \end{cases} \tag{10}$$

Eq. (10) is the same as Eq. (9), which indicates that system (10) is a self-reproducing system having the same solution except for a shift in phase space according to the period of the new introduced function.

In the following, we will consider the hyperchaotic systems [32–34] and apply the method above for constructing an infinite lattice of hyperchaotic attractors. For easy demonstration, we rewrite the equation as follows in reference [32],

$$\begin{cases} \dot{x} = y - x, \\ \dot{y} = -xz + u, \\ \dot{z} = xy - a, \\ \dot{u} = -by. \end{cases} \tag{11}$$

when $a = 2.6$, $b = 0.44$, system (11) has a hyperchaotic solution with Lyapunov exponents (0.0704, 0.0128, 0, −1.0832) and a Kaplan–Yorke dimension $D_{KY} = 3.0768$. According to the definition in [28], system (11) is a one-dimensional offset boostable system, where a constant introduced in the second dimension can revise the offset of the variable u. However, we cannot find any single constant in the right side of Eq. (11) to realize the offset boosting of any other variables. In other words, in order to realize offset control, additional separate constants must be introduced such as

$$\begin{cases} \dot{x} = (y + c_2) - (x + c_1), \\ \dot{y} = -(x + c_1)(z + c_3) + (u + c_4), \\ \dot{z} = (x + c_1)(y + c_2) - a, \\ \dot{u} = -b(y + c_2). \end{cases} \tag{12}$$

For the four offset controllers $c_1 = -4, c_2 = -6, c_3 = 5, c_4 = 4$, the hyperchaotic strange attractor is shifted in the positive direction of x and y and in the negative direction of z and u, as shown Fig. 6. All the variables become unipolar.

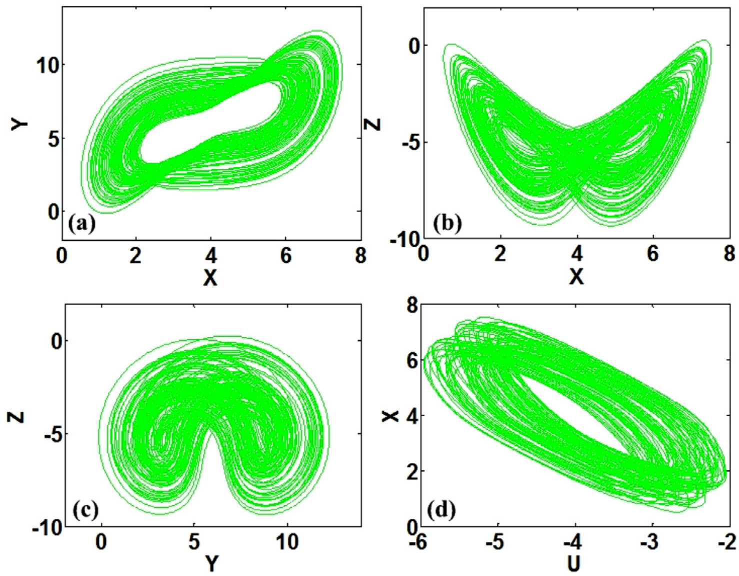

Fig. 6. Offset boosted hyperchaotic attractor for system (12) with $a=2.6$, $b=0.44$, $c_1=-4, c_2=-6, c_3=5$, $c_4=4$, and initial conditions (2, 4, 0, 0), (a) x-y plane, (b) x-z plane, (c) y-z plane and (d) u-x plane.

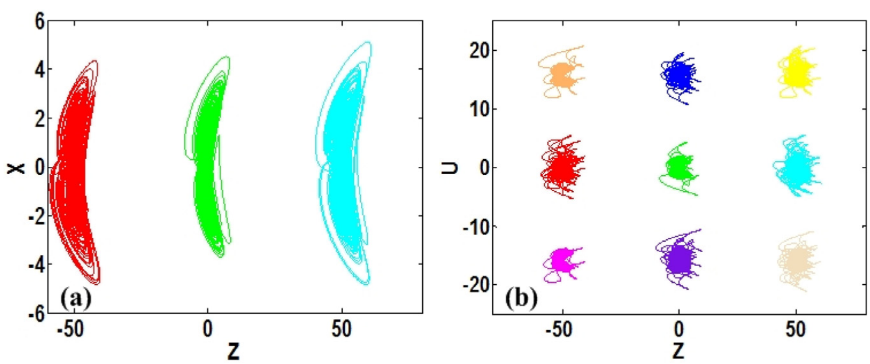

Fig. 7. 2-D lattice of hyperchaotic attractors from system (13) (a) z-x plane when the initial conditions are (2, 4, $16k\pi$, 0), (b) z-u plane when the initial conditions are (2, 4, $16k\pi$, $5k\pi$) ($k\in\{-1,\ 0,\ 1\}$).

