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Hypogenetic chaotic jerk flows

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ABSTRACT

Removing the amplitude or polarity information in the feedback loop of a jerk structure shows that special nonlinearities with partial information in the variable can also lead to chaos. Some striking properties are found for this kind of hypogenetic chaotic jerk flow, including multistability of symmetric coexisting attractors from an asymmetric structure, hidden attractors with respect to equilibria but with global attraction, easy amplitude control, and phase reversal which is convenient for chaos applications.

1. Introduction

Jerk dynamical systems have the form $\dot{x} = y$, $\dot{y} = z$, $\dot{z} = f(x, y, z)$, which can be depicted as $\dddot{x} = f(x, \dot{x}, \ddot{x})$, and therefore can be realized with a compact electrical circuit structure [1–5]. Generally, jerk circuits have four connections at the node x, where the derivative of x is determined by the amplitude and polarity of y, and the amplitude and polarity of x influence the derivative of z. In this paper, we study those chaotic flows with incomplete information transmission from the node x based on the jerk structure, and are therefore named hypogenetic chaotic jerk flows with incomplete feedback loops since the modified systems are not always conventional jerk flows.

Generally symmetric systems have a symmetric pair of coexisting attractors [6–8] and some asymmetric systems give asymmetric multistability [9,10], hidden attractors with composite basins of attraction [11–15], and amplitude control [16–18] which is realized in the circuit by a single variable resistor. Our exploration shows that some cases of hypogenetic chaotic jerk flows may have coexisting symmetric and asymmetric strange attractors. Hidden attractors from systems without equilibria may have global attraction [19]. Experimentally, amplitude control can be achieved by varying a resistor and capacitor while not altering the frequency of the signals. In Section 2, we propose such hypogenetic chaotic jerk flows based on three or two connections at the node x. In Section 3, the property of special multistability and hidden attractors with global attraction is explored. In Section 4, amplitude control and phase reversal are discussed. Conclusions are given in the last section.

2. Hypogenetic chaotic jerk flows

2.1. Hypogenetic jerk flows with three connections

A jerk system has inherent connections among its variables. For a conventional jerk system, the derivative of x is determined by the variable y, and further influences the derivative of the variable z. Typically, the derivative of the variable x is determined by the variable y through a simple governing equation $\dot{x} = y$; that is to say the derivative of the variable x is associated with the amplitude evolution and the alternate polarity change of the variable y. All the feedback information of amplitude and polarity from the variables x, y and z usually determine the time derivative of the variable z. In view of this jerk structure, since the derivative of the variable x is determined by the variable y and impacts the derivative of variable z, therefore the variable x can be regarded as an intermediate variable absorbing the amplitude and polarity information from the variable y and transmitting them to the variable z. The variable x connects the other two variables y and z, making all the nodes active in a whole feedback system.

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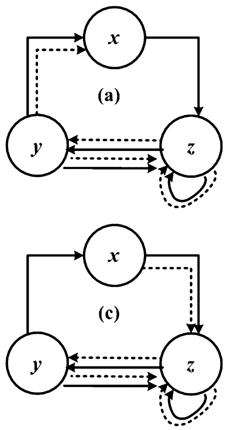

Fig. 1. Structure of hypogenetic jerk flows with three connections.

As mentioned above, each variable has two kinds of information, *i.e.* amplitude and polarity. Therefore a standard jerk system has four connections at the node of variable x. However, the "arm" of the variable x may not be so strong, which means that the intermediate variable x takes the amplitude and/or polarity information from the variable y and hands it to the variable z with incomplete information. Consequently, those hypogenetic jerk flows with incomplete information transmission can be classified in two categories, one of which has three connections, and the other one has two connections, while for the regular jerk systems there are four connections at the node of the variable x.

In Fig. 1, where solid lines mean amplitude connections while dotted lines indicate polarity connections, when the amplitude or polarity information is absent in the dynamical loop and the variable x only has three connections with other nodes, there can exist four structures for chaotic jerk systems. An existing example is the chaotic system proposed by Linz and Sprott [1], which could be the simplest chaotic system of finite size with an absolute value nonlinearity. The system is given by $\dddot{x} = -a\ddot{x} - \dot{x} + |x| - 1$, and corresponds to the structure of Fig. 1(a). Some other cases associated with the structures in Fig. 1(b)–(d) are shown in Table 1. In contrast to case HJ1, the node in case HJ2 of variable x gets the whole information of amplitude and polarity from the node of variable y, and only influences the node of variable z with its polarity information. In case HJ3, the node of variable x only gets the amplitude from the node of variable y, and influences the node of variable z with its whole information, while in case HJ4, the node of variable x only gets the polarity information from the node of variable y.

