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ORIGINAL PAPER

An infinite 2-D lattice of strange attractors

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Abstract Periodic trigonometric functions are introduced in 2-D offset-boostable chaotic flows to generate an infinite 2-D lattice of strange attractors. These 2-D offset-boostable chaotic systems are constructed based on standard jerk flows and extended to more general systems by exhaustive computer searching. Two regimes of multistability with a lattice of strange attractors are explored where the infinitely many attractors come from a 2-D offset-boostable chaotic system in cascade or in an interactive mode.

Keywords Offset boosting · Multistability · Infinitely many attractors

1 Introduction

Multistability in a dynamical system poses a threat in practical engineering applications because the behavior of the system cannot be guaranteed, since the system may unpredictably visit its various solutions depending on the initial conditions. For this reason, multistability has recently been extensively studied, including symmetric multistability [1–8], asymmetric multistability [8–10], conditional symmetric multistability [11–13], delay or hysteresis-induced multistability [14–18], driving-induced multistability [19], and extreme multistability [20–23]. On the other hand, multistability may have applications to understanding memory [15,16], and it can enhance the performance of secure communications when chaos is used to conceal information since the initial conditions can provide an additional secret key.

A multistable system can have multiple attractors, even infinitely many attractors (a special case of which is known as extreme multistability [20–23]), or hidden attractors [24–31] which cannot be found using initial conditions in the neighborhood of an equilibrium. An unbounded solution can also be regarded as an attractor. Li et al. [12] proposed a method for constructing multistable systems with conditional symmetry or with an infinite one-dimensional chain of strange attractors. In this paper, we give a general method for constructing 2-D offset-boostable chaotic flows, after which the method is extended to the production of an infinite 2-D lattice of strange attractors by the modulation of offset-boostable variables. The advantage of this class of system is that it produces infinitely many attractors in a plane, and each lattice site can contain multiple attractors.

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2 Infinite 2-D lattice of attractors from offset boosting in cascade

Definition 2.1 Define a dynamical system $\dot{X} = F(X) = (f_1(X), f_2(X), \ldots, f_N(X))(X = (x_1, x_2, \ldots, x_N))$ as a *2-D offset-boostable system* [11–13] if there exist two variables x_i, x_j where $x_i = u_i + c_i, x_j = u_j + c_j, x_m = u_m$ (here $1 \leq i, j, m \leq N \in Z^+$, $i \neq j, m \in \{1, 2, \ldots, N\} \setminus \{i, j\}$), and the corresponding system $\dot{U} = F(U)(U = (u_1, u_2, \ldots, u_N))$ recovers its governing equation except for two additional constants allowing offsets of x_i and x_j.

Theorem 2.1 *A 2-D offset-boostable system constructed from the system*

$$\begin{cases} \dot{x} = F(y), \\ \dot{y} = G(z), \\ \dot{z} = f(\dot{x}, \dot{y}, x) \end{cases} \quad (1)$$

will produce infinitely many identical attractors if system (1) has a bounded solution (an attractor) for one period and if the functions $F(y)$ and $G(z)$ are periodic.

Proof 2.1 Since the functions $F(y)$ and $G(z)$ are periodic, suppose p_1 and p_2 are their respective periods, i.e., $F(y) = F(y + p_1)$, and $G(z) = G(z + p_2)$. For $x = u, y = v + kp_1, z = w + lp_2$ $(k, l \in Z)$. Then system (1) becomes

$$\begin{cases} \dot{u} = F(v), \\ \dot{v} = G(w), \\ \dot{w} = f(\dot{u}, \dot{v}, u). \end{cases} \quad (2)$$

System (2) is identical to system (1), indicating that introducing the constants kp_1 and lp_2 does not change the dynamics of system (1) but gives corresponding offset boosting in the dimensions of y and z which consequently gives birth to infinitely many attractors on a lattice in the y–z plane. Furthermore, the attractors can merge to make one attractor that stretches to infinity, but then the individual attractor would be unbounded. The time derivatives on the right-hand side of Eqs. (1) and (2) can be easily removed, but they will be retained in what follows to emphasize the mechanism of offset boosting.

As an example, consider the chaotic memory oscillator MO4 [32] given by $\dddot{x} + 0.5\ddot{x} + \dot{x} = x(x - 1)$ which can be changed according to $\dot{x} = y + m, \dot{y} = z + n, \dot{z} + 0.5\dot{y} + \dot{x} = x(x - 1)$ to give offset boosting with an identical strange attractor displaced in y and z by distances m and n, respectively, but with the same Lyapunov exponents $(0.0938 \pm 0.0001, 0, -0.5938)$ as for $m = n = 0$. This suggests that one might replace $y + m$ with a function that is periodic in y and $z + n$ with one that is periodic in z without destroying the chaos. Indeed, it is found that the system

$$\begin{cases} \dot{x} = \sin(y), \\ \dot{y} = 1.05 \sin(z), \\ \dot{z} = -\dot{x} - 0.5\dot{y} - x + x^2 \end{cases} \quad (3)$$

gives chaos with Lyapunov exponents $(0.0890 \pm 0.0001, 0, -0.5808)$ and a Kaplan–Yorke dimension of 2.1534 in one period and with a bounded solution. Therefore, according to Theorem 2.1, system (3) will have an infinite 2-D lattice of strange attractors. Figure 1a shows nine of the coexisting attractors when the initial conditions are selected according to $(0, 0.1 - 2k\pi, 0 + 2l\pi$ $(-1 \leq k, l \in Z \leq 1))$. Figure 1b shows that the time-averaged values of y and z on the various attractors are proportional to k for $-50 \leq k = l \in Z \leq 50$ while the average of x remains unchanged as expected. Figure 2 confirms that the Lya-