Since system (11) is an offset boostable system in the u dimension, introducing a periodic function of the variable u can produce a 1-D lattice of hyperchaotic attractors. Additional periodic functions can still reproduce the attractor in other dimensions, but they will inevitably revise the dynamics more dramatically since the nonlinear periodic functions couple together at multiple positions,

$$\begin{cases} \dot{x} = g_2(y) - g_1(x), \\ \dot{y} = -g_1(x)g_3(z) + g_4(u), \\ \dot{z} = g_1(x)g_2(y) - a, \\ \dot{u} = -bg_2(y). \end{cases} \tag{13}$$

There are cross product terms of periodic functions like $g_1(x)g_3(z)$ and $g_1(x)g_2(y)$, which increases the difficulty of finding periodic functions to substitute xz and xy and retain the basic dynamics of the original system. Trigonometric functions are the common periodic functions for constructing a self-reproducing system [29-30]. As predicted, when $a=2.6$, $b=0.44$, $g_1(x)=x$, $g_2(y)=y$, $g_3(z)=z$, $g_4(u)=2.5\sin(0.4\,u)$, system (13) gives an infinite 1-D lattice of hyperchaotic strange attractors with Lyapunov exponents (0.0995,0.0118,0,−1.1113) and a Kaplan-Yorke dimension of 3.1001. An infinite 2-D lattice of hyperchaotic strange attractors can also be obtained if an additional periodic function of the z variable is introduced. When $g_1(x)=x$, $g_2(y)=y$, $g_3(z)=8\sin(0.125z)$, $g_4(u)=2.5\sin(0.4\,u)$, system (13) gives lattice of hyperchaotic attractors with Lyapunov exponents (0.1097, 0.0081, 0, −1.1175) and a Kaplan-Yorke dimension of 3.1052 as shown in Fig. 7.

Moreover, system (13) can provide an infinite 4-D lattice of attractors if the appropriate periodic functions are selected. For the parameters $a=2.6$, $b=0.44$, let $g_1(x)=8\sin(\frac{x}{8})$, $g_2(y)=8\sin(\frac{y}{8})$, $g_3(z)=8\sin(\frac{z}{8})$, $g_4(u)=8\sin(\frac{u}{8})$, system (13) is chaotic rather than hyperchaotic with Lyapunov exponents (0.0462, 0, −0.0042, −1.0214) and a Kaplan-Yorke dimension of 3.0450, and therefore system (13) can give an infinite 4-D lattice of chaotic attractors in hyperspace as shown in Fig. 8. System (13) also has no equilibria, and therefore all the coexisting attractors are hidden [35–42]. System (13) retains its rotational symmetry about the z-plane and therefore all the attractors are symmetric.

Other trigonometric functions can also be introduced for constructing self-reproducing systems such as cosine, tangent, or cotangent functions. For example, if $g_1(x)=8\sin(\frac{x}{8})$, $g_2(y)=8\cos(\frac{y}{8})$, $g_3(z)=8\cos(\frac{z}{8})$, $g_4(u)=8\sin(\frac{u}{8})$, system (13) has a chaotic solution with Lyapunov exponents (0.0462, 0, −0.0042, −1.0214) and a Kaplan-Yorke dimension of 3.0450. Since there is a phase difference between the sinusoidal and cosine functions, when the initial condition changes, a conditional symmetry can be found [27,28] as shown in Fig. 9. When the initial conditions vary as $(-16k\pi,\ 4, 16k\pi,\ 0\ (-50\le k\le 50))$, the average of the variables x and z decreases and increases accordingly, while the av-

