# Chaos in a Class of Local Interaction Models

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**Abstract**

In this note, we report the presence of long-term nonlinearities and chaos in a four-dimensional system of ordinary differential equations that model a concrete and relevant social interaction process in which agents endowed with different sentiments or different 'views of the world' coexist and influence one another. We find that intricate dynamic outcomes emerge in the characterized setting even for an unsophisticated and homogeneous network of relations.

*Key words:* Sentiment propagation, Nonlinear dynamics, Continuous-time chaos

## 1. Introduction

Contagion processes are some of the most debated and scrutinized themes in science. In a detailed survey of the voluminous literature on the propagation of epidemics, Pastor-Santorras, *et al.* [1] emphasize that the ancient epidemic modeling apparatus [2] may be interpreted as a metaphor to characterize a large number of phenomena in nature and society. The most popular applications of the epidemics framework are those concerning the spreading of infectious diseases within a population and, in a social context, the propagation of rumors in large networks.[1]

In typical propagation models, each individual agent resides in one of three different classes or categories. In an epidemiological setting, these categories are the susceptible, the infected, and the recovered. In the spreading of rumors, the categories are the ignorant, the spreaders, and the stiflers. In such cases, the individuals usually interact randomly over a given network structure and move from one category to another according to the assumed transition probabilities.

In this paper, we adapt the rumor spreading paradigm to a setting of sentiment propagation following the ideas of refs. [10, 11, 12]. Again, three types of agents are

[1] Rumor propagation theory, in particular, has witnessed a significant boost in recent years following the initial contributions of Daley and Kendall [3] and Maki and Thompson [4]. (See, e.g., refs. [5, 6, 7, 8, 9].)

assumed: in this case, the neutral, the exuberant, and the non-exuberant. However, since two antagonistic sentiments are considered (optimism and pessimism), we have five different categories because both optimists and pessimists can be either in a state of exuberance or in a state of non-exuberance.

Agents switch from one sentiment category to another according to a set of dynamic rules defined in Section 2. These rules lead to a simple continuous-time system of ordinary differential equations (ODEs) that exhibits chaotic behavior. Hence the proposed theoretical structure provides a candidate explanation for the kind of persistent irregular waves of optimism and pessimism that is frequently observed in multiple socio-economic scenarios, e.g., in scenarios involving macroeconomic data.[2]

One of the reasons the model is simple is that we consider only homogeneous network structures. As highlighted by Boccaletti, *et al.* [16] and by Zschaler [17], many real-world networks are strongly heterogeneous, with a power-law degree distribution. For the networks considered here, every node in the network is connected randomly to an identical number of other nodes. This is a convenient assumption since it limits and preserves the dimensionality of the dynamic system (the system considered will involve a set of four equations), allowing for a detailed analysis of its dynamics.

Besides describing diffusion within networks, this note also contributes to the literature that explores the presence of chaos in low-dimensional continuous-time systems (see, e.g., refs. [18, 19, 20, 21, 22]). The framework presented here fulfills the requisites Sprott [23] lists as essential for publication of new chaotic systems: it considers a substantive and meaningful phenomenon, it generates novel nonlinear dynamic results, and it involves an especially simple system of equations.

## 2. Local Interaction and Sentiment Spreading

Consider a social environment where each individual resides at every instant $t$ in one of five possible sentiment categories. These categories, and the respective densities of individuals they contain, are: neutrality ($x$), exuberant optimism ($y$), non-exuberant optimism ($z$), exuberant pessimism ($v$), and non-exuberant pessimism ($w$). The five sentiment states comprise the whole universe of agents, i.e.,

$$x + y + z + v + w = 1, \forall t. \tag{1}$$

Agents switch between categories according to a set of rules that are adapted from rumor propagation theory:

1. The interaction of a neutral agent with an exuberant agent, optimist or pessimist, will cause a transition from neutrality to the respective exuberance category with probability $\lambda \in (0, 1]$.

[2]The association between dominant sentiments in society, typically assessed through various survey measures, and observable economic aggregate fluctuations, is exhaustively documented in recent empirical literature (see, e.g., Leduc and Sill, [13], Beaudry and Portier, [14], and Girardi, [15]).

2. The interaction of a non-exuberant agent, optimist or pessimist, with an agent in the neutral state will cause a transition of the non-exuberant individual to neutrality with probability $\theta \in (0, 1]$.
3. The interaction between an exuberant individual, optimist or pessimist, with another exuberant individual or with a non-exuberant individual within the same type of sentiment will cause a shift from exuberance to non-exuberance with probability $\sigma \in [0, 1]$.

Fig. 1 outlines the set of established interaction rules. Observe that the rules impose a sequence of mental stages that agents go through following a given order: one cannot pass from neutrality to non-exuberance without first experiencing exuberance; one cannot go directly from exuberance to neutrality without first passing through the category of non-exuberance. Note, as well, that in this interpretation exuberance is a pivotal category: an exuberant individual is able to convince someone who has no opinion about the prevailing sentiment, and then the sentiment tends to fade as exuberants pass to the non-exuberance state and from this to the neutrality state. Agents in opposite categories (optimism and pessimism) are unable to influence each other because they are not in a susceptibility mode (neutrality); only susceptible individuals can be recruited to a sentiment class.