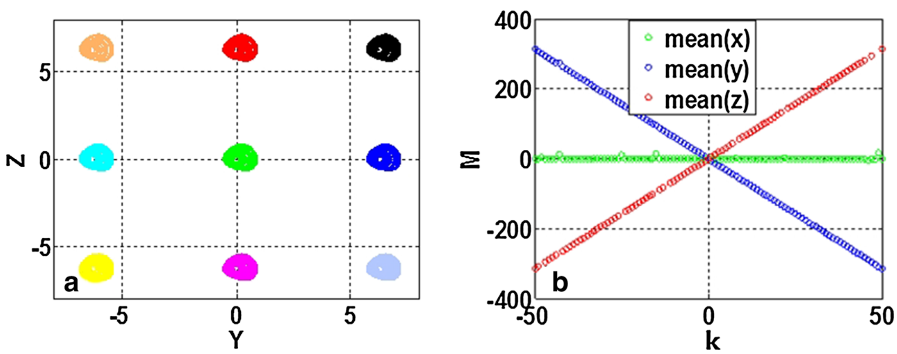

Fig. 1 Lattice of strange attractors from system (3). **a** Coexisting strange attractors when initial conditions are $(0, 0.1 + 2k\pi, 0 + 2l\pi$ $(-1 \leq k, l \in Z \leq 1))$, **b** regulated offset when initial conditions are $(0, 0.1 - 2k\pi, 0 + 2k\pi$ $(-50 \leq k \in Z \leq 50))$

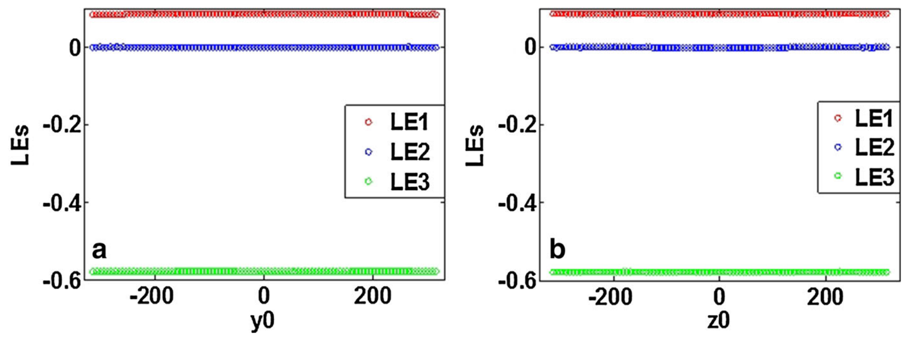

Fig. 2 Lyapunov exponents of system (3) with different initial conditions. **a** $(0, 0.1 + 2k\pi, 0$ $(-50 \le k \in N \le 50))$, **b** $(0, 0.1, 0 + 2k\pi$ $(-50 \le k \in N \le 50))$

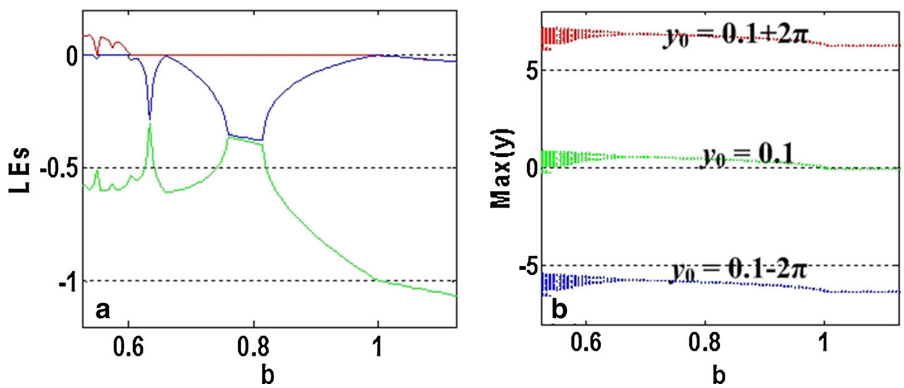

Fig. 3 Bifurcations of system (4) with initial conditions $(0, y_0, 0)$ when $a = 1.05$, b varies in [0.525, 1.125]. **a** Lyapunov exponents, **b** bifurcation diagram showing the maxima of y

punov exponents are independent of initial conditions as expected. For easy illustration, in the following we take $k = l$ when plotting the regulated offsets and corresponding Lyapunov exponents.

System (3) can be written in standard form with four sine functions as

$$\begin{cases} \dot{x} = \sin(y), \\ \dot{y} = a\sin(z), \\ \dot{z} = -\sin(y) - b\sin(z) - x + x^2 \end{cases} \tag{4}$$

where $a = 1.05$, $b = 0.525$. This system is asymmetric with a rate of volume contraction given by the Lie derivative, $\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -b\cos(z)$, indicating that the damping $b\sin(z)$ is periodic, but this is not an essential ingredient for producing infinitely many attractors as will be shown later. System (4) has two series of saddle-foci $P_1 = (0, k\pi, l\pi)$, $P_2 = (1, k\pi, l\pi)\ (k, l \in N)$. When b varies in [0.525, 1.125], system (4) experiences typical period-doubling bifurcations for different initial conditions as shown in Fig. 3. All the attractors including limit cycles and point attractors are arranged in a lattice in the 2-D phase space of y and z. At $b = 1$, the equilibrium point $P_1 = (0, k\pi, l\pi)$ $(k, l = 2N)$ changes its stability to a stable node with critical eigenvalues of $(-1, 0 \pm 1.0247i)$, showing this equilibrium undergoes a Hopf bifurcation.

