



## Simple Chaotic Hyperjerk System

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In literature many chaotic systems, based on third-order jerk equations with different nonlinear functions, are available. A jerk system is taken to be a part of dynamical systems that can exhibit regular and chaotic behavior. By extension, a hyperjerk system can be described as a dynamical system with  $n$ th-order ordinary differential equations where  $n$  is 4 or up to. Hyperjerk systems have been investigated in literature in the last decade. This paper consists of numerical studies and experimental realization on FPAA for fourth-order hyperjerk system with exponential nonlinear function.

*Keywords:* Hyperjerk system; chaos; analysis; FPAA.

### 1. Introduction

Chaos and chaos-based systems have been attractive in the literature. Chaos has occurred in systems which have at least third-order dimensional autonomous ordinary differential equations [Chua *et al.*, 1993; Kennedy, 1992; Lakshmanan & Murali, 1996; Lorenz, 1963; Rossler, 1976; Sprott, 2000a, 2000b]. Among the chaotic systems, simple chaotic systems, proposed by Sprott, have attracted considerable interest in the literature due to simplicities and rich contents [Sprott, 2000a, 2000b, 2010, 2011]. Simple chaotic systems based on “jerk systems” can be described as;

$$\frac{d^3x}{dt^3} = J\left(\frac{d^2x}{dt^2}, \frac{dx}{dt}, x\right) \quad (1)$$

where  $J$  is called as “jerk” which is the derivation of a single scalar variable  $x$ . In a Newtonian system,  $\frac{dx}{dt}$ ,  $\frac{d^2x}{dt^2}$  and  $\frac{d^3x}{dt^3}$  are the displacements of velocity,

acceleration and jerk, respectively. If a system is the fourth-order derivative system of the form  $\frac{d^4x}{dt^4} = J(\frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x)$ , it is called “hyperjerk system” or “snap system” [Chlouverakis & Sprott, 2006]. Whereas  $\frac{d^4x}{dt^4}$  is described as “jounce”, “sprite” or “surge” but it is generally labeled as “snap” in literature [Sprott, 2010]. There are several studies on hyperjerk systems in the literature [Chlouverakis & Sprott, 2006; Linz, 2008; Munmuangsaen & Srisuchinwong, 2011; Munmuangsaen *et al.*, 2011; Sprott, 2010; Vaidyanathan *et al.*, 2015]. Those works deal with numerical analyses of hyperjerk systems.

Recently, the programmable feature of analog array components-based new circuit techniques provides flexible design opportunity for chaos generators. FPAA (Field Programmable Analog Array) is a suitable programmable device to realize flexible and useful chaotic generators which are based on mathematical models.

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In this paper, we have proposed a fourth-order hyperjerk system by using exponential nonlinear function with numerical analysis results and have presented the experimental realization of this hyperjerk system on FPAA. The paper is organized as follows: In Sec. 2 the proposed hyperjerk system is analyzed using the numerical method. FPAA-based experimental realization of the hyperjerk system is given in Sec. 3. Finally some concluding remarks will be discussed in Sec. 4.

## 2. Analysis of Proposed Hyperjerk System

The hyperjerk system which is the focus of this paper is written below in generalized form.

$$\frac{d^4x}{dt^4} + a\frac{d^3x}{dt^3} + b\frac{dx}{dt} + cx = f\left(\frac{d^2x}{dt^2}\right) \quad (2)$$

while  $b$  and  $c$  are the system parameters,  $a$  is the bifurcation parameter.  $f\left(\frac{d^2x}{dt^2}\right)$  is the nonlinear function which plays an important role for chaos.

The hyperjerk system given in Eq. (2) is rearranged using phase variables.  $\dot{x}_1$ ,  $\dot{x}_2$ ,  $\dot{x}_3$  and  $\dot{x}_4$  are used instead of  $\frac{dx}{dt}$ ,  $\frac{d^2x}{dt^2}$ ,  $\frac{d^3x}{dt^3}$  and  $\frac{d^4x}{dt^4}$  respectively. The form is given below.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c & -b & 0 & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ f(x_3) \end{bmatrix} \quad (3)$$

where  $f(x_3)$  is the nonlinear function in Eq. (3) and is given below.

$$f(x_3) = -\exp(x_3) \quad (4)$$

In Eq. (3), the parameters  $a$ ,  $b$  and  $c$  are equal to 1, 3 and 1, respectively.