$$***fig.1***$$

While $\lambda$ and $\theta$ are assumed to be fixed constants, the probability $\sigma$ is allowed to change in time. We assume that exuberant individuals have the possibility, when establishing social contact, of assessing the overall sentiments of the population and altering their value of $\sigma$, which is thus a subjective probability of transition. Probability $\sigma$ will be bounded by floor and ceiling values, respectively $\sigma_L$ and $\sigma_H$ ($\sigma_L \leq \sigma_H \leq 1$), and will obey the following rule: exuberant agents will react to the inflow into the corresponding sentiment (given by the term $\lambda xy$ for optimists, and by the term $\lambda xv$ for pessimists) and to the respective outflow back to the neutrality state (given by the term $\theta zx$ for optimists, and by the term $\theta wx$ for pessimists). Whenever the share of optimists (pessimists) is increasing, i.e., $\lambda xy > \theta zx$ ($\lambda xv > \theta wx$), the respective value of $\sigma$ will be relatively low and approaches $\sigma_L$ as the inflow-outflow gap widens. In the opposite case, when the share of optimists (pessimists) is decreasing, i.e., $\lambda xy < \theta zx$ ($\lambda xv < \theta wx$), the respective value of $\sigma$ will be relatively high and approaches $\sigma_H$ as the outflow-inflow gap widens. This rule captures the idea that an observed fall in optimism (pessimism) will lead exuberant optimists (pessimists) to change at a faster pace to the non-exuberance category, while an observed rise in optimism (pessimism) will produce the opposite outcome.[3]

[3] An extreme assumption would be to restrict the analysis to case $\sigma_L = 0$ and $\sigma_H = 1$, i.e., to the case where the strong inflow and the strong outflow from the sentiment class would imply, respectively, the persistence of all individuals in the exuberant state or the complete desertion from this category, when in contact with others in the same sentiment class. This is a possibility in the formulated model,

For each of the sentiment types, the described reasoning is formally characterized by the two transition probability functions,

$$\sigma(opt) = \frac{1}{2}\left\{(\sigma_H + \sigma_L) - (\sigma_H - \sigma_L)\tanh\left[\kappa(\lambda xy - \theta zx)\right]\right\} \tag{2}$$

$$\sigma(pes) = \frac{1}{2}\left\{(\sigma_H + \sigma_L) - (\sigma_H - \sigma_L)\tanh\left[\kappa(\lambda xv - \theta wx)\right]\right\} \tag{3}$$

In the hyperbolic tangent function, factor $\kappa$ measures the rate at which $\sigma$ approaches its extreme values given the extent of sentiment changes, and is hereafter taken as $\kappa = 100$ to approximate the signum function while avoiding discontinuities in the flow that cause problems in the calculation of quantities such as the Lyapunov exponents.[4] When no change occurs in sentiment shares, the transition probabilities will be such that $\sigma(opt) = \sigma(pes) = \frac{1}{2}(\sigma_H + \sigma_L)$. Fig. 2 shows the value of $\sigma(opt)$ as a function of the net value $\lambda xy - \theta zx$.[5]

$$* * * fig.2 * **$$

Nekovee *et al.* [24] explain how a set of transition rules such as these can be translated into a system of ODEs using the law of mass action.[6] From the above rules assuming a homogeneous network of connectivity degree equal to 1,[7] the procedure gives the five-dimensional system,

$$\begin{cases} \dot{x} = -\lambda x(y+v) + \theta(z+w)x \\ \dot{y} = \lambda xy - \sigma(opt)y(y+z) \\ \dot{v} = \lambda xv - \sigma(pes)v(v+w) \\ \dot{z} = \sigma(opt)y(y+z) - \theta zx \\ \dot{w} = \sigma(pes)v(v+w) - \theta wx \end{cases} \tag{4}$$

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but by not imposing it *a priori* we keep the analysis at a more general level, allowing for a wider array of possible dynamic outcomes.

[4] The value of $\kappa$ indicates how fast $\sigma$ approaches its extreme values. Our intention is to separate two states: one in which the sentiment is rising and thus $\sigma$ should be low, and the opposite. Instead of assuming a discontinuous function reflecting this, for a matter of the model's tractability, we have chosen to work with a sigmoid function; when $\kappa$ is sufficiently large, the effect we intended is accomplished because the function shifts fast from one limit value to the other. Therefore, a large value of this parameter, as the one selected for the analysis, serves the intended purpose.

[5] A similar plot could be displayed for $\sigma(pes)$ as a function of $\lambda xv - \theta wx$.

[6] A full description of the process through which the transition rules give place to the ODEs system is beyond the scope of this paper. As in [24], we arrive to the system by conceiving the network of social relations as an interacting Markov chain (IMC). An IMC is a network in which the state of each node evolves according to an internal Markov chain but where, unlike typical Markov chains, the transition probabilities depend not only on the state of the node itself but also on the states of the nodes with which it establishes a connection.

[7] Instead of a degree 1 connectivity network, we could consider a homogeneous network of any degree $k \in \mathbb{Z}^+$. This would imply a rescaling of time, but such a rescaling would not alter, from a qualitative point of view, the model's main dynamic results.

with initial conditions $x(0), y(0), z(0), v(0), w(0)$. Using condition (1), we can suppress one of the equations in system (4) and transform it into a four-dimensional system,

$$\begin{cases} \dot{y} = \lambda \left[1 - (y+z+v+w)\right] y - \sigma(opt) y(y+z) \\ \dot{v} = \lambda \left[1 - (y+z+v+w)\right] v - \sigma(pes) v(v+w) \\ \dot{z} = \sigma(opt) y(y+z) - \theta z \left[1 - (y+z+v+w)\right] \\ \dot{w} = \sigma(pes) v(v+w) - \theta w \left[1 - (y+z+v+w)\right] \end{cases} \tag{5}$$

with initial conditions $y(0), z(0), v(0), w(0)$.

The long-term behavior of the sentiment densities that underlie system (5) will be analyzed in detail in the next sections. The richness of the dynamics will be illustrated in Section 4 for particular values of the four parameters $\lambda, \theta, \sigma_L$, and $\sigma_H$.