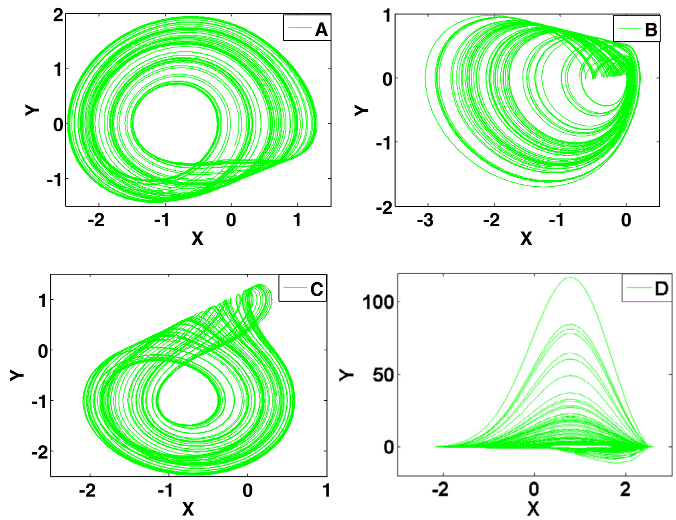

Fig. 2. Chaotic attractors in hypogenetic jerk systems with three connections, A: case HJ1 with initial condition $(-2, 0, 1)$, B: case HJ2 with initial condition $(1, 0, -1)$, C: case HJ3 with initial condition $(2, 0, -1)$, D: case HJ4 with initial condition $(3, 2, -1)$.

Note that for case HJ3, the derivative of variable x gets only the absolute value information from the variable y, and thus in the absence of the additional -1 term, the system would be unbounded unless y is identically zero, in which case x would be constant. Consequently, there is no system of that form that gives chaos. However, as with the Linz–Sprott jerk system, adding the constant -1 in system HJ3 gives chaotic solutions.

By the same method, we can design other systems using feedback of the amplitude to retain the chaos. We cannot obtain a chaotic jerk system of case HJ4 by slightly modifying the Linz–Sprott jerk system, where the node of variable x only gets the polarity information from the node of variable y. To find chaotic solutions with the form of the specified nonlinear functions shown in Eq. (1), a systematic numerical search procedure developed in [5,19,20] was employed. In this procedure, the space of control parameters embedded in the $\dot{z}$ equation and initial conditions were scanned to find a positive Lyapunov exponent, which is a signature of chaos. By this method, many more such chaotic systems were found, some simple examples of which are given in Table 1 along with the numerically calculated Lyapunov exponents and Kaplan–Yorke dimensions. To limit the complexity of the examples, we consider only quadratic nonlinearities. Some other system features are also listed in Table 1 (see Fig. 2).

Table 1
Hypogenetic chaotic jerk flows with three connections.

Model	Equations	Parameters	Equilibria	x_0 y_0 z_0	LEs	D_{KY}
HJ1	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = b\,\lvert x\rvert - y - az - 1$	$a = 0.6$ $b = 1$	(1, 0, 0) (−1, 0, 0)	−2 0 1	0.0156 0 −0.6156	2.0253
HJ2	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -b * \mathrm{sgn}(x) - y - az - 1$	$a = 0.6$ $b = 1.25$	none	1 0 −1	0.0605 0 −0.6605	2.0916
HJ3	$\dot{x} = \lvert y\rvert - 1$ $\dot{y} = z$ $\dot{z} = x - by - az$	$a = 0.6$ $b = 1$	(1, 1, 0) (−1, −1, 0)	2 0 −1	0.0305 0 −0.6305	2.0484
HJ4	$\dot{x} = \mathrm{sgn}\,(y)$ $\dot{y} = z$ $\dot{z} = -ay - x - bxy - xz$	$a = 0.2$ $b = 1.5$	(0, 0, 0)	3 2 −1	0.0429 0 −0.6372	2.0673

