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How to Bridge Attractors and Repellors

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Strange attractors have been extensively studied, but the same is not true for strange repellors. Some time-reversible systems have repellors that mirror their corresponding attractors and that exchange roles when time is reversed. In this paper, a conversion operator is introduced by which an easy transformation can be constructed between such a time-reversible system with an attractor/repellor pair and an irreversible one with a pair of attractors, or vice versa, thus expanding the list of such examples.

Keywords: Strange attractor; strange repellor; time-reversible system; symmetric system.

1. Introduction

Most systems of ordinary differential equations with dissipation are irreversible with an attractor in forward time and unbounded solutions in reversed time as expected from the Second Law of Thermodynamics. However, there are also examples of dissipative systems that are time-reversible with an attractor/repellor pair that exchange roles when time is reversed [Hoover, 1995; Hoover *et al.*, 1996; Sprott, 2015]. Many systems are also bistable with a symmetric pair of attractors in forward time [Sprott, 2014a, 2014b; Xu *et al.*, 2016; Bao *et al.*, 2016; Li & Sprott, 2014a; Lai & Chen, 2016a; Galias & Tucker, 2013; Zhang *et al.*, 2015]. Here we provide a general method whereby symmetric time-reversible systems with an attractor/repellor pair can be converted into a bistable irreversible system in which the repellor becomes an attractor, and vice versa. Practical applications of the idea include a time-reversal mirror (or phase conjugate array) implemented to spatially and temporally refocus an incident acoustic field back to its origin [Kuperman *et al.*, 1998] and the time-reversal process that recombines the temporal multipath resulting

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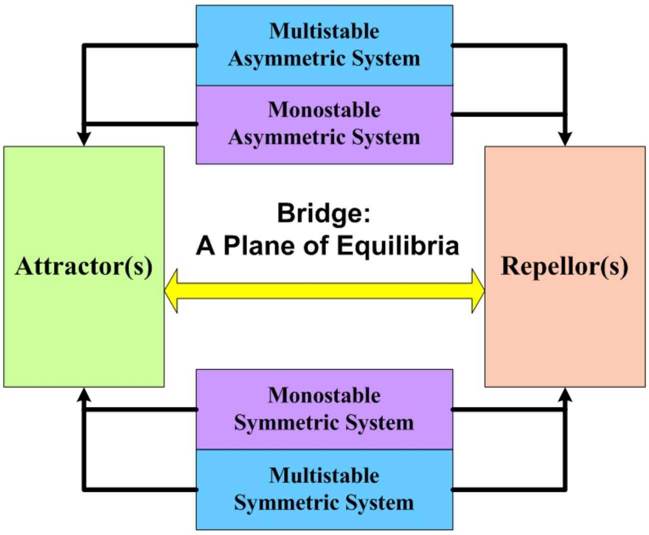


Fig. 1. Relation between attractors and repellors.

in reduced bit errors for underwater acoustic communication [Edelmann *et al.*, 2001].

We show here that there are relations between irreversible and time-reversible dissipative systems and that the introduction of a plane of equilibria can convert one type of system into the other, thereby converting an attractor into a repellor, or vice versa as shown in Fig. 1. Four types of chaotic systems are considered, including monostable and multistable asymmetric and symmetric systems.

In Sec. 2, we propose a general approach for bridging an attractor and a repellor. In Sec. 3, we show examples in which a symmetric attractor/repellor pair in a time-reversible system is converted into a symmetric pair of strange attractors and the converse in which systems with a symmetric pair of strange attractors are converted into systems with a symmetric attractor/repellor pair. In Sec. 4, we show examples in which a conditionally symmetric pair of strange attractors is converted into an attractor/repellor pair in an asymmetric system. Discussion and conclusions are given in the last section.

2. An Approach for Bridging Attractors and Repellors

An attractor in a dynamical system may correspond to a repellor in another system. Some of the attractors can be transformed into repellors when an operation is applied to the governing equations. If a dynamical system has an attractor located in a subspace, the attractor can be transformed into a repellor by a suitable plane of equilibria. For example, if a dynamical system $\dot{X} = F(X)$, $(X = (x_1, x_2, \ldots, x_N))$ has an attractor with at least one positive definite variable x_i, the derived system with a plane of equilibria $\dot{X} = F(X)p$, $(p = -\mathrm{sgn}(x_i))$ can produce a repellor, which can be proved by a variable substitution. Some (if not all) the attractors in a dynamical system can be transformed into repellors by this means. In principle, there is nothing about the method that requires the system to be symmetric. However, symmetric or asymmetric systems have different regimes of multistability, which provide different classes of coexisting attractors for this transformation. Symmetric systems may have coexisting symmetric pairs of attractors, while asymmetric systems may have conditional symmetry and coexisting conditionally symmetric attractors where conditional symmetry is defined in [Li *et al.*, 2017].

Definition 2.1. Define a dynamical system $\dot{X} = F(X) = (f_1(X), f_2(X), \ldots, f_N(X))$, $(X = (x_1, x_2, \ldots, x_N))$ as a *conditionally symmetric time-reversible system* if there exists a variable substitution: $u_1 = -x_1$, $u_2 = -x_2, \ldots, u_k = -x_k$, $u_i = x_i$, $u_j = x_j + c_j$, $s = -t$ (here $k, i \in Z^+$, $i \in \{k+1, k+2, \ldots, N\}\backslash\{j\}$) satisfying $\dot{U} = F(U)$, $(U = (u_1, u_2, \ldots, u_N))$. Here some variables are reversed along with the reversed time, and $k \le N - 1$ since there must exist at least one variable for offset boosting.