## 3. Equilibria: Existence and Stability

In this section, we enumerate the equilibria and their stability for system (5). However, to make the analysis more tractable, we limit our consideration to the special case of $\lambda = \sigma_H = 1$, which means that contact among the corresponding agents always produces a change of status. With this simplification, only two parameters become relevant in the analysis: $\sigma_L$ and $\theta$.[8]

Before proceeding with the analysis under the imposed parameter constraint, we present the general long-term equilibrium result associated with system (5).

**Proposition 1.** *The long-term dynamics of the continuous-time sentiment model is characterized by the existence of four isolated equilibrium points plus one line of infinitely many equilibria, namely:*

$$e(1) : (y^*, v^*, z^*, w^*) =$$
$$\left( \frac{\lambda\theta}{(2\lambda+\overline{\sigma})(\lambda+\theta)}; \frac{\lambda\theta}{(2\lambda+\overline{\sigma})(\lambda+\theta)}; \frac{\lambda^2}{(2\lambda+\overline{\sigma})(\lambda+\theta)}; \frac{\lambda^2}{(2\lambda+\overline{\sigma})(\lambda+\theta)} \right)$$
$$e(2) : (y^*, v^*, z^*, w^*) = \left( 0; \frac{\lambda\theta}{(\lambda+\overline{\sigma})(\lambda+\theta)}; 0; \frac{\lambda^2}{(\lambda+\overline{\sigma})(\lambda+\theta)} \right)$$
$$e(3) : (y^*, v^*, z^*, w^*) = \left( \frac{\lambda\theta}{(\lambda+\overline{\sigma})(\lambda+\theta)}; 0; \frac{\lambda^2}{(\lambda+\overline{\sigma})(\lambda+\theta)}; 0 \right)$$
$$e(4) : (y^*, v^*, z^*, w^*) = (0; 0; 0; 0)$$
$$e(5) : (y^*, v^*, z^*, w^*) = (0; 0; z^*; w^*), \ z^* + w^* = 1 \ \textit{(equilibrium line)}$$

*with* $\overline{\sigma} \equiv \frac{\sigma_H + \sigma_L}{2}$.

[8] The imposed constraints on parameter values allow to simplify the analysis in a way that we believe is reasonable from an intuitive point of view. Basically, the simplification highlights the relevant role of exuberant individuals: exuberants are capable of fully converting neutral individuals to their belief ($\lambda = 1$) and to completely abandon the exuberance state when faced with an overall desertion (a net outflow) from the respective sentiment class ($\sigma_H = 1$).

**Proof.** The equilibria are determined by solving system (5) under the conditions $\dot{y} = \dot{v} = \dot{z} = \dot{w} = 0$. Observe that, in the steady-state, $\sigma^*(opt) = \sigma^*(pes) = \overline{\sigma}$. The equilibrium solutions are derived, in the current setting, by solving the following system,

$$\begin{cases} y^* = 0 \vee 1 - (v^* + w^*) = \frac{\lambda + \overline{\sigma}}{\lambda}(y^* + z^*) \\ v^* = 0 \vee 1 - (y^* + z^*) = \frac{\lambda + \overline{\sigma}}{\lambda}(v^* + w^*) \\ (\overline{\sigma} y^* + \theta z^*)(y^* + z^*) = \theta z^* [1 - (v^* + w^*)] \\ (\overline{\sigma} v^* + \theta w^*)(v^* + w^*) = \theta w^* [1 - (y^* + z^*)] \end{cases} \tag{6}$$

System (6) has four solution points and a solution line. The first solution, $e(1)$, is obtained by solving the system for $y^* \neq 0 \wedge v^* \neq 0$; when assuming $y^* = 0 \wedge v^* = 0$, the equilibrium line $e(5)$, as well as the equilibrium point, $e(4)$, are attained. The other cases, $y^* = 0 \wedge v^* \neq 0$ and $y^* \neq 0 \wedge v^* = 0$, allow for determining the remaining equilibria, respectively $e(2)$ and $e(3)$. ■

A few comments are worth making about the proposition 1. First, note the symmetric nature of $e(1)$, the only equilibrium that allows coexisting exuberant optimists and exuberant pessimists in the long-term solution. In this case, the percentages of optimists and pessimists are identical for both exuberant and non-exuberant categories. Second, observe that besides the densities of optimists and pessimists, equilibrium points also give an indirect indication about the shares of neutral agents in each case, as follows:

$$e(1): x^* = \frac{\overline{\sigma}}{2\lambda + \overline{\sigma}};\ e(2), e(3): x^* = \frac{\overline{\sigma}}{\lambda + \overline{\sigma}};\ e(4): x^* = 1;\ e(5): x^* = 0$$

Point $e(4)$ and line $e(5)$ are corner solutions, with the population in the long-run being fully concentrated in the categories of neutrality and non-exuberance, respectively. A third remark concerning proposition 1 is that the equilibria that occur in a degree-1 network would also be present for any other connectivity degree $k$; thus, for this type of framework, the long-term outcomes are exclusively contingent on the values of the transition probabilities regardless of the intensity of contacts across the population.