The equilibrium points of the proposed hyperjerk system in Eqs. (3) and (4) are figured out as

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4) &= x_2 = 0 \\ f_2(x_1, x_2, x_3, x_4) &= x_3 = 0 \\ f_3(x_1, x_2, x_3, x_4) &= x_4 = 0 \\ f_4(x_1, x_2, x_3, x_4) &= -cx_1 - bx_2 - \exp(x_3) - ax_4 \\ &= 0 \end{aligned} \quad (5)$$

For  $a = 1$ ,  $b = 3$  and  $c = 1$ , the equilibrium points of this system are characterized by,

$$-x_1 - 1 = 0; \quad x_2 = 0; \quad x_3 = 0; \quad x_4 = 0 \quad (6)$$

In this case the corresponding equilibrium points are  $P(-1, 0, 0, 0)$ . The Jacobian matrix of the system is given by

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -3 & -\exp(x_3) & -1 \end{bmatrix} \quad (7)$$

It is known that  $|\lambda \cdot I - J| = 0$ , so the eigenvalues of the Jacobian matrix  $J$  are calculated as  $\lambda_1 = -1.4277$ ,  $\lambda_2 = -0.368$ ,  $\lambda_{3,4} = 0.3978 \pm 1.3211i$ . Here  $\lambda_1$  and  $\lambda_2$  are negative real numbers and  $\lambda_3$  and  $\lambda_4$  are complex conjugate eigenvalues with positive real parts. Therefore, the proposed hyperjerk system is unstable and the equilibrium  $P$  is a saddle-focus.

For the hyperjerk system, the initial conditions are not critical and need not be carefully chosen. The system can produce chaotic behaviors for most initial conditions which lie within the basin of attraction. For the presented system those are taken as  $(-7.4, 0, 0, 0.1)$  in this paper. In this case the Lyapunov exponents with the initial conditions are calculated as  $(0.157, 0, -0.245, -0.913)$ . The Lyapunov exponents of the hyperjerk system were calculated with  $10^6$  iterations using a fourth-order Runge–Kutta algorithm which is described by Wolf *et al.* [1985] with a step size of  $\Delta t = 0.01$ . Also, the Kaplan–Yorke dimension of the hyperjerk system is obtained as

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.6408 \quad (8)$$

For the proposed system, the chaotic dynamics and the chaotic attractor illustration, obtained from simulations are shown in Figs. 1(a) and 1(b). To verify the appearance of chaotic behavior of the system, we additionally show the bifurcation diagram of the system in Fig. 1(c). In the bifurcation diagram  $x_1$  versus  $a$ , which is the control parameter of the system that is plotted, a period-doubling route to chaos is clearly seen in the diagram when the parameter  $a$  varies between 0.95 and 1.7. The obtained diagram of the Lyapunov exponents is given in Fig. 1(d) while the parameter  $a$  varies in the same region.