We can also select other trigonometric functions such as the tangent function,

$$\begin{cases} \dot{x} = 1.1\tan(y), \\ \dot{y} = 0.9\tan(z), \\ \dot{z} = -\dot{x} - 0.5\dot{y} - x + x^2. \end{cases} \tag{5}$$

In this case, the attractors are arranged with a period of π. Comparing Fig. 4 with Fig. 1 shows that the spaces between adjacent attractors shrink, and the time-average of the variables y and z are also changed.

For a symmetric system, a symmetric pair of attractors can be obtained when the symmetry of the solution is broken [1–6]. For example, the system MO5, $\dddot{x} + 0.7\ddot{x} + \dot{x} = x(1 - x^2)$ can be changed to $\dot{x} = y + m$, $\dot{y} = z + n$, $\dot{z} + 0.7\dot{y} + \dot{x} = x(1 - x^2)$ for 2-D offset boosting with the same Lyapunov exponents $(0.1380 \pm 0.0001, 0, -0.8380)$. When two sine functions are introduced,

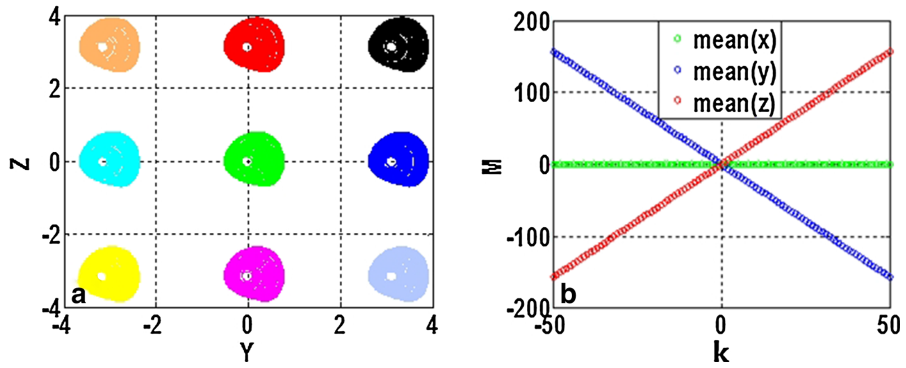

Fig. 4 Lattice of strange attractors from system (5). **a** Coexisting attractors when initial conditions are $(0, 0.1 + k\pi, 0 + l\pi$ $(-1 \le k, l \in Z \le 1))$, **b** regulated offset when initial conditions are $(0, 0.1 - k\pi, 0 + k\pi$ $(-50 \le k \in Z \le 50))$

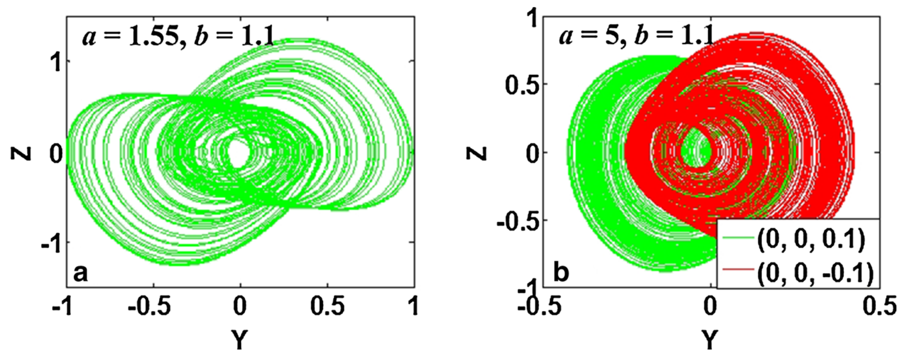

Fig. 5 The symmetric strange attractor becomes a symmetric pair in system (6). **a** $a = 1.55$, $b = 1.1$, **b** $a = 5$, $b = 1.1$

$$\begin{cases} \dot{x} = a\sin(y), \\ \dot{y} = b\sin(z), \\ \dot{z} = -\dot{x} - 0.7\dot{y} + x - x^3, \end{cases} \tag{6}$$

system (6) preserves the dynamics of the original system. For $a = 1.55$, $b = 1.1$, system (6) also gives a symmetric strange attractor with Lyapunov exponents $(0.1122 \pm 0.0001, 0, -0.8120)$ and a Kaplan–Yorke dimension of 2.1382. For $a = 5$, $b = 1.1$, the solution is a symmetric pair of strange attractors as shown in Fig. 5 with Lyapunov exponents $(0.1587 \pm 0.0001, 0, -0.8795)$ and a Kaplan–Yorke dimension of 2.1804. The Lyapunov exponents of the coexisting attractors are larger than those of the symmetric attractor, which is different from the other systems [1–6]. All the attractors can be replicated in a lattice using altered initial conditions as shown in Fig. 6. The method can be applied in other jerk systems [33–38] including those with coexisting attractors such as the models proposed by Kengne et al. [33,34] in which there are four coexisting attractors. The revised versions for giving infinitely many attractors can be $\dot{x} = 3\sin(0.2y)$, $\dot{y} = 8.7\sin(0.2z)$, $\dot{z} + \dot{y} + ar\dot{x} = ax(1 - x^2)$ and $\dot{x} = 6\sin(0.1y)$, $\dot{y} = 21\sin(0.1z)$, $\dot{z} + \dot{y} + \sigma\gamma\dot{x} = \sigma(x - \varepsilon\sinh(\rho x))$. When $a = 18$, $r = 0.725$ or $\sigma = 9.3$, $\gamma = 2$, $\rho = 4.0485$, $\varepsilon = 2.682 \times 10^{-4}$, there are four coexisting attractors at each lattice, which are symmetric pairs of limit cycles and strange attractors.