Definition 2.2. Define a dynamical system $\dot{X} = F(X) = (f_1(X), f_2(X), \ldots, f_N(X))$, $(X = (x_1, x_2, \ldots, x_N))$ as a *symmetric time-reversible system* if there exists a variable substitution: $u_1 = -x_1$, $u_2 = -x_2, \ldots, u_k = -x_k$, $u_i = x_i$, $s = -t$ (here $k, i \in Z^+$, $i \in \{k+1, k+2, \ldots, N\}$) satisfying $\dot{U} = F(U)$, $(U = (u_1, u_2, \ldots, u_N))$. Here some variables are reversed along with the reversed time, specifically, if $k = N$, all the variables will be reversed in the corresponding domain of time.

Definition 2.3. Define a dynamical system $\dot{X} = F(X) = (f_1(X), f_2(X), \ldots, f_N(X))$, $(X = (x_1, x_2, \ldots, x_N))$ as *symmetric* if there exists a variable substitution: $u_1 = -x_1$, $u_2 = -x_2, \ldots, u_k = -x_k$, $u_i = x_i$ (here $k, i \in Z^+$, $i \in \{k+1, k+2, \ldots, N\}$) satisfying $\dot{U} = F(U)$, $(U = (u_1, u_2, \ldots, u_N))$. Specifically, for a three-dimensional dynamical system,

$\dot{X} = F(X)$, $(X = (x_1, x_2, x_3))$: If $x_i = -u_i$ $(i \in \{1, 2, 3\})$ is subject to the same governing equation, the system is *reflection symmetric*; if $x_i = -u_i$, $x_j = -u_j$, $x_k = u_k$ $(i, j \in \{1, 2, 3\}, i \neq j, k \in \{1, 2, 3\} \backslash \{i, j\})$ is subject to the same governing equation, the system is *rotationally symmetric*. If $x_1 = -u_1$, $x_2 = -u_2$, $x_3 = -u_3$ is subject to the same governing equation, the system is *inversion symmetric* [Sprott, 2014a; Li & Sprott, 2016].

Theorem 2.1. *A symmetric time-reversible dynamical system can be transformed into a symmetric system by introducing a plane of equilibria with an odd function.*

Proof. Suppose there is a symmetric time-reversible dynamical system,

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n) \\ \dfrac{dx_2}{dt} = f_2(x_1, x_2, \ldots, x_n) \\ \qquad \vdots \\ \dfrac{dx_n}{dt} = f_n(x_1, x_2, \ldots, x_n). \end{cases} \tag{1}$$

According to Definition 2.1, there exists a transformation, $u_1 = -x_1$, $u_2 = -x_2, \ldots, u_m = -x_m$, $u_k = x_k$, $s = -t$ (here $m, k \in Z^+$, $k \in \{m+1, m+2, \ldots, n\}$) subject to the equation

$$\begin{cases} \dfrac{du_1}{ds} = f_1(u_1, u_2, \ldots, u_n) \\ \dfrac{du_2}{ds} = f_2(u_1, u_2, \ldots, u_n) \\ \qquad \vdots \\ \dfrac{du_n}{ds} = f_n(u_1, u_2, \ldots, u_n). \end{cases} \tag{2}$$

Thus $f_j(x_1, x_2, \ldots, x_n) = f_j(u_1, u_2, \ldots, u_n)$, $(j = 1, 2, \ldots, m)$; $f_k(x_1, x_2, \ldots, x_n) = -f_k(u_1, u_2, \ldots, u_n)$, $(k \in \{m+1, m+2, \ldots, n\})$. Suppose there exists an odd function $g(x_1, x_2, \ldots, x_m)$ that satisfies the condition $g(-x_1, -x_2, \ldots, -x_m) = -g(x_1, x_2, \ldots, x_m)$, introducing an odd function as in Eq. (3) can preserve the basic dynamics of system (1) since it only adds a plane of equilibria to the original system,

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n)g(x_1, x_2, \ldots, x_m) \\ \dfrac{dx_2}{dt} = f_2(x_1, x_2, \ldots, x_n)g(x_1, x_2, \ldots, x_m) \\ \qquad \vdots \\ \dfrac{dx_n}{dt} = f_n(x_1, x_2, \ldots, x_n)g(x_1, x_2, \ldots, x_m). \end{cases} \tag{3}$$

