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Infinite Multistability in a Self-Reproducing Chaotic System

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Multistability exists in various regimes of dynamical systems and in different combinations, among which there is a special one generated by self-reproduction. In this paper, we describe a method for constructing self-reproducing systems from a unique class of variable-boostable systems whose coexisting attractors reside in the phase space along a specific coordinate axis and any of which can be selected by choosing an initial condition in its corresponding basin of attraction.

Keywords: Offset boosting; self-reproducing system; infinite multistability.

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1. Introduction

Multistability has attracted considerable interest in neurobiological and other dynamical systems. Multistability basically means a dynamical system has multiple solutions under different initial conditions, which occurs whether in symmetric systems or asymmetric systems. Symmetric systems correspond to invariance of the equations with respect to changing the sign of some of the variables. The change of any sign of the variables will break the polarity balance of an asymmetric system. Symmetric systems generally have a symmetric pair of coexisting attractors, including equilibrium points, limit cycles, and strange attractors [Bao *et al.*, 2016a; Lai & Chen, 2016; Li & Sprott, 2013, 2014a, 2014b, 2014c; Li *et al.*, 2015a; Li *et al.*, 2015b; Sprott, 2014]. Some asymmetric systems also give coexisting asymmetric attractors [Barrio *et al.*, 2009; Jafari & Sprott, 2013; Sprott *et al.*, 2013]. Even in a simple dynamical system, a point attractor can coexist with a limit cycle and strange attractor [Sprott *et al.*, 2013]. For a system with a line of equilibrium points [Jafari & Sprott, 2013; Li *et al.*, 2015a; Li *et al.*, 2015b], different initial conditions can cause the system to be attracted to different parts of the line of equilibria. Furthermore, multistability in dynamical systems can exist in such a way that an asymmetric system can have coexisting symmetric attractors and asymmetric ones for different parameter values [Li *et al.*, 2016b, 2017].

Hens *et al.* [2012] suggested a new method to achieve extreme multistability in a system of two coupled Rössler oscillators, and later Patel *et al.* [2014] showed experimental observations of that multistability in an electronic system consisting of two coupled Rössler oscillators. This kind of extreme multistability is achieved by adding extraneous variables and using their initial conditions in place of the existing parameters or as additional parameters [Sprott & Li, 2014]. In addition to the extreme multistability mentioned in [Bao *et al.*, 2017; Bao *et al.*, 2016b; Hens *et al.*, 2012; Patel *et al.*, 2014; Yuan *et al.*, 2016], Chawanya studied the asymptotic behavior of game dynamic systems with five species and an interaction matrix and found coexistence of infinitely many attractors [Chawanya, 1996, 1997]. This kind of extreme or infinite multistability occurs in a system whose dimension is greater than three. Here we show that a simple three-dimensional system can provide infinitely many attractors by reproducing themselves along a particular dimension. We suggest an approach for breeding countless coexisting attractors based on a special regime of a chaotic system. Interestingly, all these coexisting attractors in the self-reproducing system share the same Lyapunov exponents under different initial conditions, some of which may be countless symmetric attractor pairs. In Sec. 2, a new regime of variable-boostable systems is defined, after which a new class of self-reproducing system is constructed. In Sec. 3, the infinite multistability is obtained and shown in two regimes of self-reproducing systems. Conclusions and discussions are given in the last section.

2. From Offset Boosting to Self-Reproducing

2.1. *Variable-boostable systems*

Since the time derivative of a constant is zero, a differential equation can preserve its behavior if an extra constant term is added to one of the variables. For example, when x changes to $x + c$ (where c is a constant), the equation $\dot{x} = f(y, z)$ remains true without any revision. Furthermore, if the other dimensions of the differential system include only a single term proportional to x, the introduction of a constant in that dimension will produce an offset of the variable x without otherwise altering the dynamics.

Definition 2.1. Suppose there is a differential dynamical system, $\dot{X} = F(X)$, $(X = (x_1, x_2, x_3, \ldots, x_i, \ldots, x_n))$, $(i \in N)$ with $x_i = u_i + c$ subject to the same governing equation except introducing a single constant in one of the dimensional equations, i.e. $\dot{Y} = F(Y, c)$, $(Y = (x_1, x_2, x_3, \ldots, u_i, \ldots, u_n)$ $(i \in N)$, then the system is *variable-boostable* since it has the freedom for offset boosting the variable x_i by choosing an appropriate value of c.

Variable-boostable systems [Li & Sprott, 2016] have the freedom of offset boosting any of the variables. Consequently, they have the same solution except for an offset of that variable while the other variables remain the same in the same solution space. Some elementary cases conforming to this topological structure have been found in [Li & Sprott, 2016; Sprott, 1994]. A typical example is the system VB14 [Li & Sprott, 2016],

$$\begin{cases} \dot{x} = 1 - ayz \\ \dot{y} = z^2 - z \\ \dot{z} = x - bz. \end{cases} \tag{1}$$

The variable x appears only once as a linear term in the $\dot{z}$ equation, and hence it is a candidate for boosting. When $a = 3.55$, $b = 0.5$ with initial conditions $(1, 0, 1)$, the system has a chaotic attractor with Lyapunov exponents $(0.1362, 0, -0.6362)$ and Kaplan–Yorke dimension 2.2141. The variable x is a bipolar signal since it oscillates between negative and positive in $(-3, 3)$.