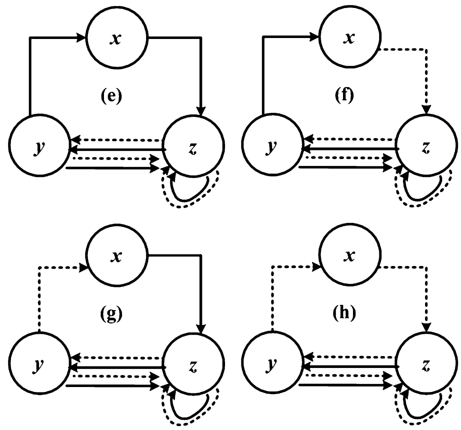

Fig. 3. Structure of hypogenetic jerk flows with two connections.

$$\begin{cases} \dot{x} = \operatorname{sgn}(y), \\ \dot{y} = z, \\ \dot{z} = a_1 y + a_2 z + a_3 y^2 + a_4 z^2 \\ \qquad + a_5 yz + a_6 x + a_7 xy + a_8 xz + a_9. \end{cases} \tag{1}$$

2.2. Hypogenetic jerk flows with two connections

Each variable has two kinds of information; the intermediate variable x may only possess two connections with the other two nodes in the closed-loop structure, where the node x absorbs the amplitude or polarity information from the variable y and only passes one of them to the node z. According to the structure of the information feedback loop in Fig. 3, we get four example cases HJ5, HJ6, HJ7, and HJ8, in which the derivative of variable x gets the polarity or amplitude information from the variable y. In case HJ5, the node of variable x gets the amplitude information from the node y, and only gives the amplitude information to the node z; in case HJ6, the node x gets the amplitude from the node y, and influences the node z with its polarity information; in case HJ7, the node x only gets the polarity information from the node y and passes its amplitude information to the node z; in case HJ8, the node x collects and transfer the polarity information.

For the same reason that non-negative absolute values lead to an increasing dissipation, in case HJ5 and HJ6, we need to introduce a new constant to balance the divergence and convergence in the node x. In this way, we get a simple example of case HJ5 from the Linz–Sprott jerk system, listed in Table 2. Furthermore, we can obtain the case HJ6 with a direct signum operation on the variable x. For the case HJ7 and HJ8, we employed a similar search from Eqs. (2) and (3). The corresponding strange attractors are shown in Fig. 4.

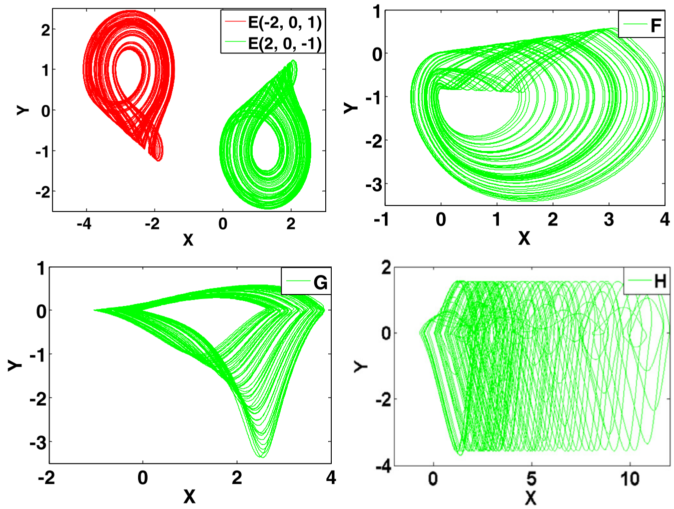

Fig. 4. Chaotic attractors in hypogenetic jerk systems with two connections: E: case HJ5 with initial condition $(\pm 2, 0, \mp 1)$, F: case HJ6 with initial condition $(-2, 0, 1)$, G: case HJ7 with initial condition $(3, -2, 2)$, H: case HJ8 with initial condition $(1, -2, -1)$.

$$\begin{cases} \dot{x} = \operatorname{sgn}(y), \\ \dot{y} = z, \\ \dot{z} = a_1 y + a_2 z + a_3 y^2 + a_4 z^2 + a_5 yz + a_6 |x| + a_7 x^2 + a_8, \end{cases} \tag{2}$$

$$\begin{cases} \dot{x} = \operatorname{sgn}(y), \\ \dot{y} = z, \\ \dot{z} = a_1 y + a_2 z + a_3 y^2 + a_4 z^2 + a_5 yz + a_6 \operatorname{sgn}(x) \\ \qquad + a_7 \operatorname{sgn}(x) y + a_8 \operatorname{sgn}(x) z + a_9. \end{cases} \tag{3}$$