Make a substitution of variables for Eq. (3): $u_1 = -x_1, u_2 = -x_2, \ldots, u_m = -x_m$, $u_k = x_k$ (here m, $k \in Z^+$, $k \in \{m+1, m+2, \ldots, n\}$). Then $\frac{du_j}{dt} = \frac{d(-x_j)}{dt} = -\frac{d(x_j)}{dt} = -f_j(x_1, x_2, \ldots, x_n)g(x_1, x_2, \ldots, x_m) = f_j(x_1, x_2, \ldots, x_n)g(-x_1, -x_2, \ldots, -x_m) = f_j(u_1, u_2, \ldots, u_n)g(u_1, u_2, \ldots, u_m)(j = 1, 2, \ldots, m)$; while $\frac{du_k}{dt} = \frac{dx_k}{dt} = f_k(x_1, x_2, \ldots, x_n)g(x_1, x_2, \ldots, x_m) = -f_k(x_1, x_2, \ldots, x_n)g(-x_1, -x_2, \ldots, -x_m) = f_k(u_1, u_2, \ldots, u_n)g(u_1, u_2, \ldots, u_m)$, $(m, k \in Z^+, k \in \{m+1, m+2, \ldots, n\})$. Therefore, the following equation is obtained,

$$\begin{cases} \dfrac{du_1}{dt} = f_1(u_1, u_2, \ldots, u_n)g(u_1, u_2, \ldots, u_m) \\ \dfrac{du_2}{dt} = f_2(u_1, u_2, \ldots, u_n)g(u_1, u_2, \ldots, u_m) \\ \qquad \vdots \\ \dfrac{du_n}{dt} = f_n(u_1, u_2, \ldots, u_n)g(u_1, u_2, \ldots, u_m), \end{cases} \tag{4}$$

which means that system (3) is a symmetric system. ■

Theorem 2.2. *A symmetric dynamical system can be transformed into a symmetric time-reversible system by introducing a plane of equilibria with an odd function.*

Proof. Suppose there is a symmetric system

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n) \\ \dfrac{dx_2}{dt} = f_2(x_1, x_2, \ldots, x_n) \\ \qquad \vdots \\ \dfrac{dx_n}{dt} = f_n(x_1, x_2, \ldots, x_n). \end{cases} \tag{5}$$

According to Definition 2.3, there exists a transformation: $u_1 = -x_1$, $u_2 = -x_2, \ldots, u_m = -x_m$, $u_k = x_k$ (here $m, k \in Z^+$, $k \in \{m+1, m+2, \ldots, n\}$) subject to the equation

$$\begin{cases} \dfrac{du_1}{dt} = f_1(u_1, u_2, \ldots, u_n) \\ \dfrac{du_2}{dt} = f_2(u_1, u_2, \ldots, u_n) \\ \qquad \vdots \\ \dfrac{du_n}{dt} = f_n(u_1, u_2, \ldots, u_n). \end{cases} \tag{6}$$

Suppose there exists an odd function $g(x_1, x_2, \ldots, x_m)$ that satisfies the condition $g(-x_1, -x_2, \ldots, -x_m) = -g(x_1, x_2, \ldots, x_m)$, when a plane of equilibria is introduced as in the following,

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n)g(x_1, x_2, \ldots, x_m) \\ \dfrac{dx_2}{dt} = f_2(x_1, x_2, \ldots, x_n)g(x_1, x_2, \ldots, x_m) \\ \qquad \vdots \\ \dfrac{dx_n}{dt} = f_n(x_1, x_2, \ldots, x_n)g(x_1, x_2, \ldots, x_m). \end{cases} \tag{7}$$

Following the proof of Theorem 1, Eq. (7) is a symmetric time-reversible system. ■

For the case of conditional symmetry, we get the same conclusion from the above proof. From the above analysis, we conclude that the difference between a symmetric irreversible system and a symmetric time-reversible system is only an operator of a plane of equilibria ($g(x_1, x_2, \ldots, x_m) = 0$). If a symmetric time-reversible system can retain its basic dynamics when a plane of equilibria is introduced, it becomes an irreversible system, and the attractor/repellor pair becomes a pair of coexisting attractors which can be selected by choosing appropriate initial conditions. The choice of where to put the plane is optional, but to preserve the symmetry, the most natural choice for $p = g(x, y, z)$ is one of the three orthogonal planes through the origin. For example, taking $p = \pm mx$ or $p = \pm m\,\mathrm{sgn}(x)$ and multiplying all the equations by p introduces a plane of equilibria at $x = 0$ into the original system. Generally, the plane of equilibria should not intersect the original attractor, and even when it does not, there is no guarantee that the attractor will survive the transformation. However, since there is a parameter m that can be adjusted, a suitable transformation can often be found. In this paper, we take the examples from [Sprott, 2015] and [Li *et al.*, 2017] for easy demonstration.

3. Symmetric Systems and Their Time-Reversible Versions

3.1. *Repellor becomes an attractor*

Case A. **Inversion-Invariant System**

A symmetric time-reversible system can be converted into a regular symmetric bistable system in which the attractor/repellor becomes a symmetric pair of attractors. A simple symmetric time-reversible inversion-invariant system is

$$\begin{cases} \dot{x} = 1 + yz, \\ \dot{y} = -xz, \\ \dot{z} = y^2 + ayz. \end{cases} \tag{8}$$

According to Theorem 2.1, when a plane of equilibria is introduced, here $p = z$, the inversion-invariant time-reversible system (8) becomes a bistable inversion-invariant system as

$$\begin{cases} \dot{x} = (1 + yz)p, \\ \dot{y} = -xzp, \\ \dot{z} = (y^2 + ayz)p. \end{cases} \tag{9}$$

Now the invisible repellor becomes an accessible attractor that coexists with the original attractor as shown in Fig. 2. We can also prove the following: system (8) is time-reversible with inversion-invariant symmetry [Sprott, 2015], and the revised system (9) has a plane of equilibria $z = 0$ when $p = z$. The transformation $(x, y, z, t) \to (-x, -y, -z, t)$, converts system (9) into an identical equation. Therefore, according to Definition 2.3, system (9) has inversion invariant symmetry.