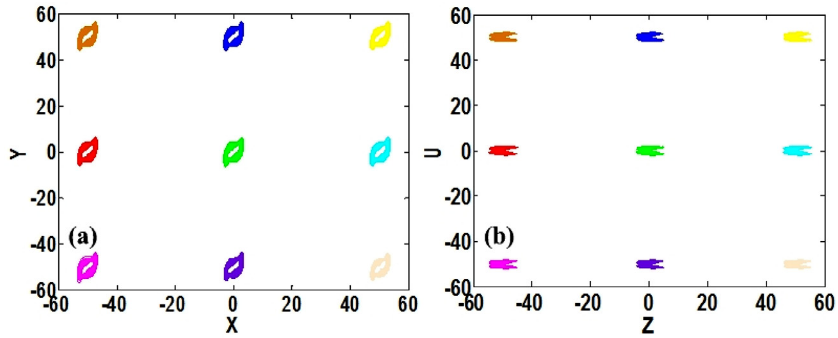

Fig. 8. 4-D Lattice of coexisting chaotic attractors from system (13) (a) *x-y* plane when the initial conditions are $(2+16k_1\pi,\ 4+16k_2\pi,\ 0,\ 0)$, (b) *z-u* plane when the initial conditions are $(2,\ 4,\ 16k_1\pi,\ 16k_2\pi)$ $(k_1, k_2 \in \{-1, 0, 1\})$.

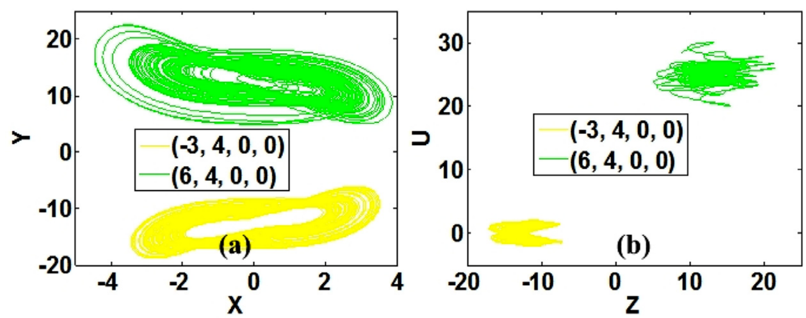

Fig. 9. Conditional symmetric coexisting attractors from system (13) when $g_2(y)$ and $g_3(z)$ are cosine functions (a) *x-y* plane, (b) *z-u* plane.

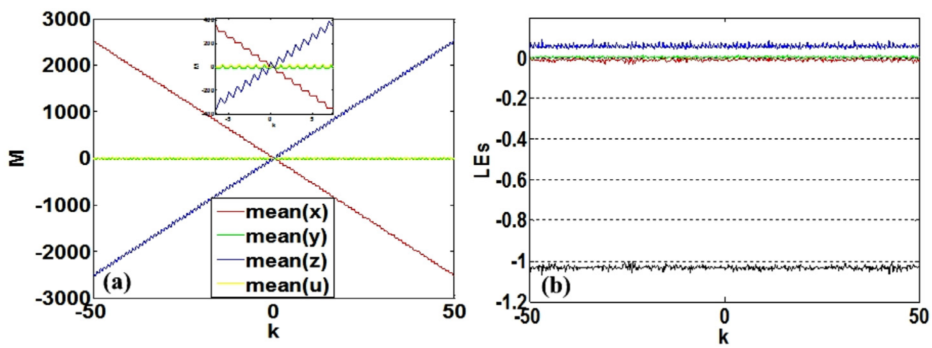

Fig. 10. Offset regulating of system (13) with invariable Lyapunov exponents (a) regulated offset when initial conditions are $(-16k\pi,\ 4,\ 16k\pi,\ 0\ (-50 \le k \le 50))$, (b) invariable Lyapunov exponents.

erage of y and u varies around zero as shown in Fig. 10, indicating that offset boosting of the variable x returns the polarity balance of the variables y, z, and u, thus producing conditional symmetry. Other combinations of sine and cosine nonlinearities can also provide a conditional symmetry when the original symmetry is broken, but a further offset boosting can restore the symmetry. Note that for a common 4-D dynamical system, it is necessary to introduce a same trigonometric function into different terms for the offset-periodization of any variable and different trigonometric functions may couple to each other, which destroys the dynamical behavior. That is why hyperchaos can be obtained from system (6) easier than from system (13). Moreover, system (11) has different regimes of multistability, correspondingly all those coexisting attractors can be reproduced by selecting appropriate trigonometric functions. For example, when $a=6$, $b=0.1$, system 11 has three coexisting attractors, *i.e.*, a torus with a symmetric pair of chaotic attractors. Take $g_1(x)=x$, $g_2(y)=y$, $g_3(z)=z$, $g_4(u)=2.5\sin(0.4u)$, system (13) gives an infinite 1-D lattice of 3-type attractors.