3. Unique multistability and hidden attractors

Even for the case with simple absolute value nonlinearities, there can be some interesting multistability. System HJ4 is such a special asymmetric system, which gives different regimes of multistability. When $b = 1$, the system has two coexisting almost symmetric attractors, as shown in Fig. 4(E), whose basin of attraction are shown in Fig. 5. Although the regions in light blue and red representing two different attractor basins are asymmetric, the strange attractors, represented in cross section by black lines, are symmetric and nearly touch their basin boundaries. Two of the

Table 2
Hypogenetic chaotic jerk flows with two connections.

Model	Equations	Parameters	Equilibria	x_0 y_0 z_0	LEs	D_{KY}
HJ5	$\dot{x} = \lvert y\rvert - b$ $\dot{y} = z$ $\dot{z} = \lvert x\rvert - y - az - c$	$a = 0.6$ $b = 1$ $c = 2$	$(\pm 3, 1, 0)$ $(\pm 1, -1, 0)$	± 2 0 ∓ 1	0.0534 0 −0.6534	2.0817
HJ6	$\dot{x} = \lvert y\rvert - 1$ $\dot{y} = z$ $\dot{z} = b * \operatorname{sgn}(x) - y - az - c$	$a = 0.65$ $b = 1.5$ $c = 2$	none	−2 0 1	0.0419 0 −0.6919	2.0606
HJ7	$\dot{x} = \operatorname{sgn}(y)$ $\dot{y} = z$ $\dot{z} = -z + ay^2 - z^2 - b\lvert x\rvert + c$	$a = 2.1$ $b = 0.54$ $c = 0.365$	$(\pm 0.6759, 0, 0)$	3 −2 2	0.0299 0 −1.0299	2.0290
HJ8	$\dot{x} = \operatorname{sgn}(y)$ $\dot{y} = z$ $\dot{z} = -ay - bz^2 + yz + \operatorname{sgn}(x)$	$a = 3$ $b = 0.4$	$(0, 0, 0)$	1 −2 −1	0.0926 0 −0.5052	2.1833

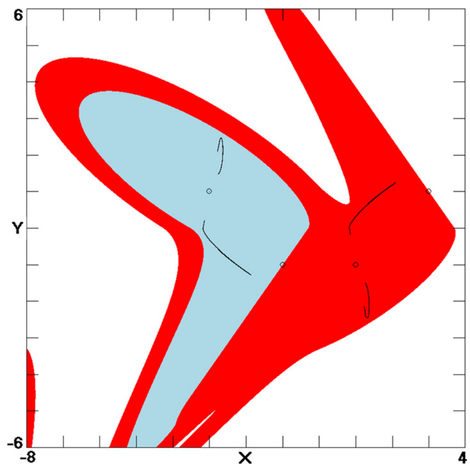

Fig. 5. Cross section for $z = 0$ of the basins of attraction for the symmetric strange attractors (light blue and red) of system HJ5 at $a = 0.6$, $b = 1$, $c = 2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

equilibrium points are within the attracting basins, while the other two are on the basin boundary.

Suppose $x = u + d$, $y = -v$, $z = -w$ (here d is another constant). If $c = (|x| + |u + d|)/2$ the new deduced system has a conditional rotational symmetry, which can be proved by the transformation $\dot{u} = |v| - b$, $\dot{v} = w$, $\dot{w} = -|u + d| - v - aw + c$. We see that in the variables u, v, w the equations are conditionally identical to the original case HJ5 system if $c - |u + d| = |x| - c$. Here the symmetry depends on the condition equation in the z dimension. We call this a degenerate conditional symmetry since the variable x should be pre-boosted with a constant shift.

Furthermore, there are other cases of asymmetric multistability. As shown in Fig. 6, when $b = 1.5$ and $b = 1.25$, one strange attractor coexists with another strange attractor or with two asymmetric limit cycles, whose attracting basins are shown in Fig. 7. Here the black lines or dots are asymmetric. For the HJ1 and HJ7 systems, since the absolute value of variable x is in the z dimension, a negative operation in the x dimension may realize a phase reversal of the variable x.