To show that the variable x can be offset-boosted, suppose $x \to x + c$, $y \to y$, $z \to z$ (c is a constant). The new system just has an additional constant term in the z dimension, $\dot{x} = 1 - ayz$, $\dot{y} = z^2 - z$, $\dot{z} = x + c - bz$. The constant c is an offset boosting controller for the variable x; specifically the added constant can easily change the signal x between unipolar and bipolar as desired for engineering applications since in many cases a specific physical circuit can only accept a unipolar signal or a bipolar signal. The broadband property of chaotic signal makes it difficult to make a polarity converter with broadband; however the offset controller here can realize this transformation directly. When $c = -3$, the signal x is up-boosted by 3, and consequently x becomes unipolar (always positive). Note that the initial condition in the boosting variable may need to be correspondingly adjusted to remain in the basin of attraction. The case in Eq. (1) was chosen because it has a global basin of attraction, and thus it can be arbitrarily boosted without concern for the initial conditions. Some other variable-boostable systems can be found in the Sprott chaotic family, such as systems P, J, L, M, N, and S [Li & Sprott, 2016; Sprott, 1994].

Another case with such a property is system HJ3 [Li *et al.*, 2016b],

$$\begin{cases} \dot{x} = |y| - 1 \\ \dot{y} = z \\ \dot{z} = x - by - az. \end{cases} \tag{2}$$

When $a = 0.6$, $b = 1$ and initial conditions $(2, 0, -1)$, this system has a chaotic attractor with Lyapunov exponents $(0.0363, 0, -0.6363)$ and Kaplan–Yorke dimension 2.0570. System (2) also has the freedom for offset boosting [Li *et al.*, 2016b, 2017; Li & Sprott, 2016] in the variable x. The substitution $x \to x + d$ will only introduce in the z dimension a new constant as $\dot{z} = (x + d) - by - az$, which means that the new introduced constant d will change the average level of the variable x without otherwise altering the dynamics. In those variable-boostable systems, the change of coordinates or shifting of the variables does not mean coexistence of attractors, but are associated with the constructing of self-reproducing system as will be discussed in the following.

2.2. *Self-reproducing systems*

As described above, adding a single additive constant to the right-hand side of a differential equation can produce an offset boosting of one of the variables in the system. We now consider the case in which the boosted variable (assumed to be x without loss of generality) is replaced by a function $F(x)$ that is periodic in x. It is reasonable to assume that the solution of the differential equation will be reproduced infinitely many times in the direction of that variable.

Definition 2.2. Suppose there is a M-dimensional variable-boostable system, $\dot{X} = F(X)$, $(X = (x_1, x_2, x_3, \ldots, x_i, \ldots, x_M)\ (i \in N))$. If the offset boosting $x_i = u_i + c$ makes the system recover its original governing equations, the system can be called a shift-invariant system. Since the system reproduces its own attractor by the offset boosting, it can be defined as a *self-reproducing system*. Self-reproducing systems keep their form when one of the variables is boosted, which means that the variable only varies in its initial space while not altering the structure of the system and not influencing its parameter space. Many functions can be introduced to transform a variable-boostable system into a self-reproducing one, a typical example of which is a periodic function.

Theorem 2.1. *For a three-dimensional self-reproducing system*,

$$\begin{cases} \dot{x} = f(y, z) \\ \dot{y} = g(y, z) \\ \dot{z} = h(y, z) + F(x) \end{cases} \tag{3}$$

if the function $G(x, y, z) = (f(y, z), g(y, z), h(y, z) + F(x))$ *satisfies Lipschitz condition, the functions*

f, g, h are all determined by the variables y, z, the function $F(x)$ in the $\dot{z}$ equation is periodic and system (3) converges to a bounded solution (an attractor) when the variable x is confined to one period, system (3) has infinitely many attractors at least some of which are equally spaced along the x axis with a period P.

Proof. Since $F(x)$ is periodic, suppose $F(x) = F(x + P)$, where $P \neq 0 \in \mathbf{R}$ is the period of the function $F(x)$. Make a substitution $x = x + P$ in system (3) leading to the same form of Eq. (3). Therefore, system (3) gives infinitely many identical attractors when the variable x varies by an integer multiple of P. ■

If a dissipative self-reproducing system remains chaotic after introducing the periodic function $F(x)$, it will generate infinitely many chaotic attractors. There is good reason to believe that an appropriately chosen sinusoidal $F(x)$ will preserve the chaos since the function $F(x) = \sin(\varepsilon x)/\varepsilon \to x$ for $\varepsilon \to 0$. Thus any chaotic system with a single linear occurrence of x on its right-hand side should remain chaotic when x is replaced by such an $F(x)$. By adjusting the parameters, it is usually possible to retain the chaos even for values of ε near unity. Self-reproducing systems may also lead to an undesired "symmetry" when some of the variables are phase-reversal invariant. In this case, the system remains invariant after the variables other than the boosted one are symmetrically transformed. We call this a conditional symmetry [Li *et al.*, 2017] since the system remains the same under the symmetry-like transformation except for one specific pre-boost. Consider a variable-boostable system $\dot{X} = F(X)$, $(X = (x_1, x_2, x_3, \ldots, x_i, \ldots)$ $(i \in N))$. If the offset boosting $u_i = x_i + c$ causes a symmetry-like transformation, we call this system a conditionally-symmetric system. Specifically, if a three-dimensional system remains the same after the transformation $x \to x + c$, $y \to -y$, $z \to -z$, the system is a conditionally rotational system. Similarly, the conditionally-symmetric system could have conditional reflection symmetry. However, there is no regime of conditional inversion symmetry since the variable-boostable system must retain at least one term that is not inversion symmetric for introducing a constant. Conditionally-symmetric, self-reproducing systems can survive offset boosting when they have a symmetric attractor.