Next, the simplifying assumption $\lambda = \sigma_H = 1$ is imposed giving

$$e(1): (y^*, v^*, z^*, w^*) =$$
$$\left( \frac{2\theta}{(5 + \sigma_L)(1 + \theta)}; \frac{2\theta}{(5 + \sigma_L)(1 + \theta)}; \frac{2}{(5 + \sigma_L)(1 + \theta)}; \frac{2}{(5 + \sigma_L)(1 + \theta)} \right)$$

$$e(2): (y^*, v^*, z^*, w^*) = \left( 0; \frac{2\theta}{(3 + \sigma_L)(1 + \theta)}; 0; \frac{2}{(3 + \sigma_L)(1 + \theta)} \right)$$

$$e(3): (y^*, v^*, z^*, w^*) = \left( \frac{2\theta}{(3 + \sigma_L)(1 + \theta)}; 0; \frac{2}{(3 + \sigma_L)(1 + \theta)}; 0 \right)$$

$$e(4): (y^*, v^*, z^*, w^*) = (0; 0; 0; 0)$$

$$e(5): (y^*, v^*, z^*, w^*) = (0; 0; z^*; w^*), \quad z^* + w^* = 1$$

The corner solutions remain identical; however, $e(1)$, $e(2)$ and $e(3)$ now depend solely on two parameters, which are the inferior bound of the transition probability between exuberance and non-exuberance, and the transition probability from non-exuberance back to neutrality. For these three equilibrium points, the share of neutral individuals becomes $x^* = \frac{1+\sigma_L}{5+\sigma_L}$ for $e(1)$, and $x^* = \frac{1+\sigma_L}{3+\sigma_L}$ for $e(2)$ and $e(3)$.

We now evaluate the stability of the equilibria with the above constraints on parameter values.

**Proposition 2.** *Let $\lambda = \sigma_H = 1$. In the sentiment propagation system (5), points $e(2)$, $e(3)$ and $e(4)$ and line $e(5)$ correspond to locally unstable equilibria. The equilibrium point $e(1)$ is stable under condition $\sigma_L > 0.830052443$; otherwise, it is unstable.*

**Proof.** The linearization of (5) in the vicinity of the equilibrium points $(y^*, v^*, z^*, w^*)$ gives the matricial system,

$$\begin{bmatrix} \dot{y} \\ \dot{v} \\ \dot{z} \\ \dot{w} \end{bmatrix} = J \cdot \begin{bmatrix} y - y^* \\ v - v^* \\ z - z^* \\ w - w^* \end{bmatrix} \tag{7}$$

where $J$ is a $4 \times 4$ Jacobian matrix. In the Appendix, we give the Jacobian matrix in its general form for each of the identified equilibria. As is well known, the local stability of each of the equilibria is dependent on the signs of the eigenvalues of matrix $J$; positive real eigenvalues or complex conjugate eigenvalues with positive real parts indicate unstable directions, while negative real eigenvalues or complex conjugate eigenvalues with negative real parts correspond to stable directions. If the eigenvalues are equal to zero, then the system is at a bifurcation point. For each of the equilibria, the four eigenvalues of matrix $J$ are, respectively,

- $e(1)$ :

$$\lambda_{1,2} = -\frac{1}{2}\frac{1}{5+\sigma_L}\left\{(1+\sigma_L)\theta - \chi \pm \sqrt{\chi\left[\chi - 2(1+\sigma_L)\theta\right] - (1+\sigma_L)^2\theta(4-\theta)}\right\}$$

$$\lambda_{3,4} = -\frac{1}{2}\frac{1}{5+\sigma_L}\left\{(1+\sigma_L)\theta - \chi \pm \sqrt{\chi\left[\chi - 2(1+\sigma_L)\theta\right] - (1+\sigma_L)\theta\left[4\left(5+\sigma_L\right) - (1+\sigma_L)\theta\right]}\right\}$$

with $\chi \equiv \frac{2(1-\sigma_L)(1+\sigma_L)\theta}{(5+\sigma_L)^2}\kappa > 0$.

- $e(2), e(3)$ :

$$\lambda_1 = -\frac{1+\sigma_L}{3+\sigma_L}\theta, \quad \lambda_2 = \frac{1+\sigma_L}{3+\sigma_L}$$

$$\lambda_{3,4} = -\frac{1}{2}\frac{1}{3+\sigma_L}\left\{(1+\sigma_L)\theta + \widetilde{\chi} \pm \sqrt{\widetilde{\chi}\left[\widetilde{\chi} + 2(1+\sigma_L)\theta\right] - (1+\sigma_L)\theta\left[4\left(3+\sigma_L\right) - (1+\sigma_L)\theta\right]}\right\}$$

with $\widetilde{\chi} \equiv \frac{4(1-\sigma_L)\theta}{(3+\sigma_L)^2}\kappa > 0$.

- $e(4)$ :

$$\lambda_{1,2} = -\theta; \lambda_{3,4} = 1$$

- $e(5)$ :

$$\lambda_1 = 0; \lambda_2 = -\frac{1}{2}(1+\sigma_L)z^*; \lambda_3 = -\frac{1}{2}(1+\sigma_L)w^*; \lambda_4 = \theta$$

From the signs of the eigenvalues, we conclude that local stability is absent, independent of parameter values, for any equilibria other than $e(1)$, since in each of those cases there is at least one positive real eigenvalue $\forall \sigma_L, \theta \in (0,1]$.

The interior equilibrium, for which exuberance of any of the two considered types exists, $e(1)$, contemplates the possibility of local stability and also the existence of a bifurcation point between stable and unstable outcomes. In this case, local dynamics are as follows: whenever the term inside the square root of $\lambda_{1,2}$ or $\lambda_{3,4}$ is positive, and thus the respective eigenvalues are real, they will display a positive sign and therefore they will represent unstable directions. In the cases in which the terms inside the square roots of both $\lambda_{1,2}$ and $\lambda_{3,4}$ are negative, two pairs of complex conjugate eigenvalues exist and the stability will be contingent on the sign of the real part of the eigenvalues; this is the same for every eigenvalue, namely $-\frac{1}{2}\frac{1}{5+\sigma_L}[(1+\sigma_L)\theta - \chi]$. Therefore stability will prevail under $(1+\sigma_L)\theta - \chi > 0$, which simplifies to $\kappa > \frac{(5+\sigma_L)^2}{2(1-\sigma_L)}$. This last inequality is equivalent to the one stated in the proposition, which specifies a lower bound for parameter $\sigma_L$, under the already selected value $\kappa = 100$. ■