Thomas introduced the concept of labyrinth chaos [39–41] to describe 3-D cyclically symmetric periodic systems that are conservative with a trajectory that wanders chaotically over the entire space. A variant of his system that is dissipative and that satisfies the conditions of Theorem 2.1 is given by

$$\begin{cases} \dot{x} = \sin(y), \\ \dot{y} = \sin(z), \\ \dot{z} = 0.442x - \sin(z), \end{cases} \tag{7}$$

and this is perhaps the simplest such example. System (7) has an infinite 2-D lattice of thin strange attractors as shown in Fig. 7 with Lyapunov exponents $(0.0263 \pm 0.0001, 0, -0.5403)$ and a Kaplan–Yorke dimension of 2.048. There are also limit cycle solutions, but for most values of the parameters, the attractors are merged into a single attractor that extends to infinity despite having zero measure in the 3-D state space in contrast to Thomas' labyrinth which is conservative and has finite measure. The colors in Fig. 7 indicate the local largest

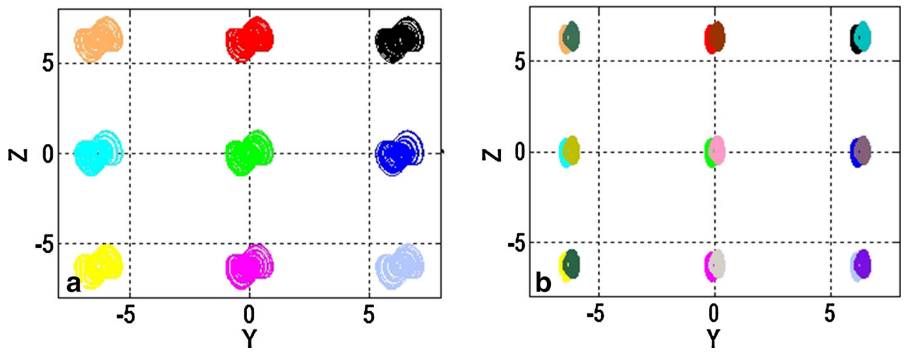

Fig. 6 Coexisting attractors of system (6) with initial conditions $(0, 0 + 2k\pi, \pm 0.1 + 2l\pi\ (-1 \le k, l \in Z \le 1)$. **a** Nine attractors when $a = 1.55, b = 1.1$, **b** eighteen attractors when $a = 5, b = 1.1$

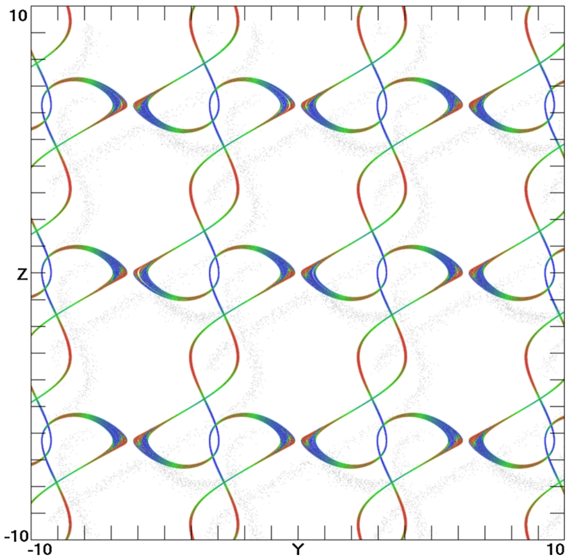

Fig. 7 Lattice of coexisting strange attractors for system (7) when the initial conditions of y and z vary in $[-10, 10]$. (Color figure online)

Lyapunov exponent with red being most positive and blue being most negative.

3 Constructing chaotic flows with 2-D offset boosting

3.1 Offset boosting in cascade

Theorem 3.1 *A jerk flow* $\dddot{x} = (x, \dot{x}, \ddot{x})$ *can be transformed to a 2-D variable-boostable system* [12,13] *by introducing two other variables* y *and* z *according to*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = f(\dot{x}, \dot{y}, x). \end{cases} \tag{8}$$

Offset boosting of the variables y *and* z *can be obtained by introducing extra constants in the first two equations.*

Proof 3.1 Let $x = u$, $y = v + m$, $z = w + n$, where u, v, w are state space variables, while m and n represent the new introduced constants,

$$\begin{cases} \dot{u} = v + m, \\ \dot{v} = w + n, \\ \dot{w} = f(\dot{u}, \dot{v}, u). \end{cases} \tag{9}$$

Since $f(\dot{u}, \dot{v}, u)$ depends only on the time derivative of u and v, which are not altered by the constants m and n, and does not depend on w, Eq. (9) in its jerk representation is identical to the jerk form of Eq. (8) and thus has the same dynamics while providing offset boosting of the variables y and z.