5. Conclusion and discussions

Periodic offset boosting is an effective means for transforming a normal dynamical system into a self-reproducing system where infinitely many attractors can be hatched. By introducing trigonometric functions into the variables for periodic offset boosting, an infinite 1-D, 2-D, 3-D lattice of hyerchaotic attractors is produced from a 4-D hyperchaotic snap system. This method can be generalized to produce higher-dimensional lattices with infinitely many attractors. By the generalized method, a hyperchaotic system with a hidden attractor was transformed for reproducing 1-D, 2-D infinite lattices of hyperchaotic attractors and a 4-D lattice of chaotic attractors. The mechanism for hatching infinitely many attractors relies on offset boosting and periodic nonlinearity, which can also be applied in rational difference equations [43,44] or discrete maps [45–47]. When the original dynamical system has multiple attractors, each lattice site can exhibit the same multiple attractors.

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References

[1] Arecchi FT, Meucci R, Puccioni G, Tredicce J. Experimental evidence of subharmonic bifurcations-multistability and turbulence in a Q-switched gas laser. Phys Rev Lett 1982;49(17):1217.
[2] Foss J, Longtin A, Mensour B, Milton J. Multistability and delayed recurrent loops. Phys Rev Lett 1996;76(4):708.
[3] Laurent M, Kellershohn N. Multistability: a major means of differentiation and evolution in biological systems. Trends Biochem Sci 1999;24(11):418–22.
[4] Joshi A, Xiao M. Optical multistability in three-level atoms inside an optical ring cavity. Phys Rev Lett 2003;91(14):143904.
[5] Ozbudak EM, Thattai M, Lim HN, Shraiman BI, Oudenaarden AV. Multistability in the lactose utilization network of escherichia coli. Nature 2004;427:737–40.
[6] Komarov A, Leblond H, Sanchez F. Multistability and hysteresis phenomena in passively mode-locked fiber lasers. Phys Rev A 2005;71(5):053809.
[7] Thomson M, Gunawardena J. Unlimited multistability in multisite phosphorylation systems. Nature 2009;460:274–7.
[8] Dafilis MP, Frascoli F, Cadusch PJ, Liley DTJ. Chaos and generalised multistability in a mesoscopic model of the electroencephalogram. Phys D 2009;238(13):1056–60.
[9] Zeng Z, Huang T, Zheng W. Multistability of recurrent neural networks with time-varying delays and the piecewise linear activation function. IEEE Trans Neural Netw 2010;21(8):1371–7.
[10] Robinson A, Calov R, Ganopolski A. Multistability and critical thresholds of the Greenland ice sheet. Nature Clim 2012;2:429–32.
[11] Ying L, Huang D, Lai YC. Multistability, chaos, and random signal generation in semiconductor superlattices. Phys Rev E 2016;93(6):062204.
[12] Kengne J, Njitacke ZT, Fotsin HB. Dynamical analysis of a simple autonomous jerk system with multiple attractors. Nonlinear Dyn 2016;83(1-2):751–65.
[13] Kengne J, Njitacke ZT, Nguomkam Negou A, Fouodji Tsostop M, Fotsin HB. Coexistence of multiple attractors and crisis route to chaos in a novel chaotic Jerk circuit. Int J Bifurcat Chaos 2016;26(5):1650081.
[14] Bao B, Li Q, Wang N, Xu Q. Multistability in Chua's circuit with two stable node-foci. Chaos 2016;26(4):043111.
[15] Xu Q, Lin Y, Bao B, Chen M. Multiple attractors in a non-ideal active voltage-controlled memristor based Chua's circuit. Chaos Soliton Fract 2016;83:186–200.
[16] Lai Q, Chen S. Research on a new 3D autonomous chaotic system with coexisting attractors. Optik 2016;127(5):3000–4.
[17] Li C, Hu W, Sprott JC, Wang X. Multistability in symmetric chaotic systems. Eur Phys J Spec Top 2015;224:1493–506.
[18] Sprott JC. Simplest chaotic flows with involutional symmetries. Int J Bifurcat Chaos 2014;24(01):1450009.
