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Simple Chaotic Flows with a Curve of Equilibria

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Using a systematic computer search, four simple chaotic flows with cubic nonlinearities were found that have the unusual feature of having a curve of equilibria. Such systems belong to a newly introduced category of chaotic systems with hidden attractors that are important and potentially problematic in engineering applications.

Keywords: Chaotic flows; curve equilibrium; hidden attractor.

1. Introduction

It is widely recognized that mathematically simple systems of nonlinear differential equations can exhibit chaos. With the advent of fast computers, it is now possible to explore the entire parameter space of these systems with the goal of finding parameters that result in some desired characteristics of the system [Sprott, 2010].

Recent research has involved categorizing periodic and chaotic attractors as either self-excited or hidden [Bragin *et al.*, 2011; Leonov & Kuznetsov, 2010, 2011, 2013a, 2013b, 2014; Leonov *et al.*, 2011, 2012; Leonov *et al.*, 2014; Leonov *et al.*, 2015a, 2015b; Sharma *et al.*, 2015a, 2015b]. A self-excited attractor has a basin of attraction that is associated with an unstable equilibrium, whereas a hidden attractor has a basin of attraction that does not intersect with small neighborhoods of any equilibrium points. The classical attractors of Lorenz, Rössler, Chua, Chen, Sprott systems (cases B to S) and other widely-known attractors are those excited from unstable equilibria. From a computational point of view this allows one to use a numerical method in which a trajectory that started from a point on the unstable manifold in the neighborhood of an unstable equilibrium, reaches an attractor and identifies it [Leonov *et al.*, 2011]. Hidden attractors cannot be found by this method and are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or an airplane wing.

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The chaotic attractors in dynamical systems without any equilibrium points, with only stable equilibria, or with a line of equilibria are hidden attractors. That is the reason such systems are rarely found, and only recently such examples have been reported in the literature [Jafari & Sprott, 2013, 2015; Jafari *et al.*, 2013; Molaie *et al.*, 2013; Jafari *et al.*, 2014; Kingni *et al.*, 2014; Lao *et al.*, 2014; Pham *et al.*, 2014a; Pham *et al.*, 2014b; Pham *et al.*, 2014c; Pham *et al.*, 2014d; Jafari *et al.*, 2015a; Jafari *et al.*, 2015b; Pham *et al.*, 2015; Shahzad *et al.*, 2015; Sprott *et al.*, 2015; Tahir *et al.*, 2015; Goudarzi *et al.*, 2016; Jafari *et al.*, 2016; Kingni *et al.*, 2016; Wang & Chen, 2013; Gotthans & Petržela, 2015; Wei, 2011; Wei *et al.*, 2014; Wei & Zhang, 2014; Wei *et al.*, 2015].

In this paper, we introduce a new category of chaotic systems with hidden attractors: systems with a curve of equilibrium. Although in such systems the basin of attraction may intersect the curve equilibrium in some sections, there are usually uncountably many points on the curve that lie well outside the basin of attraction of the chaotic attractor, and thus it is impossible to identify the chaotic attractor for sure by choosing an arbitrary initial condition in the vicinity of the unstable equilibria. In other words, from a computational point of view these attractors are hidden, and knowledge about the equilibria does not help in their localization. The goal of this paper is to describe a new category of hidden attractor and expand the list of known mathematically simple hidden chaotic attractors. Thus we perform a systematic computer search for chaos in three-dimensional autonomous systems with cubic nonlinearities which have been designed so that there will be a curve equilibrium, and we ensure that the curve equilibrium cannot be made to vanish by reduction to a system of lower dimension.