Fig. 3 shows the regions of local stability and instability for the equilibrium point $e(1)$ in $(\theta, \sigma_L)$ space. The line $\sigma_L = 0.830052443$ is the boundary between the stable and unstable regions. The instability region is separated into three different sections by a pair of dashed lines; the upper instability region, between the area of stability and the first dashed curve, corresponds to pairs of values $(\theta, \sigma_L)$ such that the four eigenvalues are two pairs of complex conjugate eigenvalues with positive real parts; the region between the two dashed curves contains points $(\theta, \sigma_L)$ for which there exists a pair of complex conjugate eigenvalues with a positive real part and two positive real eigenvalues; finally, below the lowest dashed line is a region of instability where the four eigenvalues are positive and real.

$$***fig.3***$$

The model's dynamics in the stability region is determined, as highlighted in the proof of proposition 2, by the existence of two pairs of complex conjugate eigenvalues, implying, in this case, the presence of converging oscillating dynamics – an initial state of the system located near the equilibrium point will converge to this spiral focus following an oscillating trajectory.[9]

[9] Recall that parameter value $\kappa = 100$ was adopted. Any other large value of $\kappa$ leads to a similar qualitative result regarding stability properties. A different outcome, though, is attained for low values of $\kappa$ and, in particular, for the limit case $\kappa = 0$ (which implies a unique transition probability $\sigma \equiv \sigma_H = \sigma_L$). In this case, we would have $\chi = 0$ and, as a consequence, equilibrium $e(1)$ would be locally stable regardless of the values of $\sigma$ and $\theta$ (the four eigenvalues are, in this circumstance, two pairs of complex conjugates with negative real parts).

Regardless of the local stability in the vicinity of the equilibrium points, a comprehensive understanding of the dynamics requires an analysis of the global dynamics, which must be done numerically. This analysis is undertaken in the next section and reveals significant portions of parameter space over which a chaotic attractor exists and mimics the endogenous irregular waves of optimism and pessimism typically observed in social systems.

## 4. Global Dynamic Analysis: Nonlinearities and Chaos

System (5) exhibits many regions in the four-dimensional parameter space where the solutions are chaotic. For example, $\lambda = \sigma_H = 1, \sigma_L = 0, \theta = 0.5$ gives the strange attractor shown in various views in Fig. 4 with a Class 3 basin of attraction [25] and in the time domain in Fig. 5. The Lyapunov exponents for this case are $(0.0230, 0, -0.0760, -0.1289)$, and the Kaplan-Yorke dimension is 2.3029. The chaos is robust over the full range of $0 < \theta < 1$ except for several small periodic windows as shown in Fig. 6.

$$***fig.4***$$

$$***fig.5***$$

$$***fig.6***$$

Furthermore, there are many regions where stable limit cycles exist, including small regions where there are two coexisting limit cycles, one example of which for $\lambda = 0.22, \sigma_H = 0.7, \sigma_L = 0.04, \theta = 0.68$ is shown in Fig. 7. For this case, the red attractor is obtained using initial conditions $(0.2, 0.2, 0.2, 0.2)$, and the green attractor is obtained using initial conditions $(0.2, 0.13, 0.4, 0.04)$. Further evidence that these are different limit cycles comes from the Lyapunov exponents whose values are $(0, -0.0050, -0.1338, -0.1387)$ for the former and $(0, -0.0762, -0.0849, -0.0849)$ for the latter.

$$***fig.7***$$

Otherwise, the system is unremarkable. An extensive search of the four-dimensional parameter space did not show any instances of limit cycles or strange attractors coexisting with stable equilibria or with one another, and thus the strange attractor is self-excited rather than hidden [26]. No evidence of hyperchaos (two positive Lyapunov exponents) was found.

The results in this and in the previous section suggest that exuberance is a central source of the identified bounded instability. Chaotic solutions emerge because there is always a share of exuberant optimists and a share of exuberant pessimists that guarantee the perpetual flow of individuals across sentiment categories. Without exuberance, the system would converge to a fixed-point equilibrium of full neutrality and, thus, sentiments would not matter for the decison-making process of agents in the society and in the economy. While the full neutrality scenario is convenient in many research settings where socio-economic events are approached (it is, in fact, the scenario underlying most of the mainstream economic analysis), it avoids, as revealed in our analysis, to understand the implications and consequences of exuberance as a source of instability and fluctuations in market relations (e.g., in financial markets).

## 5. Discussion and an Application

Recent literature on low-dimensional continuous-time dynamic models has unveiled the existence of chaotic solutions in a multiplicity of nonlinear systems. However, many of these models are abstract theoretical constructions that were not intended to address any real phenomena in nature, society, or the economy. By contrast, this paper considers a well defined network of social relations where sentiments of optimism and pessimism propagate through direct contagion. Although extremely simple in its structure, the network and the interaction therein generate complex dynamics, with chaotic solutions being particularly relevant in the sense that they allow for irregular endogenous waves of optimism and pessimism that originate from a set of simple and straightforward interaction rules.

The proposed framework may be employed, for instance, to justify economic processes like systematic changes in consumer and investor confidence and the consequent occurrence of recessions that alternate with periods of macroeconomic recovery. In this context, the discussed framework can be associated with the recent effort in macroeconomics to introduce a substantive behavioral component in the explanation of business cycles. Some of this literature, as De Grauwe [27], promotes the recovery of animal spirits as a key piece of the discussion about business cycles by exploring a mechanism of bounded rationality to frame the formation of expectations, resorting to discrete choice theory; in such a setting, sentiments of optimism and pessimism emerge as a direct reaction to the perceived macro performance. Other authors, like Angeletos and La'O [28], instead, pursue the typical interpretation on business cycles' literature that considers that the source of aggregate fluctuations lies on exogenous shocks (in the case, not shocks on fundamentals but shocks on human sentiments). The second approach complements the first, by recognizing that human interaction beyond the sphere of economics plays a relevant role in shaping animal spirits. We agree with this perspective.