Note that sometimes we can construct a plane of equilibria based on the existing terms. For example, in Eq. (8), three of five terms include the variable z, so we can multiply the constant term of 1 and the quadratic term of y^2 by $\mathrm{sgn}(z)$ to make the system symmetric about the $z = 0$ plane, and such a modification often preserves the chaos, perhaps requiring a readjustment of the parameters.

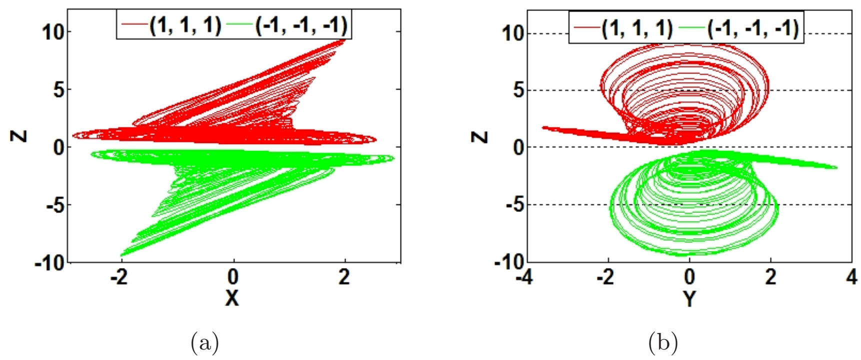


Fig. 2. Coexisting attractors of system (9) with $a = 2$ when the repellor becomes an attractor.

Case B. Rotationally-Invariant System

The simplest time-reversible rotationlly-invariant system is the Sprott D system [Sprott, 1994]. The system with a minor transformation of variables is

$$\begin{cases} \dot{x} = y + z, \\ \dot{y} = -x, \\ \dot{z} = ax^2 + yz. \end{cases} \tag{10}$$

When $a = 3$, the corresponding strange attractor has Lyapunov exponents $(0.1027, 0, -1.3198)$. System (10) is invariant under the transformation $(x, y, z, t) \rightarrow (x, -y, -z, -t)$, giving a time-reversed dynamic that is symmetric with the one for forward time but rotated by 180°. According to Theorem 2.1, introducing a plane of equilibria, i.e. $p = z$, the time-reversible rotationally-invariant system (10) becomes the following bistable rotationally-invariant system

$$\begin{cases} \dot{x} = (y + z)p, \\ \dot{y} = -xp, \\ \dot{z} = (ax^2 + yz)p. \end{cases} \tag{11}$$

System (11) has a coexisting symmetric pair of attractors with rotational symmetry as shown in Fig. 3. The symmetric repellor becomes an attractor when the plane of equilibria is introduced. The new introduced plane of equilibria can also be obtained from $p = \mathrm{sgn}(z)$.

Since the special case of the Nosé–Hoover thermostated oscillator has a strange attractor/repellor pair coexisting with a set of nested invariant tori that are symmetric about the y- and the z-axes, the plane of equilibria $y = 0$ or $z = 0$ will "cut" the attractor. Therefore, the direct introduction of a plane of equilibria cannot transform the repellor into an attractor since the repellor lies in the basin of the attractor. However, the attractor can be shifted to be positive or negative in z or y, and then introducing a plane of equilibria will transform the repellor into an attractor. When the z variable is boosted positive by $\dot{y} = -x - y(z - 8\,\mathrm{sgn}(z))$, the new introduced plane of equilibria $p = \mathrm{sgn}(z)$ will give coexisting strange attractors, one of which is from the original overlapped repellor. The newly introduced term $-8\,\mathrm{sgn}(z)$ is for offset boosting, which is applied instead of constant -8 for not destroying the property of time-reversible rotational