Note that systems HJ2 and HJ6 have hidden attractors since they have no equilibrium points, but both strange attractors are globally attracting [19]. The z dimension is $\dot{z} = -b\,\mathrm{sgn}(x) - y - az - 1$ in HJ2, and it is $\dot{z} = b\,\mathrm{sgn}(x) - y - az - c$ in HJ6. The disappearance of the equilibria is a result of the signum operation. Systems HJ4 and HJ8 have one equilibrium point, HJ1, HJ3, and HJ7 have two equilibrium points, and HJ5 has four equilibrium points, but none of those systems are globally attracting, and they have initial conditions that give unbounded solutions.

4. Amplitude control and phase reversal

Since most of the cases give chaos by the absolute value or a signum nonlinearity, it is relatively convenient to obtain total amplitude control, for instance using the constant in systems HJ1 and HJ3. System HJ3 is a special system for its freedom of amplitude control. We see that setting $x = u + c$ will introduce in the z dimension a new constant c which will change the average value of the variable x. Thus it is convenient to offset the bipolar signal x by a unipolar DC voltage in the circuit, or vice versa. It is important to revise the initial conditions to guarantee the shifted attractors are in the corresponding basins of attraction. Some other unique cases with the same property of variable boosting can be found in the Sprott J, L, M, N, P and S systems [21]. Note that HJ3 is a system with a striking property for amplitude control. The constant in

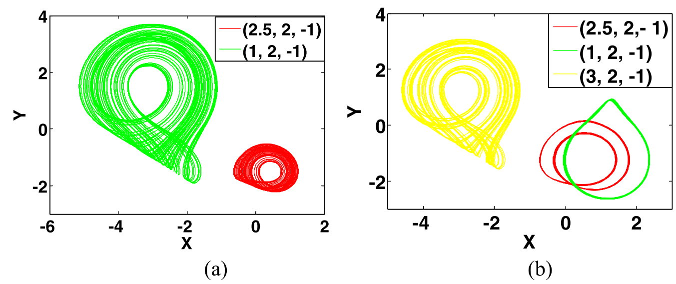

Fig. 6. Asymmetric coexisting attractors for system HJ5 (a) two coexisting strange attractors for $b = 1.5$, (b) three coexisting attractors for $b = 1.25$.

the x dimension is a total amplitude controller. Therefore the system has the freedoms of amplitude control and variable boosting, which are convenient for engineering applications.

Some coefficients can control the amplitude of two variables. The system HJ4 can realize such partial amplitude control for the variables y and z. As shown in the following equations, a new introduced coefficient n in the z dimension is such a partial amplitude controller,

$$\dot{x} = \mathrm{sgn}(y)$$
$$\dot{y} = z$$
$$\dot{z} = -ay - nx - bxy - xz. \tag{4}$$

To show this, let $x = u$, $y = nv$, $z = nw$ to obtain new equations in the variables u, v, w that are identical to the system HJ4. Therefore, the coefficient n controls the amplitude of variables y and z according to n. The signum operation in the x dimension makes the derivative of variable x independent of the magnitude of the variable y.

For circuit implementation, besides the traditional amplitude control, some cases have special partial amplitude control where the time constant provides amplitude control, such as systems HJ2 and HJ8. The time constant in the x dimension controls the amplitude of the variable x because its amplitude is removed by the signum operation without influencing the balance in the z dimension. As shown in the following equations, a new introduced coefficient m in the x dimension is a partial amplitude controller for the variable x,

$$\dot{x} = my$$
$$\dot{y} = z$$
$$\dot{z} = -b * \mathrm{sgn}(x) - y - az - 1. \tag{5}$$

To show this, let $x = mu$, $y = v$, $z = w$ to obtain new equations in the variables u, v, w that are identical to system HJ2. Therefore, the coefficient m controls the amplitude of variable x according to m. Note that here the coefficient m in the x dimension can be realized in a circuit with a resistor or a capacitor, which means that the time constant in the first integration circuit controls the amplitude of variable x. Usually, the capacitor controls the time scale or frequency of the variable. Here the capacitor is another amplitude scaler. For system HJ8, there is a similar transformation to give partial amplitude control.

$$\dot{x} = m * \mathrm{sgn}(y)$$
$$\dot{y} = z$$
$$\dot{z} = -ay - bz^2 + yz + \mathrm{sgn}(x). \tag{6}$$

The coefficient m in the x dimension also provides a time constant in the physical circuit, which means both the capacitor and the resistor in the x dimension determine the size of the variable x.