In jerk flows, the offset boosting is produced in cascade, where the offset boosting is based on the preceding variable. The simplest case of this type is JD0 [42], $\dddot{x} = -2.02\ddot{x} + \dot{x}^2 - x$ which can be modified to be a three-dimensional system with 2-D offset boosting of y and $z : \dot{x} = y + m, \dot{y} = z + n, \dot{z} = -2.02\dot{y} + \dot{x}^2 - x$. As another example, the simplest cubic jerk system proposed by Malasoma [32,38], $\dddot{x} = -2.03\ddot{x} + x\dot{x}^2 - x$ can be modified to be $\dot{x} = y + m, \dot{y} = z + n, \dot{z} = -2.03\dot{y} + x\dot{x}^2 - x$. Chaotic memory oscillators [32] can be transformed for offset boosting as well. For example, MO0, $\dddot{x} + 0.6\ddot{x} + \dot{x} = |x| - 1$ can be modified to be $\dot{x} = y + m, \dot{y} = z + n, \dot{z} + 0.6\dot{y} + \dot{x} = |x| - 1$. In general, when controlling the offset by introducing a constant, the initial conditions must be adjusted accordingly to remaining the basin of the attractor since most of the chaotic systems are not global attracting.

Although conservative systems do not have attractors, they can exhibit chaos. It is proved that the Nosé–Hoover oscillator [42,43], $\dot{x} = y, \dot{y} = yz - x, \dot{z} =$

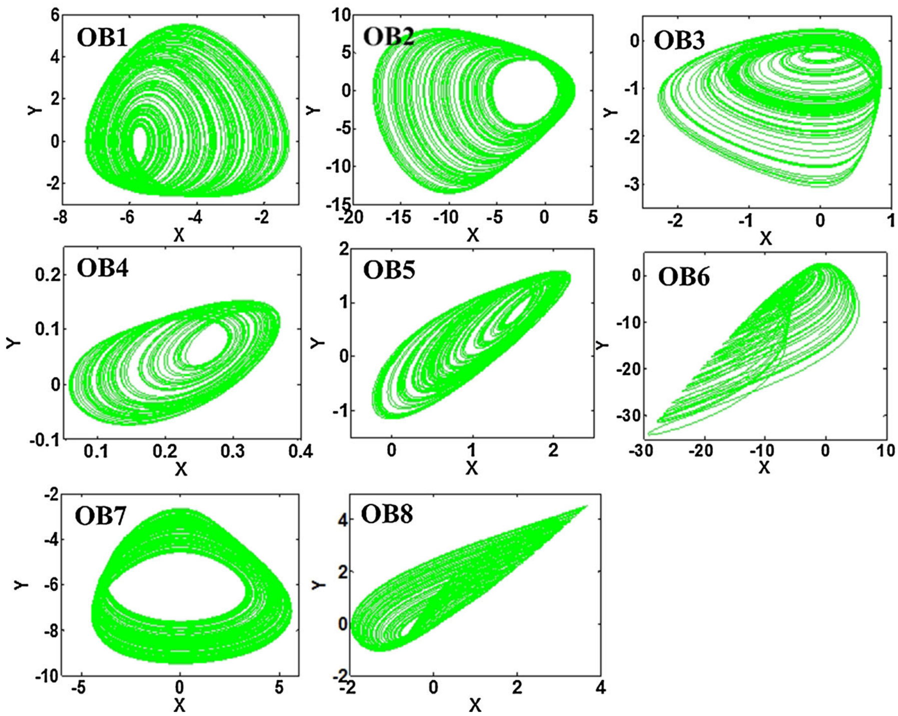

Fig. 8 2-D offset-boostable strange attractors

$1 - y^2$ can be written in jerk form [44] and thus admits 2-D offset boosting in this type of conservative system, but this will not be further discussed.

3.2 Offset boosting in interactive mode

The above offset boosting is of the cascade type, where the boosting of y is realized in the x dimension while the boosting of z is realized in the y dimension. Another regime called offset boosting in interactive mode can be explored by computer search, where the offset boosting of two variables depends on a constant in the other variable. Specifically, if the offset boosting of x is obtained in the y dimension while the offset boosting of y needs to be realized in the x dimension, the general simplified equation is

$$\begin{cases} \dot{x} = a_1 y + a_2 z + a_3 z^2, \\ \dot{y} = a_4 x + a_5 z + a_6 z^2, \\ \dot{z} = a_7 \dot{x} + a_8 \dot{y} + a_9 z + a_{10} z^2 + a_{11}. \end{cases} \tag{10}$$

After an exhaustive computer search, eight simple chaotic cases restricted to no more than seven terms were found with attractors as shown in Fig. 8. Four cases (OB3, OB4, OB8) have a parameter for total amplitude control, where there is a single quadratic term [45–48]. The basic properties including equilibria, eigenvalues, and Lyapunov exponents are shown in Table 1. All the cases are asymmetric in x and y, which makes it difficult to introduce symmetric trigonometric functions for replicating the attractors.

4 Infinite 2-D lattice of attractors from offset boosting in interactive mode

Two-dimensional offset-boostable chaotic systems in interactive mode can also produce an infinite 2-D lattice of attractors when the time derivatives on the right-hand side are preserved, which follows the same proof as Theorem 2.1. Take OB4 for example,

$$\begin{cases} \dot{x} = y - az, \\ \dot{y} = x - z, \\ \dot{z} = b\dot{x} - cz + z^2. \end{cases} \tag{11}$$

When $a = 0.26$, $b = 1.64$, $c = 0.3$, system (11) has chaotic solutions as shown in Table 1. When the first dimension is substituted into the last dimension, system (11) becomes