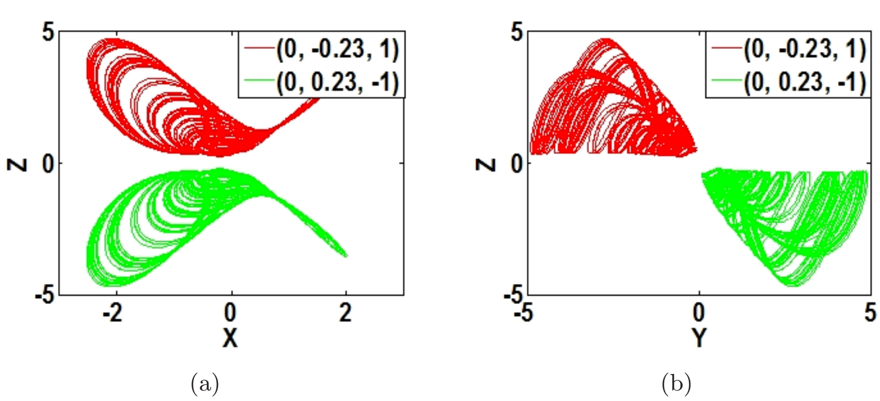
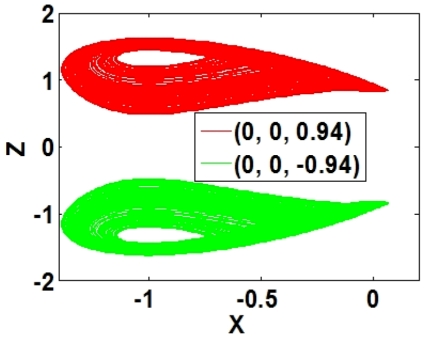
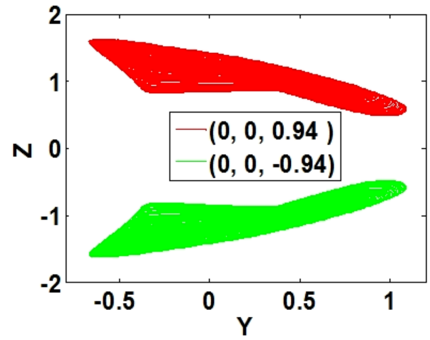


Fig. 3. Coexisting attractors of system (11) with $a = 3$ when the repellor becomes an attractor.



(a)



(b)

Fig. 4. Coexisting attractors of system (13) with $a = 2$ when the repellor becomes an attractor.

symmetry. Note that since the rate of volume contraction is revised to be $-(z - 8\,\mathrm{sgn}(z))$ whose average is not zero, the coexisting torus disappears.

***Case C.* Reflection-Invariant System**

A time-reversible symmetric system was proposed [Sprott, 2015], which is invariant under the transformation $(x, y, z, t) \to (x, y, -z, -t)$ showing a dynamic that is symmetric with the one for forward time but reflected about the $z = 0$ plane,

$$\begin{cases} \dot{x} = -yz, \\ \dot{y} = (ax + y + z^2)z, \\ \dot{z} = x - x^3. \end{cases} \tag{12}$$

When $a = 2$, system (12) has a strange attractor/repellor pair with Lyapunov exponents $(0.0892, 0, -1.2270)$. According to Theorem 2.1, introducing a plane of equilibria, i.e. $p = -\mathrm{sgn}(z)$, the time-reversible reflection-invariant system (12) becomes the bistable reflection-invariant system,

$$\begin{cases} \dot{x} = -yzp, \\ \dot{y} = (ax + y + z^2)zp, \\ \dot{z} = (x - x^3)p. \end{cases} \tag{13}$$

System (13) has a symmetric pair of coexisting attractors with reflection symmetry as shown in Fig. 4, whose basins of attraction are shown in Fig. 5. The symmetric repellor returns when the plane of equilibria is introduced. Note that the basins have a simple symmetric structure rather than fractal.

3.2. *Attractor becomes a repellor*

Similarly, a bistable symmetric system can be transformed into a symmetric time-reversible system when a proper plane of equilibria is introduced. To demonstrate this idea, we introduce a new plane of equilibria into three systems of different symmetry. Note that since a plane of equilibria will "cut" the attractor and consequently destroy the original strange attractor, the parameter should be adjusted before introducing the equilibria so that the dynamics of the system in forward time is preserved.

***Case A.* Inversion-Invariant System**

A system with inversion-invariant symmetry was studied by Coullet *et al.* [1979],

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -z - ay - x^3 + bx. \end{cases} \tag{14}$$

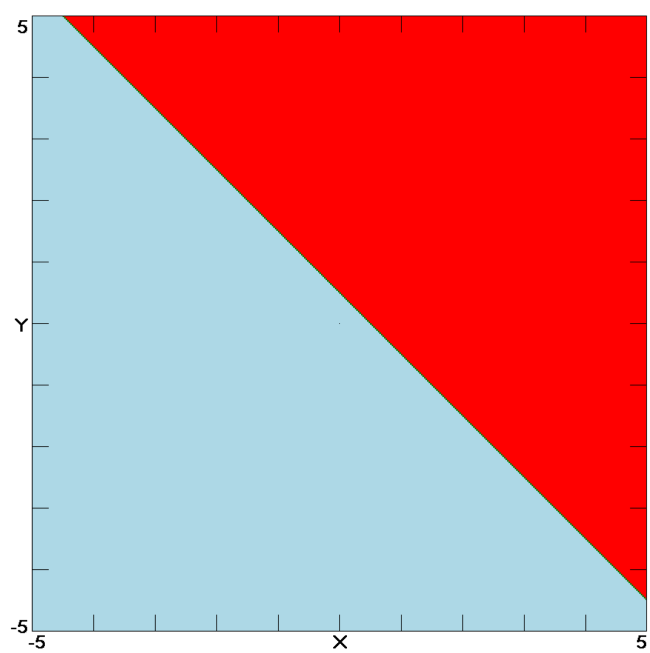


Fig. 5. Basins of attractors at $z = 0$ for system (13).