3. Self-Reproducing Breeds Infinite Multistability

3.1. *Self-reproducing breeds infinitely many attractors*

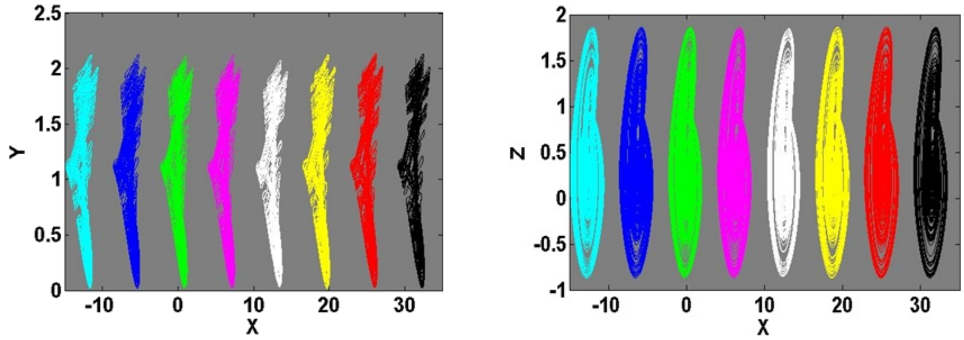
As an example, consider a self-reproducing modification of system (1) given by

$$\begin{cases} \dot{x} = 1 - ayz \\ \dot{y} = z^2 - z \\ \dot{z} = F(x) - bz. \end{cases} \tag{4}$$

Set $F(x) = A\sin(x)$, here A is a constant coefficient. Let $x = u + 2\pi k$ $(k \in N)$, $y = v$, $z = w$ ($2\pi k$ is the constant for offset boosting). Since $F(x) = A\sin(x) = F(u + 2\pi k) = A\sin(u) = F(u)$, the new equations in the variables u, v, w are identical to system (4), indicating that system (4) is a self-reproducing system along the x-axis.

When $a = 3.55$, $b = 0.5$, $A = 2$, system (4) is chaotic with infinitely many coexisting attractors depending on the initial conditions. Eight attractors are shown in Fig. 1 when the initial condition x_0 varies from -5 to 4 with $y_0 = 0$ and $z_0 = -1$, and the basins of attraction are shown in Fig. 2. The basins for $y = 1$ being tangled and the basin plot shows why that is, although the basin structure is relatively simple. Note that only a relatively small change in x_0 is required to access some of these various attractors that are more widely spread along the x-direction because of the complicated basins of attraction. Figure 3(a) indicates that all the attractors have the same Lyapunov exponents $(0.1644, 0, -0.6644)$, and the values are different from those for Eq. (1). The staircase waveform of the average x shown in Fig. 3(b) shows which of the eight attractors is accessed by each initial condition. The basins of attraction are not arranged in a simple array, but are tangled together.

To observe how many attractors are reproduced, we consider the bifurcation of the initial condition x_0 in a wider range. When the initial condition of x varies in $[-15, 35]$, the bifurcations of system (4) show that there are more than $\mathrm{ceil}(\frac{m}{n}) = \mathrm{ceil}(\frac{50}{4.25}) = 12$ attractors residing in the limited region shown in Fig. 4 (Here m is the range of the initial conditions x_0, while n is the size of the attractor in the x-dimension). The number of coexisting attractors depends on their size and the size of their projections. Note that different selections of the initial condition (x_0, y_0, z_0) lead to different



(a) (b)

Fig. 1. Coexisting attractors for system (3) with different initial conditions: cyan for $(-5,0,-1)$, pink for $(-4,0,-1)$, yellow for $(-3,0,-1)$, red for $(-2,0,-1)$, green for $(-1,0,-1)$, blue for $(1,0,-1)$, white for $(2,0,-1)$, black for $(4,0,-1)$.

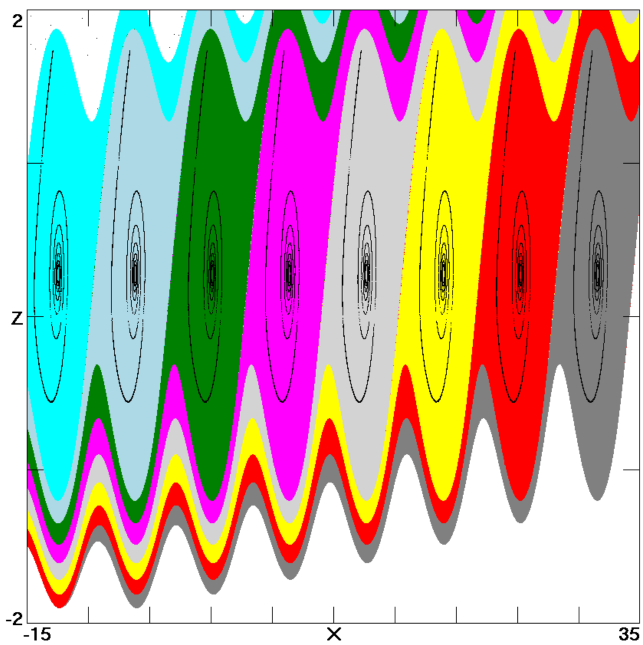
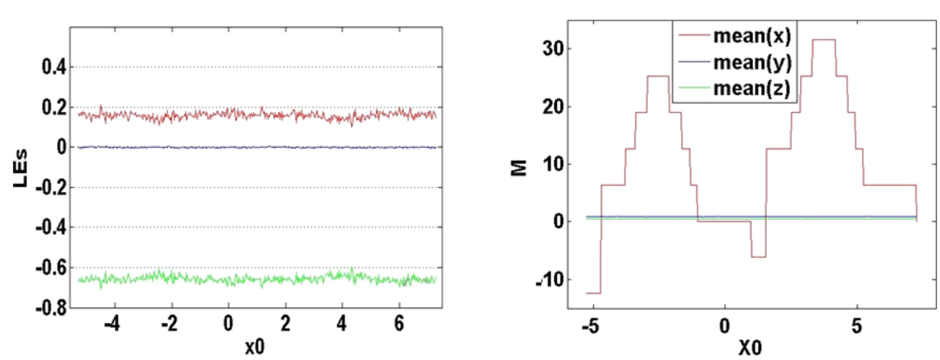