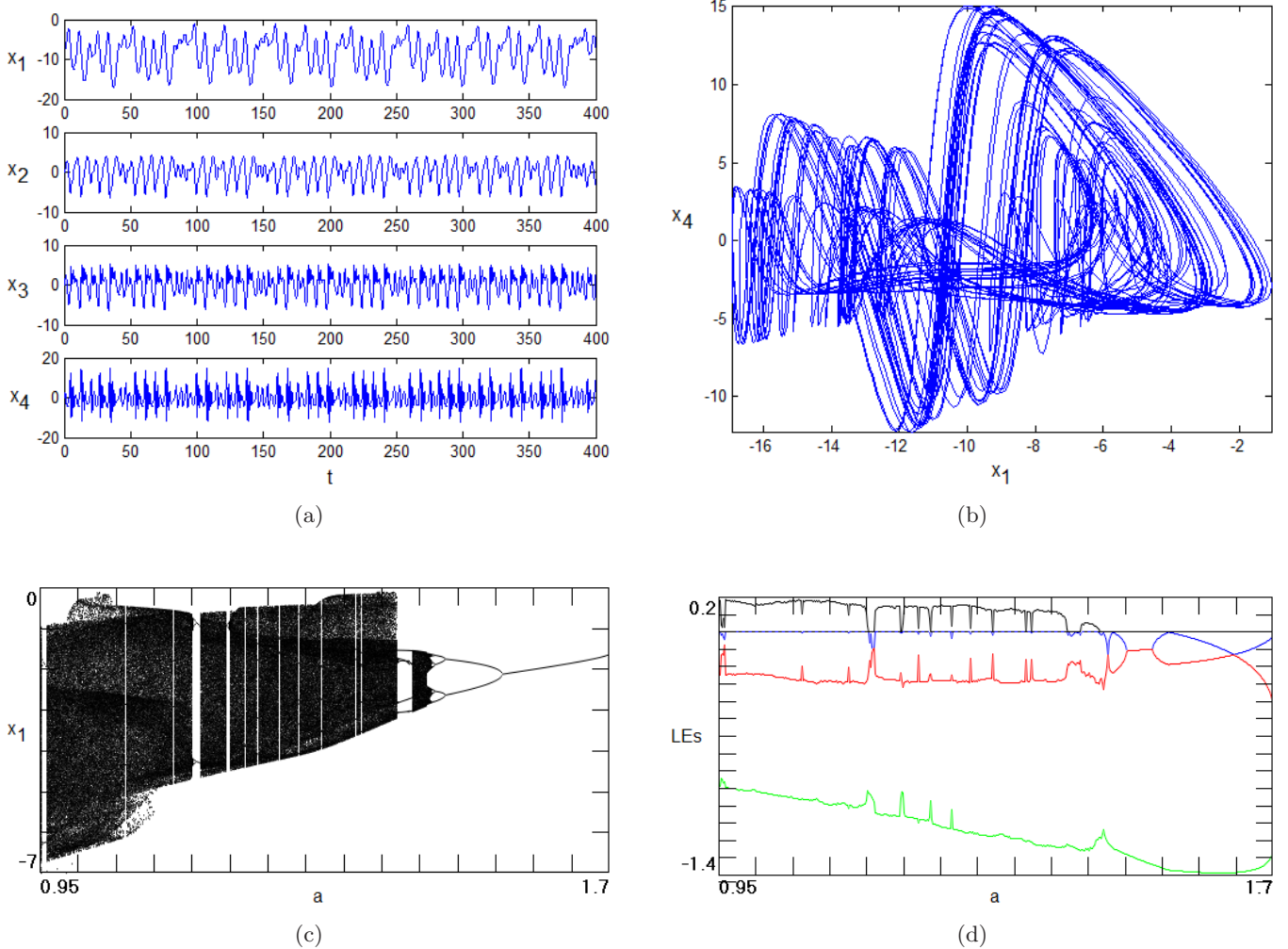


Fig. 1. Numerical analysis results of the hyperjerk system: (a) The chaotic dynamics, (b) the chaotic attractor in  $(x_1-x_4)$  plane, (c) bifurcation diagram and (d) the diagram of the Lyapunov exponents.

### 3. Experimental Realization of Proposed Hyperjerk System on FPAA

FPAA (Field Programmable Analog Array) is a programmable IC to implement a rich variety of systems including analog functions via dynamic reconfiguration. This means that a new design or a modification on the available design can be easily downloaded to an FPAA. In addition, FPAAs provide more efficient and economical solutions in a much smaller footprint and with increased reliability to design analog dynamical systems [Anadigm, 2016; Callegari *et al.*, 2005; Caponetto *et al.*, 2005; Kilic & Dalkiran, 2009, 2010].

The flow diagram of a typical FPAA implementation is shown in Fig. 2. It is clearly seen in the diagram that the mathematical definitions of a system are tested with numerical simulations before

implementing them on an FPAA. Due to the simulation results, rescaling process may be required when any mismatches between the system parameters and FPAA characteristics occur. After completing the rescaling process the system can be modeled and downloaded to the FPAA development board using FPAA interface software. The experimental results, obtained from programmed hardware, are compared with simulation results. If there is any error between hardware and software, it is eliminated by modifying the system model using FPAA interface software. The implementation is completed when all the errors are eliminated.

Before programming the FPAA board to model the hyperjerk system, it is tested in SIMULINK. Due to the fact that SIMULINK is structurally similar to FPAA interface software, numerical simulations in FPAA flow diagram are realized using