2. Simple Chaotic Flows with a Curve Equilibrium

In the search for chaotic flows with a curve equilibrium, we were inspired by the structure of the Gotthans–Petržela system which has a circle of equilibria [Gotthans & Petržela, 2015], which is the first chaotic system with a curve of equilibria

$$\begin{aligned} &\dot{x} = az, \quad \dot{y} = z(bx + cz^2), \\ &\dot{z} = x^2 + y^2 - r^2 + z(dx). \end{aligned} \tag{1}$$

We consider a general parametric form of Eq. (1) with cubic nonlinearities of the form

$$\begin{aligned} &\dot{x} = a_1 z, \quad \dot{y} = z f_1(x,y,z), \\ &\dot{z} = g(x,y,z) + z f_2(x,y,z), \\ &f_1 = (a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2 \\ &\qquad + a_8 xy + a_9 xz + a_{10} yz + a_{11}) \\ &f_2 = (a_{12} x + a_{13} y + a_{14} z + a_{15} x^2 + a_{16} y^2 \\ &\qquad + a_{17} z^2 + a_{18} xy + a_{19} xz + a_{20} yz + a_{21}). \end{aligned} \tag{2}$$

Table 1. Four simple chaotic flows with curve equilibrium.

Case	Equations	Parameters	Equilibrium	LEs	D_{KY}	(x_0, y_0, z_0)
CE_1	$\dot{x} = z$	$a = 2$	Circle	0.0653	2.0794	0
	$\dot{y} = -z(y^2 + xz)$		$x^2 + y^2 = 1$	0		0.8
	$\dot{z} = x^2 + y^2 - 1 + z(y^2 - az^2 + x)$		$z = 0$	−0.8227		0.61
CE_2	$\dot{x} = -z$		Hyperbola	0.1250	2.1658	0.75
	$\dot{y} = z(z^2 - 1)$		$x^2 - y^2 = 1$	0		−0.9
	$\dot{z} = x^2 - y^2 - 1 + z(y^2 - z^2)$		$z = 0$	−0.7538		0
CE_3	$\dot{x} = az$	$a = 0.6$	Two parallel lines	0.0345	2.0391	0
	$\dot{y} = z(by^2 + cxz)$	$b = 0.3$	$y = \pm 1$	0		0.7
	$\dot{z} = y^2 - 1 - xyz$	$c = 0.5$	$z = 0$	−0.8817		−1.3
CE_4	$\dot{x} = -az$	$a = 2$	Parabola	0.1433	2.0164	0.23
	$\dot{y} = -z^3$		$y = -x^2$	0		3.89
	$\dot{z} = x^2 + y + z(z - xy)$		$z = 0$	−8.7143		2

As can be seen, this system can have an equilibrium in $g(x, y, 0) = 0$ which can be set as any desired curve in the xy-plane.

An exhaustive computer search was done considering millions of combinations of different curves $(g(x, y) = 0)$, coefficients a_1 through a_{21} and initial conditions, seeking dissipative cases for which the largest Lyapunov exponent is greater than 0.001. For each case that was found, the space of coefficients was searched for values that are deemed "elegant" [Sprott, 2010], by which we mean that as many coefficients as possible are set to zero with the others set to ± 1 if possible or otherwise to a small integer or decimal fraction with the fewest possible digits. Cases CE_1–CE_4 in Table 1 are four simple cases found in this way.

In addition to the cases in the table, dozens of additional cases were found, but they were either equivalent to one of the cases listed by some linear transformation of variables, or they were more complicated cases with additional inessential terms.