Independently of the strict economic evaluation, agents will switch from optimism to pessimism, and vice-versa, because they are social actors that are constantly communicating and changing their perception of reality as the process of communication unfolds. Most of the 'mood swings' we are subject to go beyond pure economic motivation: they are intrinsic to human nature; it is because of what we are and who we interact with, and not only because of what we consciously choose to do, that our sentiments change. One can consider this an exogenous process, as in the Angeletos-La'O framework, or one can give a step forward and try to capture it through a simple modeling structure like the one we propose. The model, as presented, implies interpreting optimism and pessimism as much more than 'economic moods'; they are attitudes or 'views of the world' that push individuals towards action. Many stories of success and failure in human history are the result of periods of collective euphoria or collective discouragement, and many times these aggregate sentiments are the result of nothing more than the structure of the interactions that are generated in the context of the existing social network.

Is it reasonable to completely detach sentiment fluctuations from economic events? Aren't they jointly determined? It is our conviction that a significant portion of the sentiment changes that influence economic decisions originate elsewhere, outside the realm of economics. This does not mean, however, that they are unimportant to explain economic phenomena. A passage from the book by Akerlof and Shiller [29] helps in advocating our view and in understanding why economists offer so much resistance in recognizing that non-economic forces determine economic events:

*"Economic textbooks are supposed to be about economics, not about psychology, anthropology, sociology, philosophy, or whatever branch of knowledge teaches us about fairness. Those who assign the economics textbooks want to teach their special expertise. Pure economic theory is indisputably valuable in a wide range of applications, and so there is a natural tendency to focus on that magnificent theory – even if it doesn't fit some other very important applications. Focusing exclusively on the rational theory leads to an elegant presentation. It would violate the etiquette of the textbooks to mention that some other factor, outside the formal discipline of economics, is the fundamental cause of certain major economic phenomena. It would be like burping loudly at a fancy dinner. It is just not done."* (p. 21).

To summarize, once we contemplate the possibility of interaction among agents with different sentiments, the eventuality of sentiment contagion and the potential existence of a class of exuberant individuals with peculiar behavioral features, we open the door for a long-term equilibrium where each agent systematically changes from one sentiment state to another in such a way that aggregate waves of optimism and pessimism emerge and persist. These systematic fluctuations influence the decisions of agents, and thus they will necessarily be passed on to the economy. This does not mean that shocks on fundamentals or feedback economic processes should be treated as irrelevant; it means that they impact an economy that is already subject to an underlying fluctuating structure that has its origins in the propagation of human sentiments.

## Appendix - Jacobian Matrix

To derive the transition probability functions (2) and (3), the derivatives of each of the functions with respect to each endogenous variable are given as follows:

$$\frac{\partial\sigma(opt)}{\partial y} = -\frac{\sigma_H-\sigma_L}{2}\frac{\partial\tanh\left[\kappa(\lambda xy-\theta zx)\right]}{\partial y} = -\frac{\sigma_H-\sigma_L}{2}\frac{\frac{\partial[\kappa(\lambda xy-\theta zx)]}{\partial y}}{\cosh^2\left[\kappa(\lambda xy-\theta zx)\right]}$$
$$= -\frac{\sigma_H-\sigma_L}{2}\frac{\kappa\left\{\lambda\left[1-(v+w)\right]-2\lambda y-(\lambda-\theta)z\right\}}{\cosh^2\left[\kappa(\lambda xy-\theta zx)\right]}$$

$$\frac{\partial\sigma(opt)}{\partial v} = -\frac{\sigma_H-\sigma_L}{2}\frac{\partial\tanh\left[\kappa(\lambda xy-\theta zx)\right]}{\partial v} = -\frac{\sigma_H-\sigma_L}{2}\frac{\frac{\partial[\kappa(\lambda xy-\theta zx)]}{\partial v}}{\cosh^2\left[\kappa(\lambda xy-\theta zx)\right]}$$
$$= \frac{\sigma_H-\sigma_L}{2}\frac{\kappa\left(\lambda y-\theta z\right)}{\cosh^2\left[\kappa(\lambda xy-\theta zx)\right]}$$

$$\frac{\partial\sigma(opt)}{\partial z} = -\frac{\sigma_H-\sigma_L}{2}\frac{\partial\tanh\left[\kappa(\lambda xy-\theta zx)\right]}{\partial z} = -\frac{\sigma_H-\sigma_L}{2}\frac{\frac{\partial[\kappa(\lambda xy-\theta zx)]}{\partial z}}{\cosh^2\left[\kappa(\lambda xy-\theta zx)\right]}$$
$$= \frac{\sigma_H-\sigma_L}{2}\frac{\kappa\left\{\theta\left[1-(v+w)\right]-2\theta z+(\lambda-\theta)y\right\}}{\cosh^2\left[\kappa(\lambda xy-\theta zx)\right]}$$

$$\frac{\partial\sigma(opt)}{\partial w} = -\frac{\sigma_H-\sigma_L}{2}\frac{\partial\tanh\left[\kappa(\lambda xy-\theta zx)\right]}{\partial w} = -\frac{\sigma_H-\sigma_L}{2}\frac{\frac{\partial[\kappa(\lambda xy-\theta zx)]}{\partial w}}{\cosh^2\left[\kappa(\lambda xy-\theta zx)\right]}$$
$$= \frac{\sigma_H-\sigma_L}{2}\frac{\kappa\left(\lambda y-\theta z\right)}{\cosh^2\left[\kappa(\lambda xy-\theta zx)\right]}$$