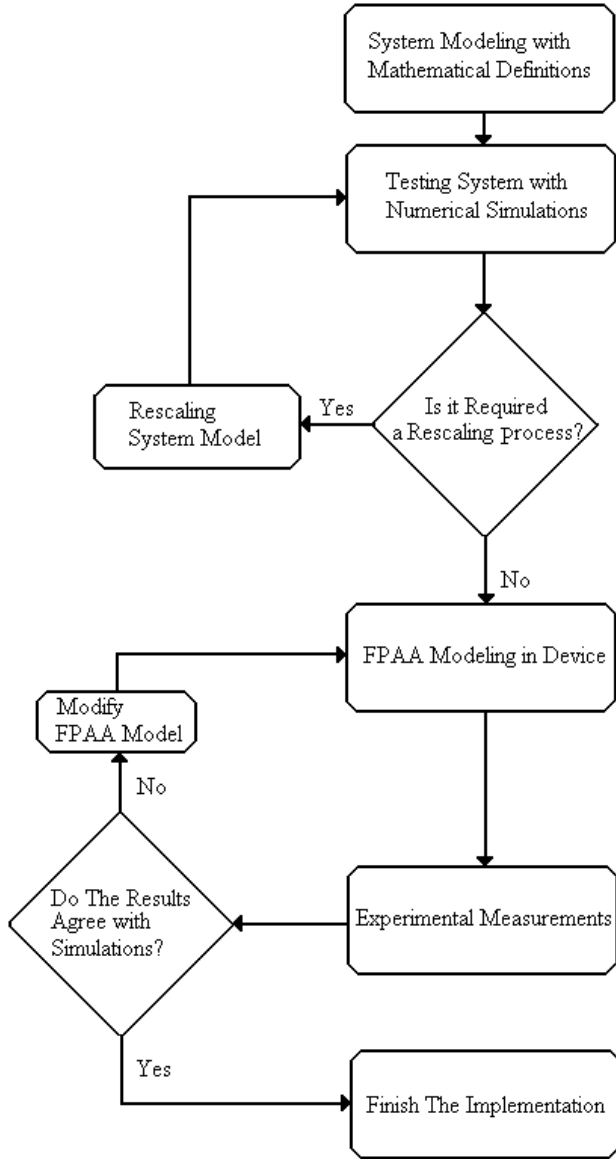


Fig. 2. FPAAs-based design and implementation procedure.

SIMULINK [1999]. In this study we used the AN231K04 type FPAAs development board produced by Anadigm. The FPAAs board has  $\pm 1.5$  V saturation level. Due to the fact that the voltage ranges of  $x_1, x_2, x_3$  and  $x_4$  state variables of the proposed system exceed saturation level, the equations of the hyperjerk system are rescaled. The following rescaling factors were chosen:

$$X_1 = \frac{x_1}{k_{x_1}}; \quad X_2 = \frac{x_2}{k_{x_2}}; \quad X_3 = \frac{x_3}{k_{x_3}}; \quad X_4 = \frac{x_4}{k_{x_4}};$$

$$k_{x_1} = 16; \quad k_{x_2} = 7; \quad k_{x_3} = 6; \quad k_{x_4} = 15.$$

After rescaling process the system described by Eqs. (3) and (4) becomes as follows.

$$\begin{aligned} \dot{X}_1 &= \frac{k_{x_2}}{k_{x_1}} X_2 \\ \dot{X}_2 &= \frac{k_{x_3}}{k_{x_2}} X_3 \\ \dot{X}_3 &= \frac{k_{x_4}}{k_{x_3}} X_4 \\ \dot{X}_4 &= -aX_4 + \frac{1}{k_{x_4}} f(k_{x_3} X_3) \\ &\quad - b \frac{k_{x_2}}{k_{x_4}} X_2 - c \frac{k_{x_1}}{k_{x_4}} X_1 \end{aligned} \tag{9}$$

$$f(k_{x_3} X_3) = -\exp(k_{x_3} X_3)$$

The FPAAs implementation scheme of the rescaled hyperjerk system, modeled in FPAAs software, is depicted in Fig. 3. The state-variables of the system namely  $X_1, X_2, X_3$  and  $X_4$  are obtained from the outputs of SUMFILTER blocks as shown in the FPAAs implementation scheme. A SUMFILTER block consists of a summing stage with up to three inputs and a single pole low pass filter. Circuit gains are implemented by the gains of SUMFILTER and SUMDIFF blocks. A SUMDIFF block is a summing stage with up to four inputs without filtering. A user-defined TRANSFER FUNCTION block and GAINHALF block are used to implement the nonlinear function  $f(k_{x_3} X_3)$ . This transfer function