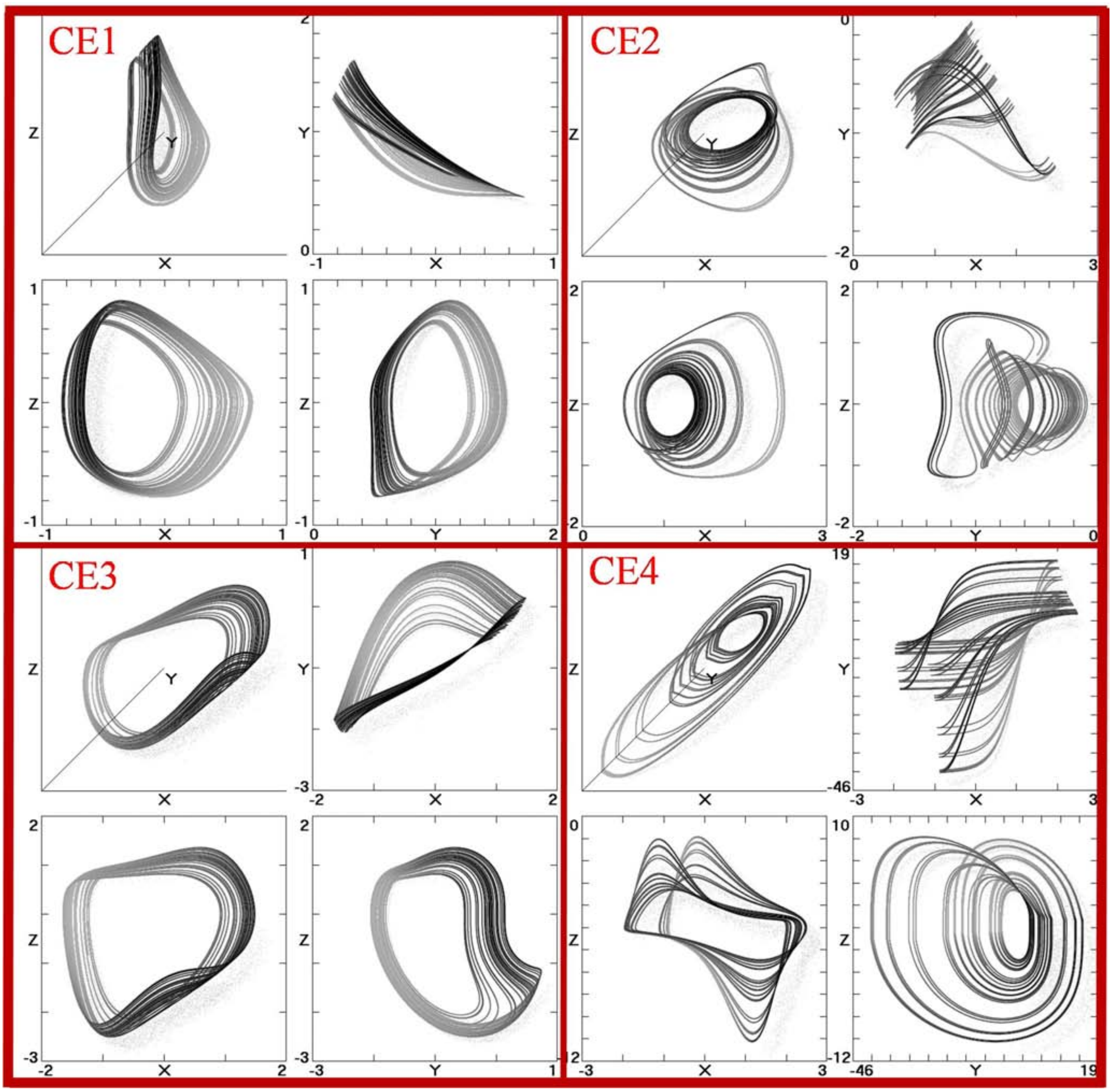

Fig. 1. State space plots of the cases in Table 1.

All these cases are dissipative with attractors projected onto various planes as shown in Fig. 1. The equilibria, Lyapunov exponents, and Kaplan–Yorke dimensions are shown in Table 1 along with initial conditions that are close to the attractor. As is usual for strange attractors from three-dimensional autonomous systems, the attractor dimension is only slightly greater than 2.0, the largest of which is CE_2 with $D_{KY} = 2.1658$, although no effort was made to tune the parameters for maximum chaos. All the cases appear to approach chaos through a succession of period-doubling limit cycles, a typical example of which is shown in Fig. 2.

Figure 3 shows a cross-section in the xy-plane at $z = 0$ of the basin of attraction for the two attractors for the typical case CE_2. Initial conditions in the light blue region are attracted to the strange attractor, and those in the red region are attracted to a point on the hyperbolic curve of equilibrium shown in yellow, only the portion of which for $y > -0.2$ is stable. Those points on the yellow curve that lie in the red region are outside the basin of the strange attractor as required for the strange attractor to be hidden.

Note that the two parallel lines equilibria in CE_3 is a limiting case of a hyperbola curve.