$$\frac{\partial\sigma(pes)}{\partial y} = -\frac{\sigma_H-\sigma_L}{2}\frac{\partial\tanh\left[\kappa(\lambda xv-\theta wx)\right]}{\partial y} = -\frac{\sigma_H-\sigma_L}{2}\frac{\frac{\partial[\kappa(\lambda xv-\theta wx)]}{\partial y}}{\cosh^2\left[\kappa(\lambda xv-\theta wx)\right]}$$
$$= \frac{\sigma_H-\sigma_L}{2}\frac{\kappa\left(\lambda v-\theta w\right)}{\cosh^2\left[\kappa(\lambda xv-\theta wx)\right]}$$

$$\frac{\partial\sigma(pes)}{\partial v} = -\frac{\sigma_H-\sigma_L}{2}\frac{\partial\tanh\left[\kappa(\lambda xv-\theta wx)\right]}{\partial v} = -\frac{\sigma_H-\sigma_L}{2}\frac{\frac{\partial[\kappa(\lambda xv-\theta wx)]}{\partial v}}{\cosh^2\left[\kappa(\lambda xv-\theta wx)\right]}$$
$$= -\frac{\sigma_H-\sigma_L}{2}\frac{\kappa\left\{\lambda\left[1-(y+z)\right]-2\lambda v-(\lambda-\theta)w\right\}}{\cosh^2\left[\kappa(\lambda xv-\theta wx)\right]}$$

$$\frac{\partial\sigma(pes)}{\partial z} = -\frac{\sigma_H-\sigma_L}{2}\frac{\partial\tanh\left[\kappa(\lambda xv-\theta wx)\right]}{\partial z} = -\frac{\sigma_H-\sigma_L}{2}\frac{\frac{\partial[\kappa(\lambda xv-\theta wx)]}{\partial z}}{\cosh^2\left[\kappa(\lambda xv-\theta wx)\right]}$$
$$= \frac{\sigma_H-\sigma_L}{2}\frac{\kappa\left(\lambda v-\theta w\right)}{\cosh^2\left[\kappa(\lambda xv-\theta wx)\right]}$$

$$
\begin{aligned}
\frac{\partial\sigma(pes)}{\partial w} &= -\frac{\sigma_H-\sigma_L}{2}\frac{\partial \tanh\left[\kappa(\lambda x v-\theta w x)\right]}{\partial w} = -\frac{\sigma_H-\sigma_L}{2}\frac{\frac{\partial[\kappa(\lambda x v-\theta w x)]}{\partial w}}{\cosh^2\left[\kappa(\lambda x v-\theta w x)\right]} \\
&= \frac{\sigma_H-\sigma_L}{2}\frac{\kappa\left\{\lambda\left[1-(y+z)\right]-2\theta w+(\lambda-\theta)v\right\}}{\cosh^2\left[\kappa(\lambda x v-\theta w x)\right]}
\end{aligned}
$$

To evaluate the above derivatives at the equilibrium position, note that $\lambda x^* y^* - \theta z^* x^* = \lambda x^* v^* - \theta w^* x^* = 0$ and $\cosh(0) = 1$. Therefore,

$$
\begin{aligned}
\sigma_y^*(opt) &\equiv \left.\frac{\partial\sigma(opt)}{\partial y}\right|_{(y^*,v^*,z^*,w^*)} \\
&= -\frac{\sigma_H-\sigma_L}{2}\kappa\left\{\lambda\left[1-(v^*+w^*)\right]-2\lambda y^*-(\lambda-\theta)z^*\right\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_v^*(opt) &\equiv \left.\frac{\partial\sigma(opt)}{\partial v}\right|_{(y^*,v^*,z^*,w^*)} \\
&= \frac{\sigma_H-\sigma_L}{2}\kappa\left(\lambda y^*-\theta z^*\right)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_z^*(opt) &\equiv \left.\frac{\partial\sigma(opt)}{\partial z}\right|_{(y^*,v^*,z^*,w^*)} \\
&= \frac{\sigma_H-\sigma_L}{2}\kappa\left\{\theta\left[1-(v^*+w^*)\right]-2\theta z^*+(\lambda-\theta)y^*\right\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_w^*(opt) &\equiv \left.\frac{\partial\sigma(opt)}{\partial w}\right|_{(y^*,v^*,z^*,w^*)} \\
&= \frac{\sigma_H-\sigma_L}{2}\kappa\left(\lambda y^*-\theta z^*\right)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_y^*(pes) &\equiv \left.\frac{\partial\sigma(pes)}{\partial y}\right|_{(y^*,v^*,z^*,w^*)} \\
&= \frac{\sigma_H-\sigma_L}{2}\kappa\left(\lambda v^*-\theta w^*\right)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_v^*(pes) &\equiv \left.\frac{\partial\sigma(pes)}{\partial v}\right|_{(y^*,v^*,z^*,w^*)} \\
&= -\frac{\sigma_H-\sigma_L}{2}\kappa\left\{\lambda\left[1-(y^*+z^*)\right]-2\lambda v^*-(\lambda-\theta)w^*\right\}
\end{aligned}
$$

$$\sigma_z^*(pes) \equiv \left.\frac{\partial \sigma(pes)}{\partial z}\right|_{(y^*,v^*,z^*,w^*)} = \frac{\sigma_H - \sigma_L}{2}\kappa\left(\lambda v^* - \theta w^*\right)$$

$$\sigma_w^*(pes) \equiv \left.\frac{\partial \sigma(pes)}{\partial w}\right|_{(y^*,v^*,z^*,w^*)} = \frac{\sigma_H - \sigma_L}{2}\kappa\left\{\theta\left[1-(y^*+z^*)\right] - 2\theta w^* + (\lambda-\theta)v^*\right\}$$