Table 1 2-D offset-boostable chaotic flows in interactive mode

Cases	Equations	Parameters	Initial conditions	Equilibria	Eigenvalues	Lyapunov Exponents
OB1	$\begin{cases}\dot{x}=y,\\ \dot{y}=x+az,\\ \dot{z}=-\dot{x}-b\dot{y}+z^2-c\end{cases}$	$a=4.98$	−0.86	−7.0428, 0, 1.4142	−0.6367, 0.3631 ± 2.0762i	0.0942
		$b=0.55$	−6.4			0
		$c=2$	1.46	7.0428, 0, −1.4142	0.4309, −4.5581, −1.4402	−0.8376
OB2	$\begin{cases}\dot{x}=y,\\ \dot{y}=x-az+bz^2,\\ \dot{z}=c\dot{x}+\dot{y}+z\end{cases}$	$a=1.65$	−2	0, 0, 0	−0.7940, 0.0720 ± 1.1199i	0.0626
		$b=0.1$	1.4			0
		$c=1.3$	−2.12			−0.7125
OB3	$\begin{cases}\dot{x}=y+az,\\ \dot{y}=x,\\ \dot{z}=-\dot{x}-b\dot{y}+cz-z^2\end{cases}$	$a=2$	0.3	0, 0, 0	−0.6591, 0.3296 ± 1.7105i	0.0371
		$b=1.8$	−2			0
		$c=2$	1	0, −4, 2	0.4395, −1.6064, −2.8331	−0.8769
OB4	$\begin{cases}\dot{x}=y-az,\\ \dot{y}=x-z,\\ \dot{z}=b\dot{x}-cz+z^2\end{cases}$	$a=0.26$	−0.1	0, 0, 0	0.3114, −0.5189 ± 0.8330i	0.0488
		$b=1.64$	0	0.3, 0.078,		0
		$c=0.3$	−0.25	0.3	−0.4002, 0.1369 ± 0.8549i	−0.3115
OB5	$\begin{cases}\dot{x}=y-z,\\ \dot{y}=x-az,\\ \dot{z}=b\dot{x}+cz^2-1\end{cases}$	$a=1.9$	0.35	1.9, 1, 1	−0.6265, 0.3132 ± 1.7591i	0.0876
		$b=2$	1.25	−1.9, −1, −1		0
		$c=1$	−0.74		0.4266, −2.6719, −1.7547	−0.7573
OB6	$\begin{cases}\dot{x}=y+az^2,\\ \dot{y}=x+bz,\\ \dot{z}=-c\dot{y}+z-1\end{cases}$	$a=1.64$	−0.3	−5.82, −1.64, 1	−1.5450, 0.1085 ± 0.7972i	0.0937
		$b=5.82$	−6.18			0
		$c=0.4$	0.97			−1.4216
OB7	$\begin{cases}\dot{x}=y+z+az^2,\\ \dot{y}=x,\\ \dot{z}=-b\dot{y}-z+cz^2\end{cases}$	$a=2.1$	0.05	0, 0, 0	0.8266, −0.9133 ± 0.6130i	0.0737
		$b=0.7$	−6.82	0, −15.625, 2.5		0
		$c=0.4$	2.56		−0.1387, 0.5694 ± 2.6237i	−0.3489
OB8	$\begin{cases}\dot{x}=y+az-z^2,\\ \dot{y}=x+bz,\\ \dot{z}=-\dot{y}+cz\end{cases}$	$a=1.38$	0	0, 0, 0	−1.5078, 0.0939 ± 0.8089i	0.0331
		$b=2.32$	1			0
		$c=1$	1			−1.3531

$$\begin{cases}\dot{x}=y-az,\\ \dot{y}=x-z,\\ \dot{z}=by-(ab+c)z+z^2.\end{cases}\tag{12}$$

However, system (12) is not a regular 2-D offset-boostable system since offset boosting of the variable y now introduces an additional constant in the last equation in addition to the first equation. However, Eq. (11) can be changed to give an infinite lattice of attractors,

$$\begin{cases}\dot{x}=d\sin(y)-az,\\ \dot{y}=e\sin(x)-z,\\ \dot{z}=b\dot{x}-cz+z^2.\end{cases}\tag{13}$$

When $a=0.26$, $b=1.64$, $c=0.3$, $d=5$, $e=1.2$, system (13) produces an infinite lattice of strange attractors in the x–y plane with Lyapunov exponents of $(0.0606\pm0.0001, 0, -0.3674)$ and a Kaplan–Yorke dimension of 2.1650. System (13) has four types of equilibria $P=(k\pi, l\pi, 0)\ (k, l\in N)$: a series of saddle-foci of index-1 with eigenvalues $(0.6006, -0.6635\pm1.5990i)$ when $k=2M, l=2M\ (k, l, M\in N)$, a series of saddle-foci of index-2 with eigenvalues $(-2.1441, 0.7088\pm0.5806i)$ when $k=2M, l=2M+1\ (k, l, M\in N)$, a series of spiral nodes of index-0 with eigenvalues $(-0.1274, -0.2995\pm3.7462i)$ when $k=2M+1, l=2M\ (k, l, M\in N)$

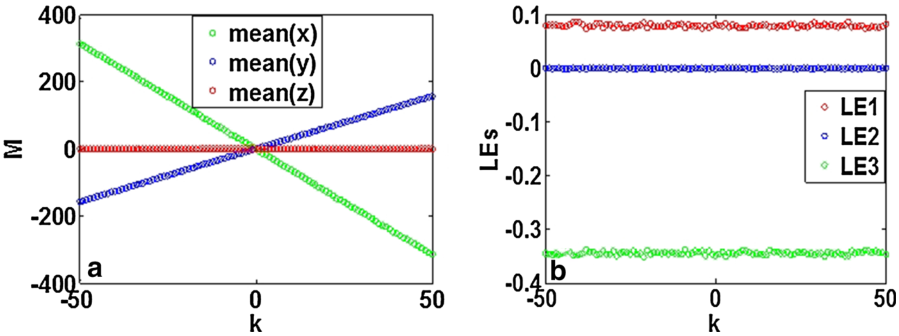

Fig. 9 Regulated offset of system (15) with initial conditions $(-0.1-2k\pi, 0+k\pi, -0.25$ $(-50 \le k \in Z \le 50)$. **a** Regulated offset, **b** invariant Lyapunov exponents

and a series of saddle nodes of index-1 with eigenvalues $(3.4901, -0.1261, -4.0904)$ when $k = 2M+1, l = 2M+1$ $(k, l, M \in N)$. System (13) is also asymmetric with a rate of volume contraction $\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(ab+c)+2z$, indicating that the damping is independent of x and y, which is different from the periodic damping in the cascade case. Apparently the damping must either be periodic in the plane of the lattice, or it must be constant in the plane, which means that the state space contraction is either periodic in the plane or perpendicular to it.