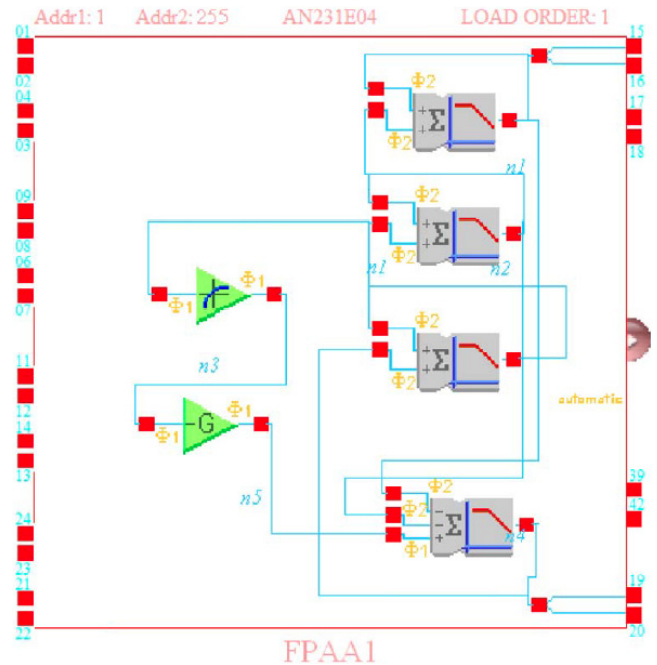


Fig. 3. FPAAs implementation schemes of proposed hyperjerk system.

module produces an output voltage related with 256 quantization steps according to a lookup table, defined by the user. GAINHALF block provides a gain which can be adjusted between 0.01 and 100 and is run inverting and noninverting the mode. In this study this block is used in inverting the

mode. The chaotic dynamics and the chaotic attractor illustration produced by experimental realization of the system are shown in Fig. 4.

#### 4. Conclusion

In conclusion, a simple hyperjerk system which is exactly simple and elegant is presented with numerical simulations and experimental results. Some fundamental properties of the hyperjerk system have been investigated with respect to chaotic dynamics, chaotic attractor, bifurcation diagram, the Lyapunov exponents, the eigenvalues of the Jacobian matrix while studying numerical simulations. Later experimental studies are realized using an FPAA application board. It is clearly seen that experimental results obtained from FPAA board correspond to numerical simulation results. The proposed FPAA-based system design can be effectively used as an easily programmable and reconfigurable chaos generator in practical applications. This study can be accepted as an alternative investigation of fourth-order hyperjerk systems. Furthermore chaotic behaviors, generated by the hyperjerk system, can be used in chaos-based applications and this system can be electronically implemented.

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#### References

- Anadigm [2016]; [www.anadigm.com](http://www.anadigm.com).  
 Callegari, S., Rovatti, R. & Setti, G. [2005] "First direct implementation of a true random source on programmable hardware," *Int. J. Circuit Th. Appl.* **33**, 1–16.  
 Caponetto, R., Mauro, A. D., Fortuna, L. & Frasca, M. [2005] "Field programmable analog array to implement a programmable Chua's circuit," *Int. J. Bifurcation and Chaos* **15**, 1829–1836.  
 Chlouverakis, K. E. & Sprott, J. C. [2006] "Chaotic hyperjerk systems," *Chaos Solit. Fract.* **28**, 739–746.  
 Chua, L. O., Wu, C. W., Huang, A. & Zhong, G.-Q. [1993] "A universal circuit for studying and generating chaos-Part I: Routes to chaos," *IEEE Trans. Circuits Syst.-I: Fund. Th. Appl.* **40**, 732–745.  
 Kennedy, M. P. [1992] "Robust op-amp realization of Chua's circuit," *Frequenz* **46**, 66–80.  
 Kilic, R. & Dalkiran, F. Y. [2009] "Reconfigurable implementations of Chua's circuit," *Int. J. Bifurcation and Chaos* **19**, 1339–1350.