Fig. 2. Basins of attraction of the coexisting attractors in the $y = 1$ plane.

bifurcations as shown in Figs. 4(a) and 4(b), indicating the complicated structure of the basins of attraction. Separate coexisting strange attractors can be triggered by initial conditions in a discrete and nonlinear way. Since the periodic function is slowly varying, the coexisting attractors occur in a periodic way, which is consistent with the change in the average value of the variable x as shown in Fig. 3(b). For initial conditions of x beyond the range of one period, the strange attractors continue to reproduce forever along the x-axis.

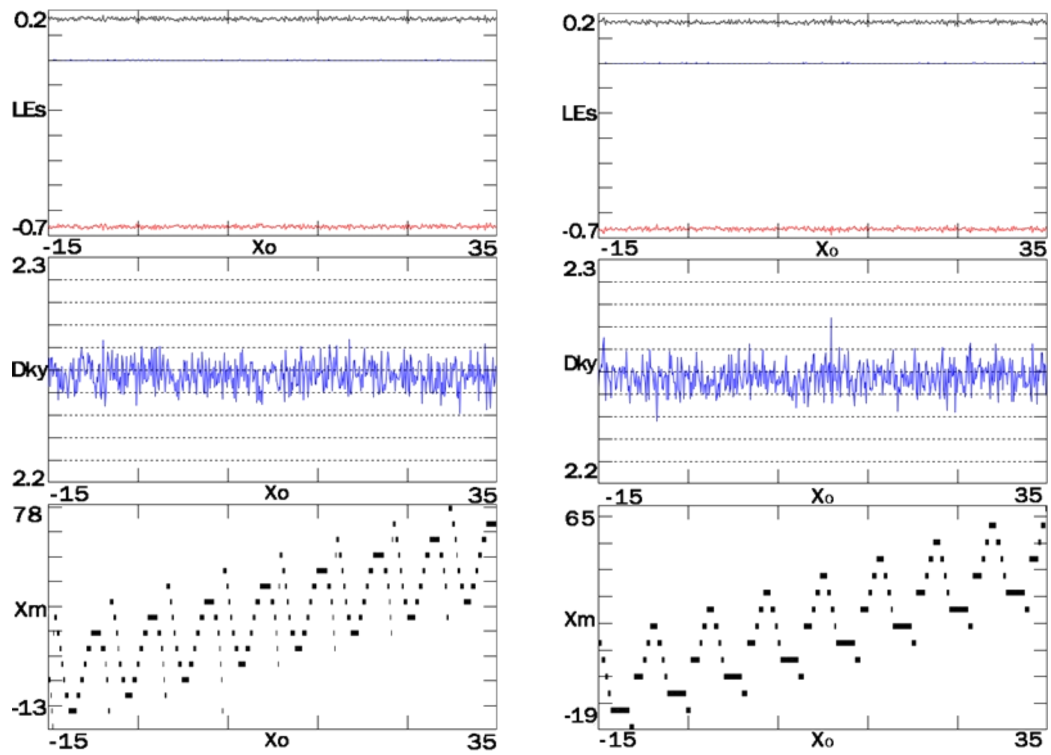
All the coexisting chaotic attractors are identical in the sense that they have the same Lyapunov exponents, but the detailed trajectories differ because of sensitive dependence on initial conditions. Moreover, different values of the parameter b give different attractors including limit cycles as shown in Fig. 5, and they are also infinitely reproduced by the selected periodic function.