Similarly, other trigonometric functions such as the tangent function can be applied for reproducing attractors,

$$\begin{cases} \dot{x} = d\tan(y) - az, \\ \dot{y} = e\tan(x) - z, \\ \dot{z} = b\dot{x} - cz + z^2. \end{cases} \tag{14}$$

$$\begin{cases} \dot{x} = d\tan(y) - az, \\ \dot{y} = e\sin(x) - z, \\ \dot{z} = b\dot{x} - cz + z^2. \end{cases} \tag{15}$$

When $a = 0.26$, $b = 1.64$, $c = 0.3$, $d = 1.75$, $e = 1$, system (14) give strange attractors with Lyapunov exponents of $(0.0700 \pm 0.0001, 0, -0.3409)$ and a Kaplan–Yorke dimension of 2.2053. When $a = 0.26$, $b = 1.64$, $c = 0.3$, $d = 1.75$, $e = 1.1$, system (15) gives strange attractors with Lyapunov exponents of $(0.0829 \pm 0.0001, 0, -0.3488)$ and a Kaplan–Yorke dimension of 2.2377. When the initial conditions vary in $(-0.1-k\pi, 0+k\pi, -0.25$ $(-50 \le k \in Z \le 50)$, the time-average of x and y change accordingly, while system (15) has the same strange attractor with invariant Lyapunov exponents as shown in Fig. 9. In system (15) two kinds of trigonometric functions with different period are applied for producing the infinite 2-D lattice of strange attractors. The structure is different from those where only a single trigonometric function is used. In this case, the coexisting attractors are arranged in two dimensions with different intervals.

5 Circuit simulations

The above systems can be realized electronically using operational amplifiers and some special signal converters. The coexisting attractors occur in Pspice simulations [35,36]. The summator, integrator, and phase inverter can be realized with operational amplifiers, and the nonlinear function can be produced by a signal generator which is a packaged unit in Pspice. In the following we select a tangent signal generator to reproduce the coexisting attractors. From Eq. (5), the analog circuit shown in Fig. 10 is governed by the equations

$$\begin{cases} \dot{x} = \frac{1}{R_1C_1}\tan(y), \\ \dot{y} = \frac{1}{R_2C_2}\tan(z), \\ \dot{z} = -\frac{1}{R_4C_3}\tan(y) - \frac{1}{R_3C_3}\tan(z) - \frac{1}{R_5C_3}x - \frac{1}{R_6C_3}x^2, \end{cases} \tag{16}$$

The circuit has three lines to realize the integration, addition, inversion, and subtraction of the state variables x, y, and z, which in Eq. (16) correspond to the state voltages of the three operational amplifies, respectively. We select OPA404/BB as the operational amplifier, and the analog multiplier AD633/AD performs the nonlinear product operation. For the system (5), the circuit element values are $R_1 = R_4 = 362\,\text{k}\Omega$, $R_2 = 443\,\text{k}\Omega$, $R_3 = 886\,\text{k}\Omega$, $R_5 = 400\,\text{k}\Omega$, $R_6 = 40\,\text{k}\Omega$ and $R_7 = R_8 = R_9 = R_{10} = 100\,\text{k}\Omega$, $V_{dd} = 15\,\text{V}$. We select the capacitor $C_1 = C_2 = C_3 = 1\,\text{nF}$ to obtain a stable phase portrait which only affects the time scale of the oscillation. Figure 11 shows the coexisting attractors from Pspice, which are the same as shown in Fig. 4.