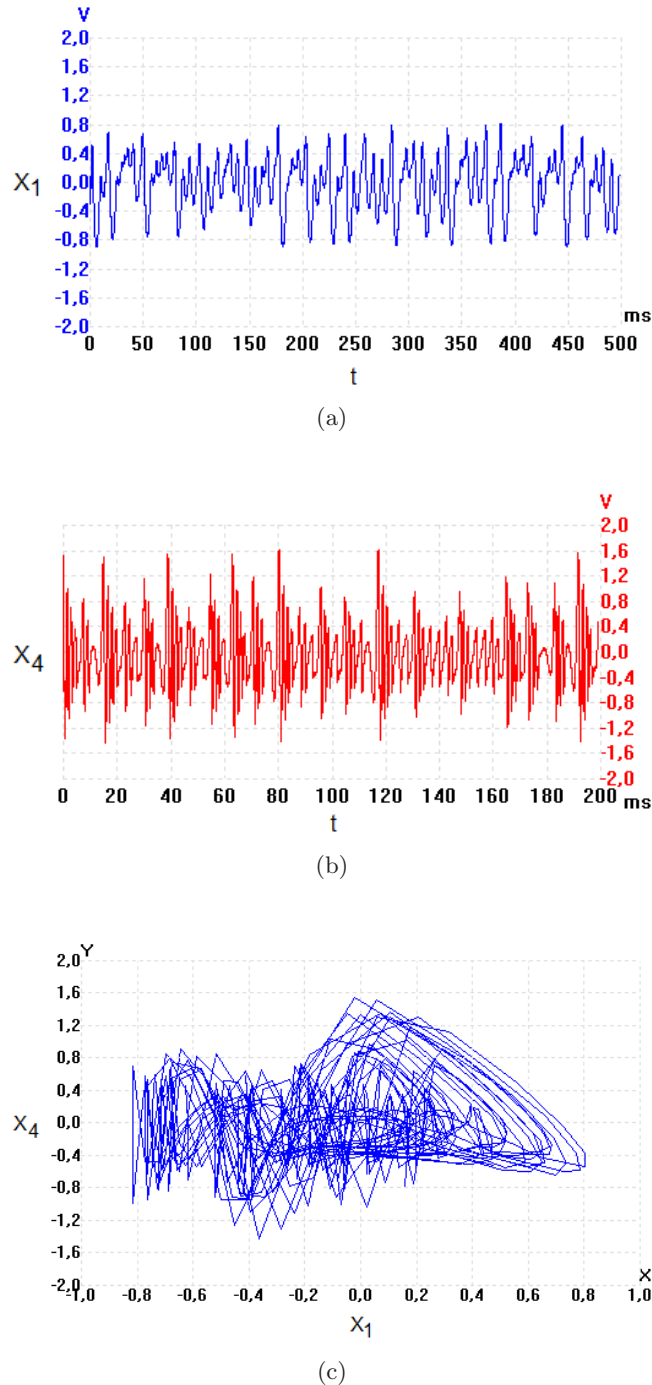


Fig. 4. Experimental results obtained from FPAA-based implementation of the hyperjerk system: (a) and (b) Time responses of  $X_1(t)$  and  $X_4(t)$  chaotic dynamics and (c) chaotic attractor projection in  $(X_1-X_4)$  plane.

- Kilic, R. & Dalkiran, F. Y. [2010] “Programmable design and implementation of a chaotic system utilizing multiple nonlinear functions,” *Turkish J. Electric. Engin. Comput. Sci.* **18**, 647–655.
- Lakshmanan, M. & Murali, K. [1996] *Chaos in Nonlinear Oscillators, Controlling and Synchronization* (World Scientific, Singapore).
- Linz, S. J. [2008] “On hyperjerky systems,” *Chaos Solit. Fract.* **37**, 741–747.
- Lorenz, E. N. [1963] “Deterministic nonperiodic flow,” *J. Atmosph. Sci.* **20**, 130–141.
- Munmuangsaen, B. & Srisuchinwong, B. [2011] “Elementary chaotic snap flows,” *Chaos Solit. Fract.* **44**, 995–1003.
- Munmuangsaen, B., Srisuchinwong, B. & Sprott, J. C. [2011] “Generalization of the simplest autonomous chaotic circuit,” *Phys. Lett. A* **375**, 1445–1450.
- Rossler, O. E. [1976] “An equation for continuous chaos,” *Phys. Lett. A* **57**, 397–398.
- SIMULINK [1999] *User’s Guide* (The MathWorks, Inc., Natick, MA).
- Sprott, J. C. [2000a] “A new class of chaotic circuit,” *Phys. Lett. A* **266**, 19–23.
- Sprott, J. C. [2000b] “Simple chaotic systems and circuits,” *Am. J. Phys.* **68**, 758–763.
- Sprott, J. C. [2010] *Elegant Chaos: Algebraically Simple Chaotic Flows* (World Scientific, Singapore).
- Sprott, J. C. [2011] “A new chaotic jerk circuit,” *IEEE Trans. Circuits Syst.-II: Expr. Briefs* **58**, 240–243.
- Vaidyanathan, S., Volos, C., Pham, V. T. & Madhavan, K. [2015] “Analysis, adaptive control and synchronization of a novel 4-D hyperchaotic hyperjerk system and its SPICE implementation,” *Arch. Contr. Sci.* **25**, 135–158.
- Wolf, A., Swift, J. B., Swinney, H. L. & Vastano, J. A. [1985] “Determining Lyapunov exponents from a time series,” *Physica D* **16**, 285–317.