The above derivatives are required for the computation of a general Jacobian matrix for system (5). Matrix $J$ will be a $4 \times 4$ square matrix containing the following sixteen elements:

$$j_{11} = \lambda(1 - 2y^* - z^* - v^* - w^*) - \overline{\sigma}(2y^* + z^*) - \sigma_y^*(opt)y^*\left(y^* + z^*\right)$$
$$j_{12} = -\lambda y^* - \sigma_v^*(opt)y^*\left(y^* + z^*\right)$$
$$j_{13} = -(\lambda + \overline{\sigma})y^* - \sigma_z^*(opt)y^*\left(y^* + z^*\right)$$
$$j_{14} = -\lambda y^* - \sigma_w^*(opt)y^*\left(y^* + z^*\right)$$
$$j_{21} = -\lambda v^* - \sigma_y^*(pes)v^*\left(v^* + w^*\right)$$
$$j_{22} = \lambda(1 - y^* - z^* - 2v^* - w^*) - \overline{\sigma}(2v^* + w^*) - \sigma_v^*(pes)v^*\left(v^* + w^*\right)$$
$$j_{23} = -\lambda v^* - \sigma_z^*(pes)v^*\left(v^* + w^*\right)$$
$$j_{24} = -(\lambda + \overline{\sigma})v^* - \sigma_w^*(pes)v^*\left(v^* + w^*\right)$$
$$j_{31} = \overline{\sigma}(2y^* + z^*) + \theta z^* + \sigma_y^*(opt)y^*\left(y^* + z^*\right)$$
$$j_{32} = \theta z^* + \sigma_v^*(opt)y^*\left(y^* + z^*\right)$$
$$j_{33} = \overline{\sigma}y^* - \theta(1 - y^* - 2z^* - v^* - w^*) + \sigma_z^*(opt)y^*\left(y^* + z^*\right)$$
$$j_{34} = \theta z^* + \sigma_w^*(opt)y^*\left(y^* + z^*\right)$$
$$j_{41} = \theta w^* + \sigma_y^*(pes)v^*\left(v^* + w^*\right)$$
$$j_{42} = \overline{\sigma}(2v^* + w^*) + \theta w^* + \sigma_v^*(pes)v^*\left(v^* + w^*\right)$$
$$j_{43} = \theta w^* + \sigma_z^*(pes)v^*\left(v^* + w^*\right)$$
$$j_{44} = \overline{\sigma}v^* - \theta(1 - y^* - z^* - v^* - 2w^*) + \sigma_w^*(pes)v^*\left(v^* + w^*\right)$$

The Jacobian matrix will then take a particular form for each of the equilibrium points and under the imposed condition $\lambda = \sigma_H = 1$:

- $e(1)$ :

$$J = \frac{1}{(5+\sigma_L)(1+\theta)} \begin{bmatrix} -(3+\sigma_L)\theta+\chi & -2\theta & -(3+\sigma_L)\theta-\chi\theta & -2\theta \\ -2\theta & -(3+\sigma_L)\theta+\chi & -2\theta & -(3+\sigma_L)\theta-\chi\theta \\ 1+\sigma_L+2(2+\sigma_L)\theta-\chi & 2\theta & [2-(1+\sigma_L)\theta]\theta+\chi\theta & 2\theta \\ 2\theta & 1+\sigma_L+2(2+\sigma_L)\theta-\chi & 2\theta & [2-(1+\sigma_L)\theta]\theta+\chi\theta \end{bmatrix}$$

with $\chi$ as defined in the proof of proposition 2.

- $e(2)$ :

$$J = \frac{1}{(3+\sigma_L)(1+\theta)} \begin{bmatrix} (1+\sigma_L)(1+\theta) & 0 & 0 & 0 \\ -2\theta & -(3+\sigma_L)\theta-\widetilde{\chi} & -2\theta & -(3+\sigma_L)\theta+\widetilde{\chi}\theta \\ 0 & 0 & -(1+\sigma_L)(1+\theta)\theta & 0 \\ 2\theta & 1+\sigma_L+2(2+\sigma_L)\theta+\widetilde{\chi} & 2\theta & [2-(1+\sigma_L)\theta]\theta-\widetilde{\chi}\theta \end{bmatrix}$$

with $\widetilde{\chi}$ as defined in the proof of proposition 2.

- $e(3)$ :

$$J = \frac{1}{(3+\sigma_L)(1+\theta)} \begin{bmatrix} -(3+\sigma_L)\theta-\widetilde{\chi} & -2\theta & -(3+\sigma_L)\theta+\widetilde{\chi}\theta & -2\theta \\ 0 & (1+\sigma_L)(1+\theta) & 0 & 0 \\ 1+\sigma_L+2(2+\sigma_L)\theta+\widetilde{\chi} & 2\theta & [2-(1+\sigma_L)\theta]\theta-\widetilde{\chi}\theta & 2\theta \\ 0 & 0 & 0 & -(1+\sigma_L)(1+\theta)\theta \end{bmatrix}$$

with $\widetilde{\chi}$ as defined in the proof of proposition 2.

- $e(4)$ :

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\theta & 0 \\ 0 & 0 & 0 & -\theta \end{bmatrix}$$

- $e(5)$ :

$$J = \begin{bmatrix} -\frac{1+\sigma_L}{2} z^* & 0 & 0 & 0 \\ 0 & -\frac{1+\sigma_L}{2} w^* & 0 & 0 \\ \left(\frac{1+\sigma_L}{2} + \theta\right) z^* & \theta z^* & \theta z^* & \theta z^* \\ \theta w^* & \left(\frac{1+\sigma_L}{2} + \theta\right) w^* & \theta w^* & \theta w^* \end{bmatrix}$$

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