Almost all 24 VB cases in [Li & Sprott, 2016] can be converted into self-reproducing systems by



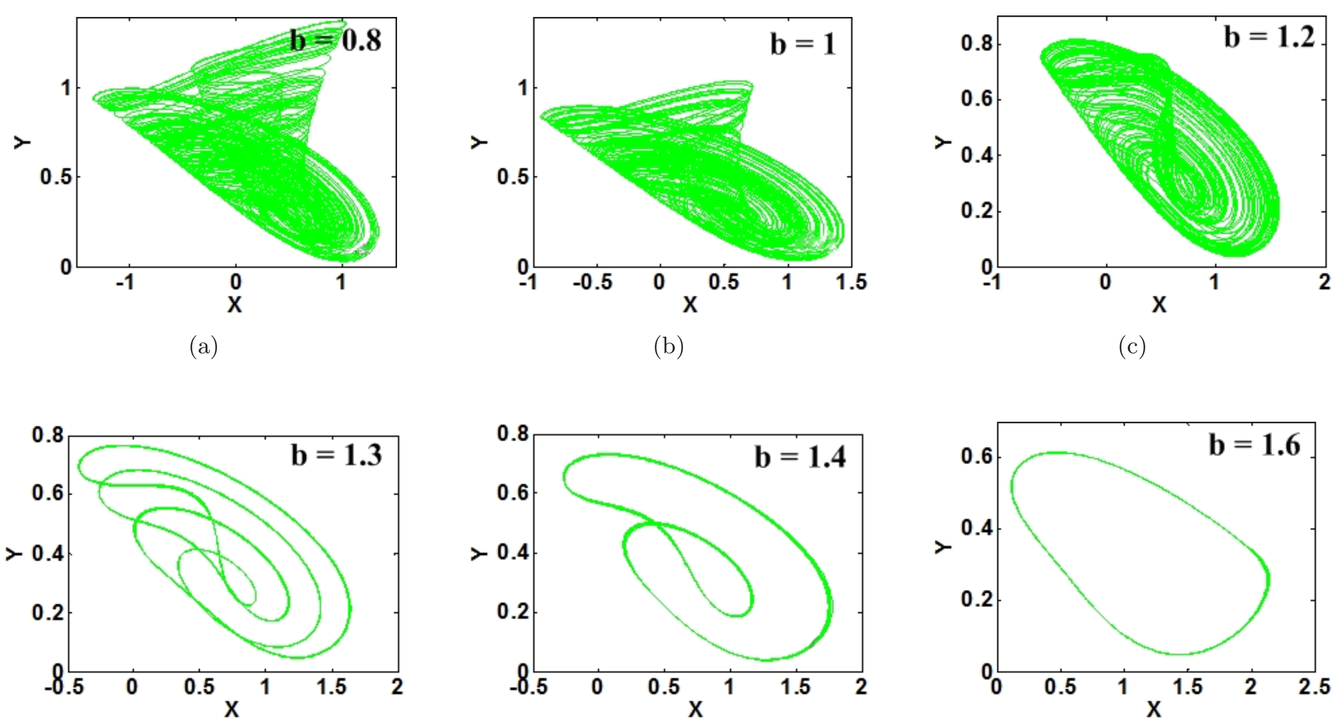
(a) (b)

Fig. 3. Lyapunov exponents and the average values of the variables in system (3) when $F(x) = 2\sin(x)$ for initial conditions $(x_0, 0, -1)$, x_0 varies from -6 to 7.



(a) (b)

Fig. 4. Dynamical behavior of system (4) with $F = 2\sin(x)$, $a = 3.55$, $b = 0.5$ when $x_0 \in [-15, 35]$: (a) initial condition $(x_0, -1, 1)$ and (b) initial condition $(x_0, 0, -1)$.



(a) (b) (c)

(d) (e) (f)

Fig. 5. Typical attractors of system (3) with $F = 2\sin(x)$, $a = 3.55$ with initial conditions $(1, 0, 1)$ for various values of b.

introducing a trigonometric function if the parameters are adjusted appropriately. One might object that in cylindrical coordinates, all these attractors are the same, but they represent a different number of rotations around the cylinder relative to the initial condition and are thus distinguishable in that sense. Furthermore, from Figs. 2 and 8, we see that basins of attractions of those coexisting attractors usually get entangled, the output attractor cannot be captured directly by a mod 2π reverberation of a single value. Furthermore, for the butterfly effect of a chaotic attractor, those coexisting attractors do not overlap exactly when the coordinate gets modulus by the period.

The period of the function $F(x)$ will change the spacing of the coexisting attractors. For $F(x) = 2\sin(fx)$, Fig. 6(a) shows how the parameter $f = P/2\pi$ compresses or expands the attractors. The $\sin(x)$ function can also be replaced by $\cos(x)$ or a combination of the two as shown in Fig. 6(b). Furthermore, for system (4) with $F(x) = 2\tan(x)$ and $a = 3.55$, $b = 0.5$, the system gives infinitely many coexisting attractors with Lyapunov exponents $(0.1855, 0, -0.6855)$. The distribution of the basins of attraction in system (3) is determined by the chosen trigonometric function. Note that since $\tan(\pm\pi/2)$ or $\tan((2k+1)\pi/2)$ $(k \in N)$ equal $\pm\infty$, the solution of the system in this neighborhood is unbounded, which results in "death" of the attractor. Therefore, we refer to the neighborhood of the initial conditions leading to unbounded solutions as the "dead sea" in initial condition space.