[19] Li C, Sprott JC. Multistability in the Lorenz system: a broken butterfly. Int J Bifurcat Chaos 2014;24(10):1450131.
[20] Li C, Sprott JC, Yuan Z, Li H. Constructing chaotic systems with total amplitude control. Int J Bifurcat Chaos 2015;25(10):1530025.
[21] Kapitaniak T, Leonov GA. Multistability: Uncovering hidden attractors. Eur Phys J Spec Top 2015;224:1405–8.
[22] Sprott JC. A dynamical system with a strange attractor and invariant tori. Phys Lett A 2014;378(20):1361–3.
[23] Barrio R, Blesa F, Serrano S. Qualitative analysis of the Rössler equations: bifurcations of limit cycles and chaotic attractors. Phys D 2009;238(13):1087–100.
[24] Sprott JC, Wang X, Chen G. Coexistence of point, periodic and strange attractors. Int J Bifurcat Chaos 2013;23(05):1350093.
[25] Sprott JC, Li C. Asymmetric bistability in the Rössler system. Acta Phys Pol B 2017;48:97–107.
[26] Li C, Sprott JC, Xing H. Hypogenetic chaotic Jerk flows. Phys Lett A 2016;380(11–12):1172–7.
[27] Li C, Sprott JC, Xing H. Constructing chaotic systems with conditional symmetry. Nonlinear Dyn 2017;87:1351–8.
[28] Li C, Sprott JC. Variable-boostable chaotic flows. Optik 2016;127(22):10389–98.
[29] Li C, Sprott JC, Mei Y. An infinite 2-D lattice of strange attractors. Nonlinear Dyn 2017;89:2629–39.
[30] Li C, Sprott JC, Hu W, Xu Y. Infinite multistability in a self-reproducing chaotic system. Int J Bifurcat Chaos 2017;27(10):1750160.
[31] Sprott JC. Elegant Chaos: algebraically simple chaotic flows, Singapore: World Scientific; 2010. QA614.82 S67 2010, ISBN 978-981-283-881-0.
[32] Li C, Sprott JC. Coexisting hidden attractors in a 4-D simplified Lorenz system. Int J Bifurcat Chaos 2014;24(03):1450034.
[33] Yanchuk S, Kapitaniak T. Chaos-Hyperchaos Transition in Coupled Rossler Systems. Phys Lett A 2001;290(3–4):139–44.
[34] Yanchuk S, Kapitaniak T. Symmetry-increasing bifurcation as a predictor of a chaos-hyperchaos. Phys Rev E 2001;64:056235.
[35] Leonov GA, Vagaitsev VI, Kuznetsov NV. Localization of hidden Chua's attractors. Phys Lett A 2011;375(23):2230–3.
[36] Leonov GA, Vagaitsev VI, Kuznetsov NV. Hidden attractor in smooth Chua systems. Phys. D 2012;241(18):1482–6.
[37] Leonov GA, Kuznetsov NV. Hidden attractors in dynamical systems. From hidden oscillations in Hilbert–Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits. Int J Bifurcat Chaos 2013;23(01):1330002.
[38] Leonov GA, Kuznetsov NV. Algorithms for searching for hidden oscillations in the Aizerman and Kalman problems. Doklady Math 2011;84(1):475–81.
[39] Jafari S, Sprott JC, Nazarimehr F. Recent new examples of hidden attractors. Eur Phys J Spec Top 2015;224:1469–76.
[40] Wei Z, Wang R, Liu A. A new finding of the existence of hidden hyperchaotic attractors with no equilibria. Math Comput Simul 2014;100:13–23.
[41] Dudkowski D, Jafari S, Kapitaniak T, Kuznetsov NV, Leonovc GA, Prasad A. Hidden attractors in dynamical systems. Phys Rep 2016;637:1–50.
[42] Jiang H, Liu Y, Wei Z, Zhang L. Hidden chaotic attractors in a class of two-dimensional maps. Nonlinear Dyn 2016;85(4):2719–27.
[43] Elsayed EM, Ahmed AM. Dynamics of a three-dimensional system of rational difference equations. Math Method Appl Sci 2016;39(5):1026–38.
[44] Elsayed EM, Ibrahim TF. Solutions and periodicity of a rational recursive sequences of order five. Bull Malaysian Math Sci Soc 2015;38(1):95–112.
[45] Jiang H, Li T, Zeng X, Zhang L. Bifurcation analysis of the Logistic map via two periodic impulsive forces. Chin Phys B 2014;23(1):010501.
[46] Liu W, Sun K, Zhu C. A fast image encryption algorithm based on chaotic map. Opt Lasers Eng 2016;84:26–36.
[47] Liu W, Sun K, He S. SF-SIMM high-dimensional hyperchaotic map and its performance analysis. Nonlinear Dyn 2017;89(4):2521–32.