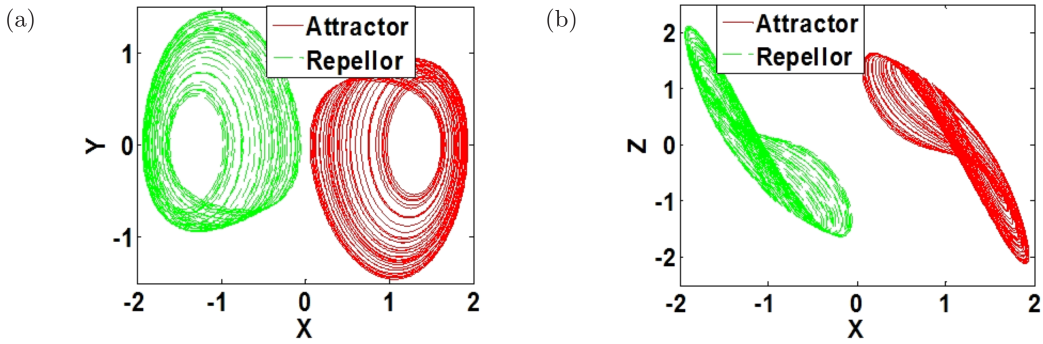


Fig. 6. The repellor returns in place of one of the attractors of system (15) with $a = 2.1$ and $b = 2$.

When $a = 2.1$, $b = 2$, system (14) has a symmetric pair of coexisting strange attractors [Sprott, 2014]. According to Theorem 2.2, introducing a plane of equilibria $p = x$ into system (14) gives a time-reversible system with inversion-invariant symmetry, as can be proved by an invariant transformation $(x, y, z, t) \to (-x, -y, -z, -t)$,

$$\begin{cases} \dot{x} = yp, \\ \dot{y} = zp, \\ \dot{z} = (-z - ay - x^3 + bx)p. \end{cases} \tag{15}$$

The corresponding attractor and repellor are shown in Fig. 6. When $p = -x$, the attractor will become a repellor, and the repellor becomes an attractor. The new introduced function can also be $p = \pm\text{sgn}(x)$.

Case B. Rotationally-Invariant System

A classic rotationally-invariant system is the Lorenz system [Lorenz, 1963; Li & Sprott, 2014a],

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = -xz + cx - y, \\ \dot{z} = xy - bz. \end{cases} \tag{16}$$

This system is invariant under the transformation $(x, y, z) \to (-x, -y, z)$, corresponding to a 180° rotation about the z-axis. When $a = 0.12$, $b = -0.6$, $c = 0$, system (16) has a symmetric pair of coexisting attractors. According to Theorem 2.2, introduction of a plane of equilibria $p = x$ in system (16) will separate the coexisting attractors into an attractor and a repellor as shown in Fig. 7, and the new system (17) is a time-reversible system with rotationally-invariant symmetry. Similarly, when $p = -x$, the attractor and the repellor will exchange. The newly introduced function can also be $p = \pm\text{sgn}(x)$,

$$\begin{cases} \dot{x} = a(y - x)p, \\ \dot{y} = (-xz + cx - y)p, \\ \dot{z} = (xy - bz)p. \end{cases} \tag{17}$$

Case C. Reflection-Invariant System

A variant of the Rössler attractor with reflection symmetry [Li *et al.*, 2015] is invariant under the transformation $(x, y, z) \to (x, y, -z)$, corresponding to symmetry about the $z = 0$ plane,

$$\begin{cases} \dot{x} = -y - z^2, \\ \dot{y} = x + ay, \\ \dot{z} = b\,\text{sgn}(z) + z(x - c). \end{cases} \tag{18}$$

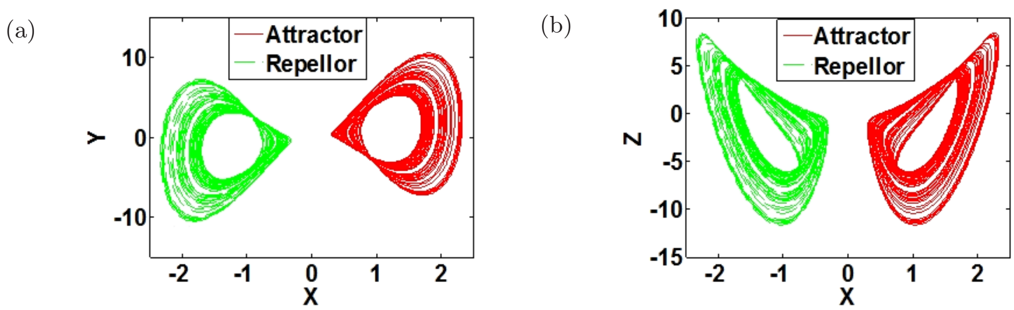
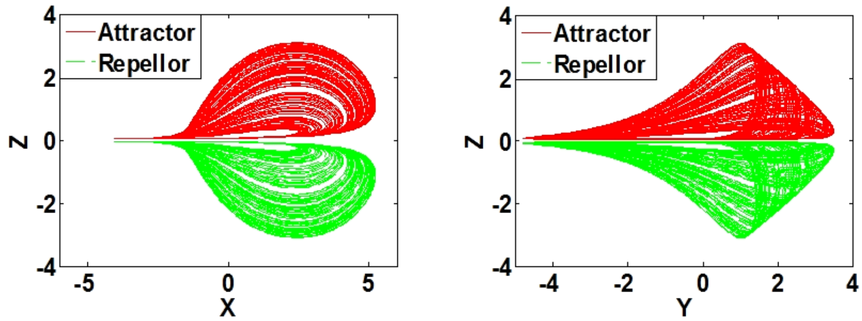


Fig. 7. The repellor returns in place of one of the attractors of system (17). Here $a = 0.12$, $b = -0.6$, $c = 0$, and the initial condition is $(0.8, -3, 0)$.