3.2. *Self-reproducing admits conditional symmetry*

Self-reproducing systems can also lead to a conditional symmetry giving infinitely many coexisting attractors, some of which are symmetric to one another. As mentioned above, system (2) is a unique system that can be revised to be conditionally symmetric if the odd variable x in the z dimension is replaced with a trigonometric function of x. According to Theorem 1, system (2) has the self-reproducing version,

$$\begin{cases} \dot{x} = |y| - 1 \\ \dot{y} = z \\ \dot{z} = F(x) - by - az. \end{cases} \tag{5}$$

Set $F(x) = A\cos(x)$ and $x = u + (2k+1)\pi (k \in N)$, $y = -v$, $z = -w$ ($(2k+1)\pi$ is a constant for offset boosting). Then system (5) becomes,

$$\begin{cases} \dot{u} = |v| - 1 \\ \dot{v} = w \\ \dot{w} = F(u) - bv - aw. \end{cases} \tag{6}$$

Here $F(x) = A\cos(x) = F(u + (2k+1)\pi) = A\cos(u + (2k+1)\pi) = -A\cos(u) = -F(u)$. The variables u, v, and w in the new equations are identical to system (4), which indicates that system (5) is a self-reproducing system with conditionally rotational symmetry. However, the function $\cos(x)$ can be changed to $\sin(x)$ to give a self-reproducing attractor since $\sin(x + \pi/2) = \cos(x)$, which just

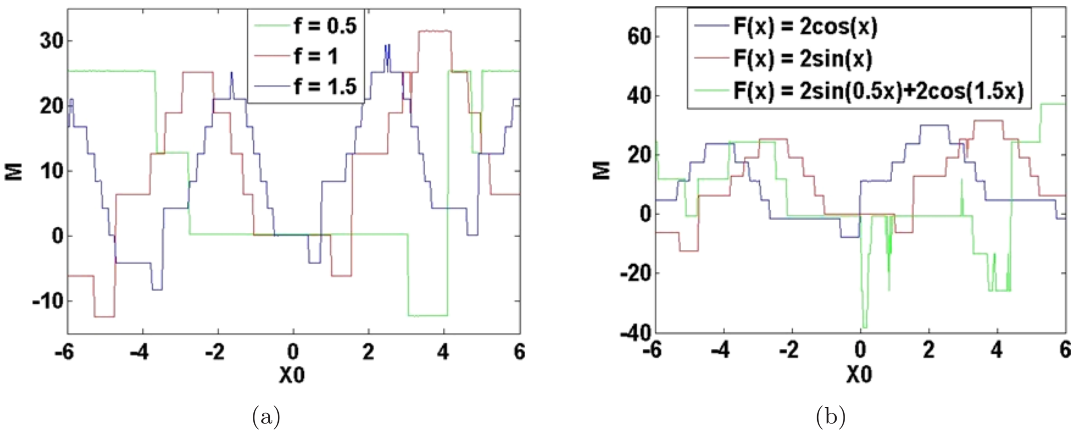


Fig. 6. Average value of x in system (3) for initial conditions $(x_0, 0, -1)$, x_0 varies from -6 to 6: (a) when $F(x) = 2\sin(fx)$, $f = 0.5$, 1, 1.5 and (b) when $F(x) = 2\sin(x)$, $2\cos(x)$ and $2\sin(0.5x) + 2\cos(1.5x)$.

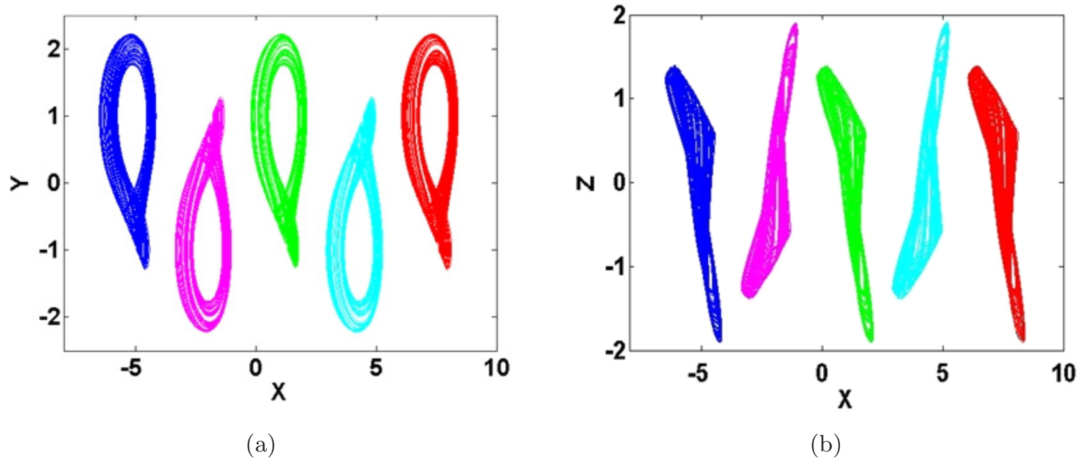


Fig. 7. Coexisting attractors for system (5), green for initial conditions $(2, 0, -1)$, blue for initial conditions $(2 - 2\pi, 0, -1)$, red for initial conditions $(2 + 2\pi, 0, -1)$, pink for initial conditions $(2 - \pi, 0, -1)$, cyan for initial conditions $(2 + \pi, 0, -1)$. (a) x–y plane and (b) x–z plane.

indicates a different boosting process. Since $\sin(x + (2k + 1)\pi) = -\sin(x)$, the above system (5) retains its conditional symmetry.