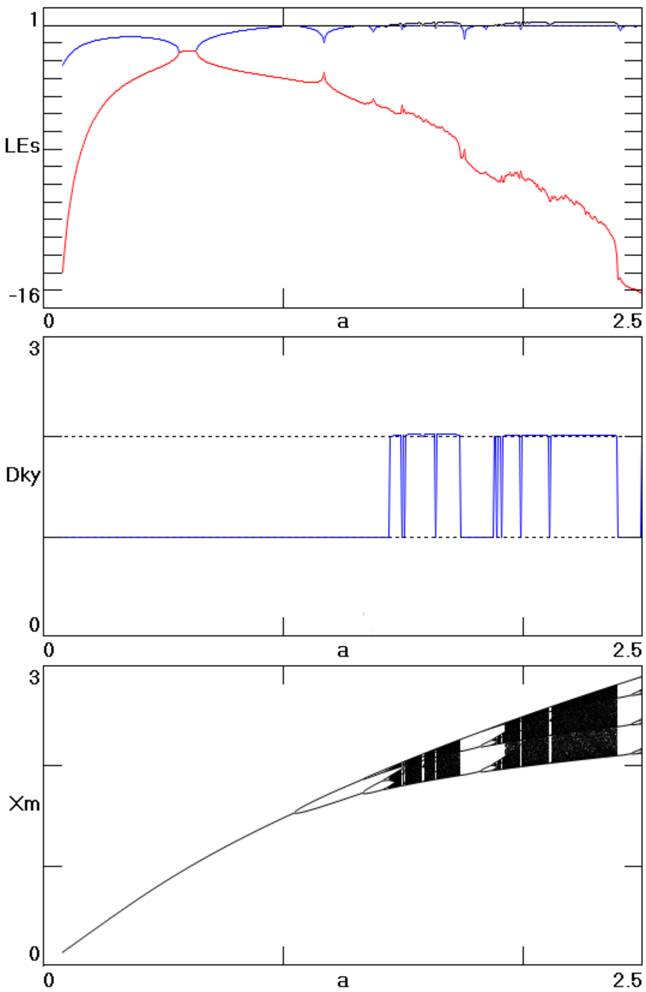

Fig. 2. Lyapunov exponents, Kaplan–Yorke dimension, and local maxima of x for case CE_4 showing a typical period-doubling route to chaos.

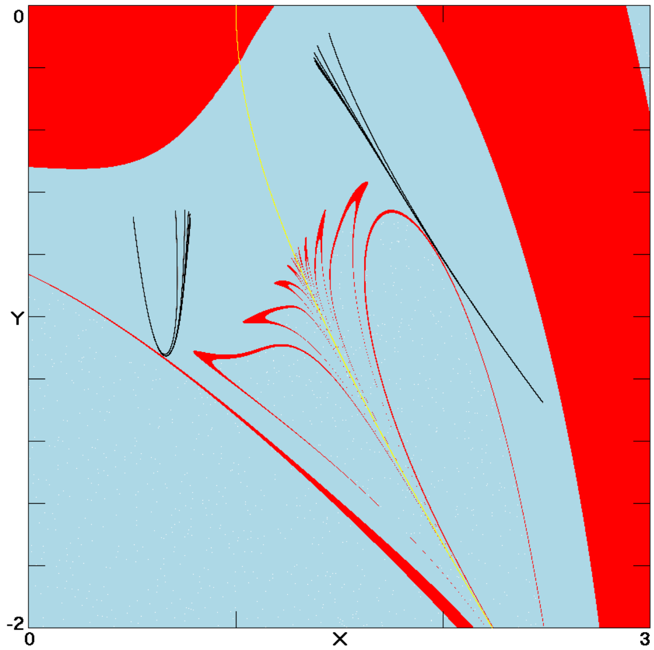

Fig. 3. Cross-section of the basins of attraction of the two attractors in the xy-plane at $z = 0$ for case CE_2. Initial conditions in the light blue region lead to the strange attractor, and those in the red region lead to the hyperbolic curve of equilibrium shown in yellow, a portion of which for $y > -0.2$ is stable.

3. Conclusion

In conclusion, it is apparent that simple chaotic systems with a curve equilibrium that seem to be rare, may in fact be rather common. These systems belong to the newly introduced class of chaotic systems with hidden attractors.

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