(a) (b)

Fig. 8. The repellor returns in place of one of the coexisting attractors of system (19). Here $a = b = 0.2$, $c = 2.5$, and the initial condition is $(0, 0, 1)$.

When $a = b = 0.2$, $c = 2.5$, system (18) has a symmetric pair of coexisting single-scroll strange attractors. According to Theorem 2.2, the newly introduced plane of equilibria $p = \mathrm{sgn}(z)$ will separate the coexisting attractors into an attractor and a repellor as shown in Fig. 8. If $p = z$, both the attractor and the repellor are limit cycles. Revising the plane of equilibria to $p = -\mathrm{sgn}(z)$ or $p = -z$ causes the attractor and the repellor to interchange,

$$\begin{cases} \dot{x} = (-y - z^2)p, \\ \dot{y} = (x + ay)p, \\ \dot{z} = (b\,\mathrm{sgn}(z) + z(x - c))p. \end{cases} \tag{19}$$

When a symmetric system only has a single symmetric attractor, the introduction of a plane of equilibria can convert the attractor into a repellor. The chaotic system VB15 is such an example [Li & Sprott, 2016],

$$\begin{cases} \dot{x} = (az - yz)p, \\ \dot{y} = (z^2 - by)p, \\ \dot{z} = (x - z)p. \end{cases} \tag{20}$$

When $a = 4$, $b = 0.4$, $p = 1$, system (20) has a single chaotic attractor with rotational symmetry located in the subspace with positive y. When a plane of equilibria is introduced, here $p = -\mathrm{sgn}(y)$, system (20) has an invisible repellor.

Note that this method depends on whether the repellor lies in the basin of the attractor. As mentioned above, the method works for cases where the attractor and repellor are sufficiently isolated since introducing the plane of equilibria between them precludes the possibility of a single orbit connecting the repellor to the attractor because the orbit cannot cross the plane [Jafari *et al.*, 2016a; Jafari *et al.*, 2016b]. The dynamical behavior is complicated since the attractor can be hidden if it cannot be found from the neighborhood of any equilibria [Leonov *et al.*, 2011, 2012; Pham *et al.*, 2016a; Pham *et al.*, 2016b; Wei *et al.*, 2015]. Here a newly introduced plane of equilibria can exchange the role of a repellor and an attractor and make the system multistable or monostable. We can construct additional time-reversible symmetric systems by starting with a system having a symmetric pair of coexisting attractors and adding a multiplicative term in all three equations to reverse the sign of time without destroying the chaos.

4. Repellor in Asymmetric Systems

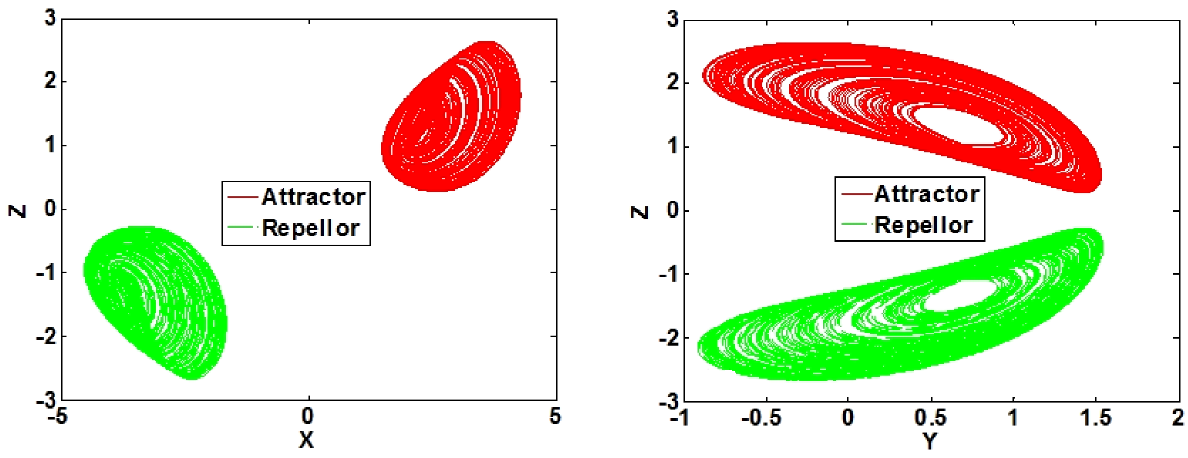
The introduction of a plane of equilibria can convert an asymmetric system into a time-reversible version, in which the attractor becomes a repellor. Take the chaotic system VB6 as an example [Li & Sprott, 2016],

$$\begin{cases} \dot{x} = 1 - yz, \\ \dot{y} = az^2 - yz, \\ \dot{z} = x. \end{cases} \tag{21}$$

When $a = 0.22$, system (21) has a strange attractor located in the subspace with positive y.