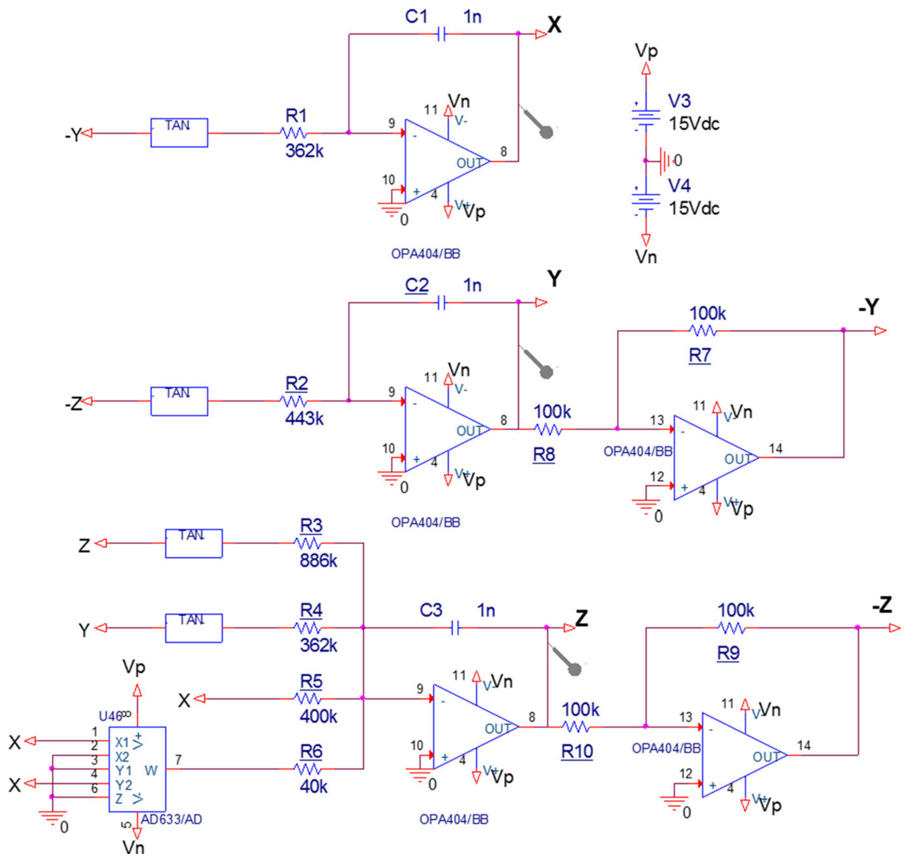

Fig. 10 Electronic circuit schematic of the system (5)

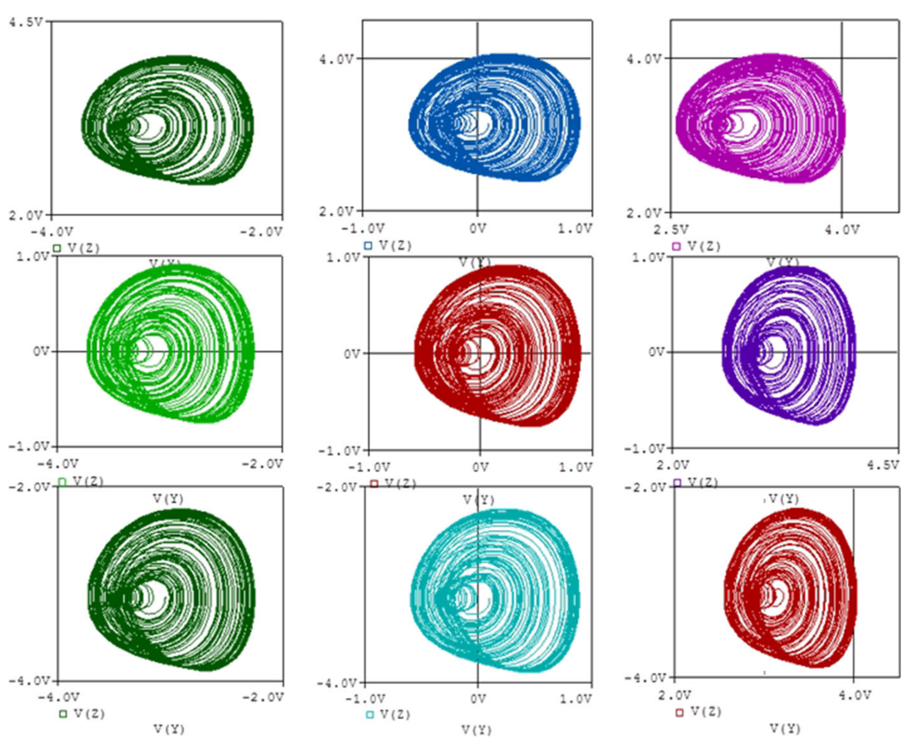

Fig. 11 Lattice of strange attractors from system (5) in Pspice when initial conditions $(0, 0.1 + k\pi, 0 + l\pi$ $(-1 \leq k, l \in Z \leq 1))$

6 Conclusions and discussion

The method for constructing infinite 2-D lattices of strange attractors has potential application in chaos-based engineering such as secure communication and weak signal detection, where initial conditions are important for determining the dynamics of the systems. In chaos-based secure communication, the unpredictability of the initial condition in the driving part of the synchronization system can additionally enhance the security of communication [49,50]. Chaos also has potential application in weak signal detection because chaotic systems are sensitive to certain signals and immune to noise at the same time, while the multistability with infinitely many attractors provides the possibility of intermittent transitions between order and chaos which is helpful for signal detection [51,52].

Here infinite 2-D lattices of strange attractors are obtained by 2-D offset boosting. Any of the attractors can be extracted from the honeycomb-like array only when a proper initial condition is given. Chaotic flows with 2-D offset boosting are classified into two regimes, where the 2-D offset boosting is realized either in cascade or in interactive mode. Two-dimensional offset-boostable chaotic flows in the cascade mode are constructed based on standard jerk flows. New regimes of 2-D offset-boostable chaotic flows in the interactive mode are explored through exhaustive computer searching.

Periodic trigonometric functions are introduced in these systems giving infinitely many attractors in the same sense as does the chaotic pendulum where they are all the same attractor when viewed in cylindrical coordinates. However, other aperiodic functions (like truncated functions) or different combinations of periodic functions can be introduced into the offset-boostable variables in different dimensions so that the lattice of attractors will be arranged in a finite way in any/or both of the dimensions or with unequal intervals. Furthermore, the method can be extended to lattices in dimensions higher than 2 and/or with higher-dimensional attractors based on systems with higher-dimensional offset boosting. Furthermore, specific aperiodic functions can be introduced to construct an infinite lattice of different attractors which will be reported later. Finally, Hamilton energy is a new index for the description of a dynamic system [53,54]. Since the coexisting attractors have the same shape but are located in different regions of space depending on initial conditions, an exhaustive analysis of the Hamilton energy will be deferred to a later publication.

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