When $F(x) = A\cos(x)$, system (5) gives infinitely many equilibrium points $(2\pi k + \arccos \times (\pm b/A), \pm 1, 0)$. Specifically, when $a = 0.6$, $b = 1$, $A = 1.55$, the equilibrium points are $(2\pi k + 0.8696, 1, 0)$ and $(2\pi k + 2.272, -1, 0)$. Five attractors are found as shown in Fig. 7 when the initial conditions of the variable x are in the region $[2 - 2\pi, 2 + 2\pi]$. However, there are infinitely many coexisting attractors since boostable initial conditions exist, some of which are symmetric like the attractors in the region shown in Fig. 7. Basins of attraction in the plane $z = -1$ are shown in Fig. 8. The basins also have a nested conditionally-symmetric structure but are arranged separately along with the line of $y = 0$, $z = -1$. The distribution of regions of attraction is related to the periodic nonlinear function, but dominantly determined by the structure of the original offset-boostable system [Sprott & Xiong, 2015]. The period of $\cos(x)$ determines the distance between attractors of the same type, while the half-period determines the distance between symmetric attractors, which are 2π and π, respectively. This is interesting because it provides a new way to calculate the value of π. Lyapunov exponents for system (5) under different initial conditions are shown in Fig. 9(a). We see that even the symmetric attractors have the same Lyapunov exponents $(0.0285, 0, -0.6285)$. Since the basins of attraction for system (5) are fully assigned to different coexisting attractors, the lines of average values smoothly evolve according to the initial condition x_0. As shown in Fig. 9(b), the average of the variable z is zero, while the average of the variable y switches between a positive and a negative value, showing that system (5) has a symmetric pair of coexisting attractors, while the average x position of the attractor increases monotonically with x_0. When the initial condition of x varies in $[-15, 35]$, the bifurcations of system (5) show that there are more than $\text{ceil}(\frac{m}{n}) = \text{ceil}(\frac{50}{3}) = 17$ attractors residing in this limited region as shown in Fig. 10. Here two transverse lines, short lines and long lines, represent the two symmetric attractors.

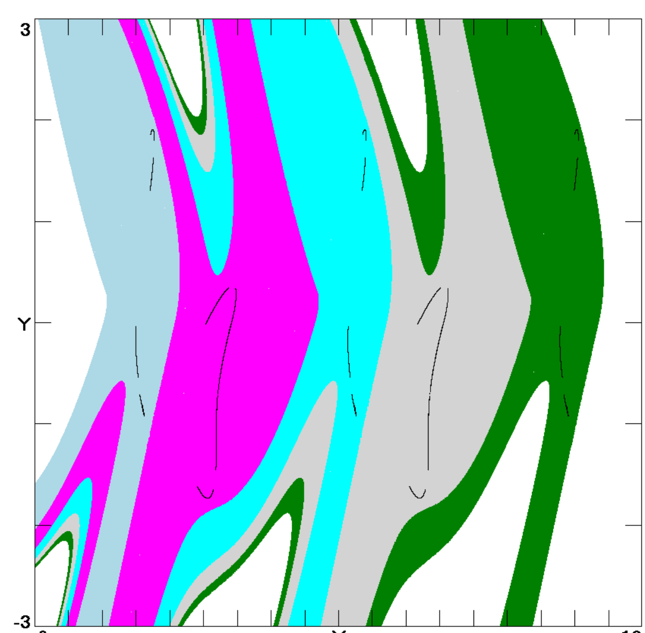
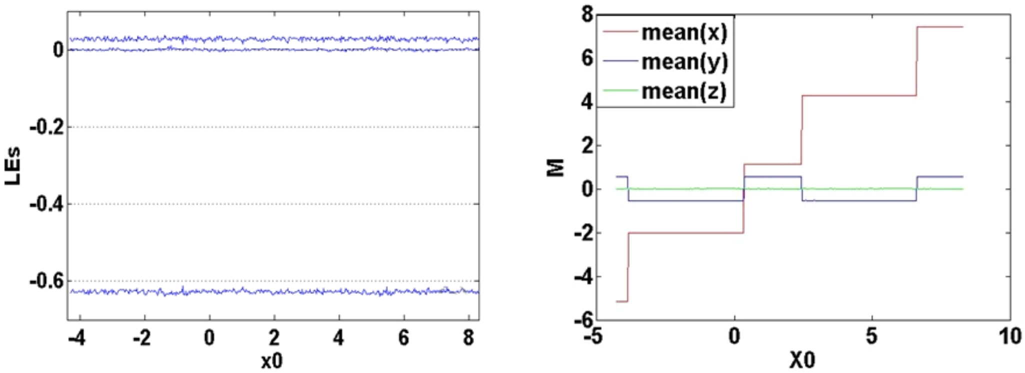
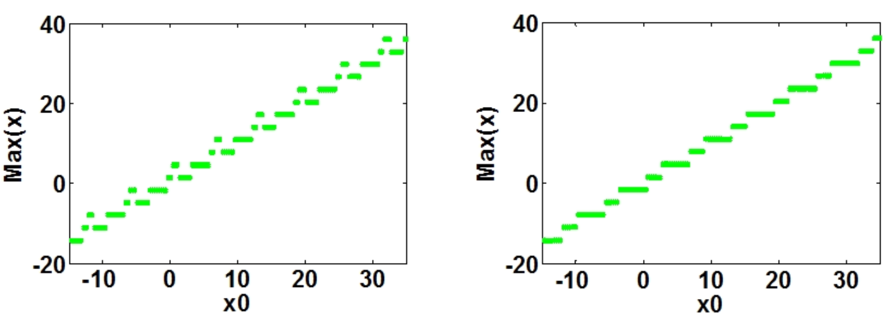


Fig. 8. Basins of attraction of the coexisting attractors in the $z = -1$ plane.