$$\begin{cases} \dot{x} = (1 - yz)p, \\ \dot{y} = (az^2 - yz)p, \\ \dot{z} = xp. \end{cases} \tag{22}$$

When a plane of equilibria is introduced, $p = -\mathrm{sgn}(y)$, the time-reversible version [Eq. (22)] has an invisible repellor. Note that the plane of equilibria introduced by $p = \mathrm{sgn}(y)$ can retain the attractor, which is unlike other cases mentioned in this



(a) (b)

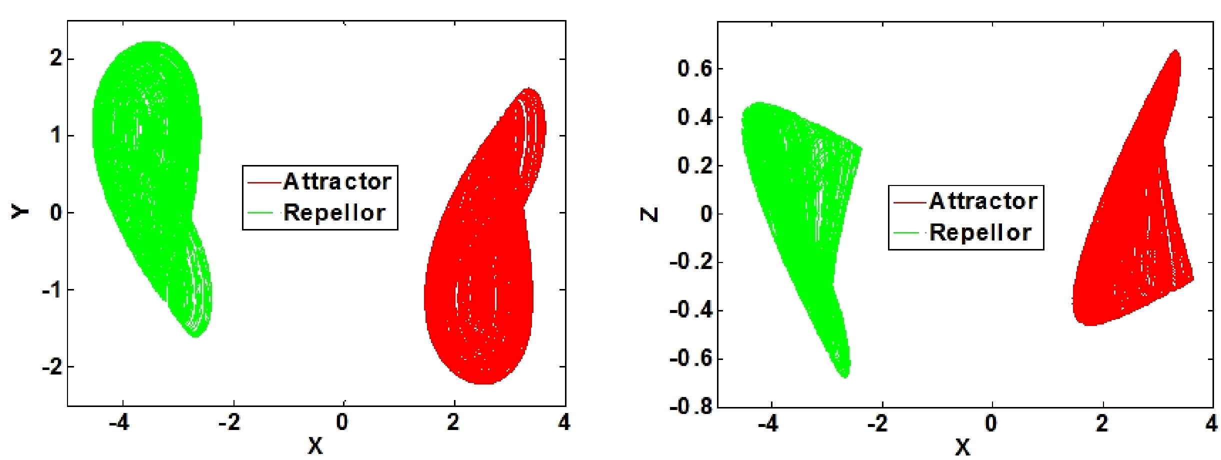
Fig. 9. The repellor returns in place of one of the coexisting attractors of system (23), initial condition $(3, -1.5, 1)$ is for the attractor.

paper with an attractor in both forward and backward time.

When an asymmetric chaotic system has coexisting attractors in the same subspace (for example, a subspace with a positive variable x), all these attractors can be converted into repellors by a plane of equilibria (correspondingly from the multiplicative term $p = -\text{sgn}(x)$). However, if an asymmetric chaotic system has coexisting attractors on both sides of the introduced plane, the introduction of a plane of equilibria will retain the attractors on one side while forcing the attractors from the other side to become repellors. Like the case with the symmetric system and the symmetric time-reversible system, we can also obtain conditional symmetric time-reversible systems. To see this clearly, take the chaotic systems with conditional symmetry (a special case of asymmetry) in [Li *et al.*, 2017] as examples. The case of conditional reflection symmetric system and its time-reversible version can be expressed as,

$$\begin{cases} \dot{x} = (y^2 - 0.4z^2)p, \\ \dot{y} = (-z^2 - 1.75y + 3)p, \\ \dot{z} = (yz + (|x| - 3))p. \end{cases} \tag{23}$$

When $p = 1$, system (23) is the original system with conditional reflection symmetry, but when a plane of equilibria is introduced by $p = \text{sgn}(z)$, the attractor above the $z = 0$ plane can also be accessible while the other coexisting attractor becomes a repellor as shown in Fig. 9. When $p = -\text{sgn}(z)$, the



(a) (b)

Fig. 10. The repellor returns in place of one of the coexisting attractors of system (24), initial condition $(3, 1, 0.5)$ is for the attractor.

attractor and the repellor will exchange. A similar process can be introduced for the case of a conditional rotationally symmetric system,

$$\begin{cases} \dot{x} = (y^2 - 1.22)p, \\ \dot{y} = 8.48zp, \\ \dot{z} = (-y - z + (|x| - 3))p. \end{cases} \quad (24)$$

Here $p = \text{sgn}(x)$ introduces a plane of equilibria for time reversal, and the attractor above the $x = 0$ plane survives while the other coexisting attractor becomes a repellor as shown in Fig. 10.

5. Conclusions and Discussion

Repellors in dissipative dynamical systems are usually invisible and intangible. However, they can often be revealed by introducing a plane of equilibria that converts them into attractors, thereby making the system multistable. Eight examples were given to show how the transformations and their inverse are executed. The method for converting a system with coexisting attractors and repellors into a system with coexisting attractors and vice versa by introducing a plane of equilibria requires that the attractor and repellor be sufficiently isolated, and it requires choosing a suitable function for constructing the plane of equilibria. Otherwise, the newly introduced function may change the dynamics of the original system. Fortunately, the signum function can be applied and works in almost all cases since it only provides a polarity change without any amplitude distortion.

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