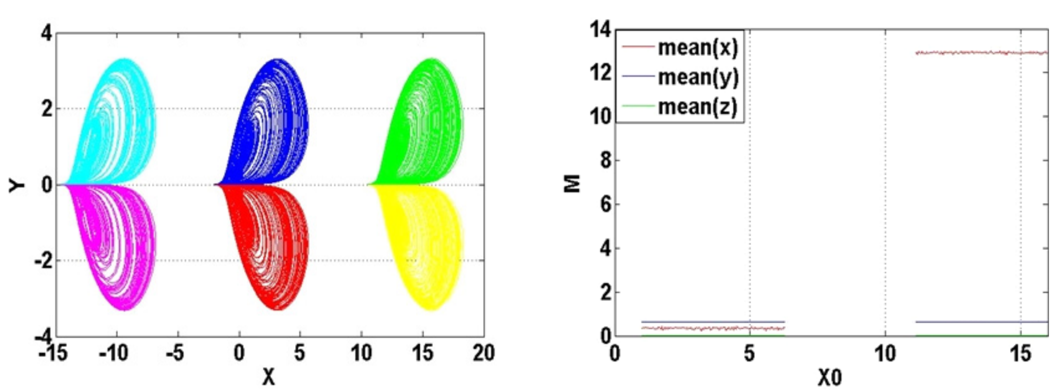
(a) (b)

Fig. 9. (a) Lyapunov exponents and (b) the average values of system (5) for initial conditions $(x_0, 0, -1)$, x_0 varies from $2 - 2\pi$ to $2 + 2\pi$.



(a) (b)

Fig. 10. Bifurcation of system (5) with $F(x) = 1.55\cos(x)$, $a = 0.6$, $b = 1$ when $x_0 \in [-15, 35]$: (a) initial condition $(x_0, -1, 1)$ and (b) initial condition $(x_0, 0, -1)$.



(a) (b)

Fig. 11. Coexisting attractors for system (8) with different initial conditions: cyan for $(-13, 1, 0)$, pink for $(-13, -1, 0)$, blue for $(1, 1, 0)$, red for $(1, -1, 0)$, green for $(13, 1, 0)$, yellow for $(13, -1, 0)$. (a) x–y plane and (b) the average values of system (8) at initial conditions $(x_0, -1, 0)$, x_0 varies from 1 to 17.

We can construct conditional reflection symmetry in the variable-boostable system VB18 [Li & Sprott, 2016; Li *et al.*, 2017]. The following system is variable-boostable with the variable x and has reflection symmetry for the variable y,

$$\begin{cases} \dot{x} = az + y^2 - 1 \\ \dot{y} = byz \\ \dot{z} = -x - z. \end{cases} \tag{7}$$

When $a = 2.8$, $b = 4$, the system has a chaotic attractor with Lyapunov exponents $(0.1149, 0, -1.1149)$ and Kaplan–Yorke dimension 2.1031. System (7) is invariant under the transformation $x \to x$, $y \to -y$, $z \to z$, corresponding to symmetry about the $y = 0$ plane. A version of conditional symmetry can be obtained by introducing a periodic function in the boosting dimension x,

$$\begin{cases} \dot{x} = az + y^2 - 1 \\ \dot{y} = byz \\ \dot{z} = -A\sin(Bx) - z. \end{cases} \tag{8}$$

When $a = 2.8$, $b = 4$, $A = 2.2$, $B = 0.5$, system (8) has infinitely many strange attractors with the same Lyapunov exponents $(0.1101, 0, -1.1101)$ as shown in Fig. 11(a). Let $x = u + 4k\pi$ $(k \in N)$, $y = -v$, $z = w$ ($4k\pi$ is a constant for offset boosting). The new system is identical to Eq. (8), and thus system (8) is conditionally symmetric with reflection invariance. As shown in Fig. 11(b), the average of the variables y and z are constant, while the average of the variable x increases with the initial condition of the variable x giving strange attractors climbing along the x-axis. The discontinuous parts of the line correspond to crossing a basin boundary.

4. Discussions and Conclusions

If any of the variables in a differential system has the freedom of offset boosting by introducing a single constant without changing the fundamental structure, the variable-boostable system can be further revised to be a self-reproducing system, where infinite multistability is obtained when a periodic trigonometric function of that variable is introduced. Two regimes of self-reproducing system were constructed, where countless attractors or countless symmetric pairs of attractors were found with uniform Lyapunov exponents. Note that as shown in the above analysis, since the attractor reproducing is based on the periodically offset boosting and the newly introduced periodic functions may destroy the basic dynamics, consequently there is a risk to reproduce the desired attractor. The construction of a self-reproducing system needs additional parameter modification in the periodic function for the main dynamics remaining in one period. It is reasonable to conclude that if the periodic trigonometric function changes to some other nonperiodic functions with slow modulation, the corresponding self-reproducing systems may probably give some if not infinitely many coexisting attractors with unequal intervals. This translation is not restricted to one variable, and corresponding n-dimensional shifted multistability can be created by the proposed method. Furthermore, self-reproducing systems can also admit conditional symmetry giving countless symmetric attractor pairs, which means that a system with the same Lyapunov exponents under different initial conditions may hide the existence of infinite multistability. Self-reproducing system provides infinitely many coexisting attractors for engineering application, which used to be generated by switchable chaotic systems or compound chaotic systems. From the view of implementation of electronic circuit, chaotic signal of different polarity becomes available from corresponding attractors, which can be probed out with an initial condition without needing an extra polarity converter.

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