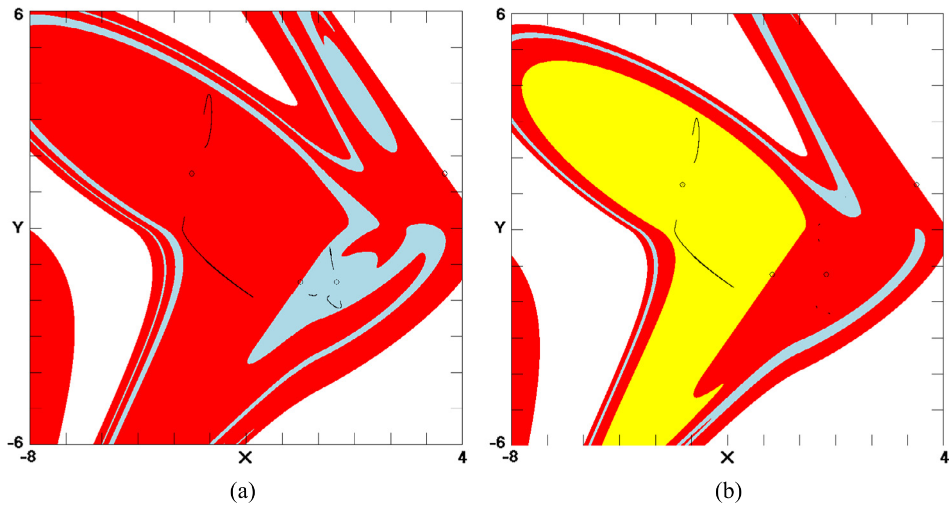

Fig. 7. Cross section for $z = 0$ of the basins of attraction for the asymmetric attractors of system HJ5 at (a) $a = 0.6$, $b = 1.5$, $c = 2$, light blue and red for two coexisting attractors (b) $a = 0.6$, $b = 1.25$, $c = 2$, yellow, light blue and red for three coexisting attractors. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Almost all of the cases have their mirror image versions, whose polarities are reversed from the original ones. The image equations can be deduced easily by a transformation $x = -u$, $y = -v$, $z = -w$. For example, the image system for system HJ2 is $\dot{u} = v$, $\dot{v} = w$, $\dot{w} = -b * \mathrm{sgn}(u) - v - aw + 1$. We see that an opposite constant changes the polarities of all three variables. For system HJ3, the image system only needs to adjust the first dimension x to be $\dot{x} = 1 - |\ y\ |$. For system HJ1, the z dimension must be revised as $\dot{z} = 1 - b\,|x| - y - az$. For system HJ4, the z dimension needs to be revised as $\dot{z} = -ay - x + bxy + xz$. For system HJ8, the z dimension needs to be revised as $\dot{z} = -ay + bz^2 - yz + \mathrm{sgn}(x)$. Image versions of other cases are obtained by changing the sign of a few of the terms. Phase trajectories from the image systems are shown in Fig. 8.

All of the cases can be realized with a common three dimensional circuit structure. Some of them can be implemented in a traditional compact jerk electrical circuit. For example, system HJ1 can be transformed into $\dddot{x} = b\,|x| - \dot{x} - a\ddot{x} - 1$, system HJ2 can be transformed into $\dddot{x} = -b * \mathrm{sgn}(x) - \dot{x} - a\ddot{x} - 1$ and system HJ3 can be transformed into $\dddot{y} = |y| - b\dot{y} - a\ddot{y} - 1$. The circuit implementation for hypogenetic chaotic jerk flows is being considered for future work.

5. Conclusions

A new class of hypogenetic jerk flow with weak feedback is explored. By the incomplete information transportation with an absolute value function or a signum operation, eight cases of hypogenetic chaotic jerk flows are obtained. It turns out that the absolute value function and the signum operation still have a strong nonlinearity, and therefore preserve the chaos. Multistability of coexisting symmetric attractors in an asymmetric system is found, and unique parameters for amplitude control are described, which can be realized in an electrical circuit with a resistor or a capacitor. Moreover, simple polarity selection by absolute value and signum operations makes it easy to revise the polarity of chaotic signals in circuits.

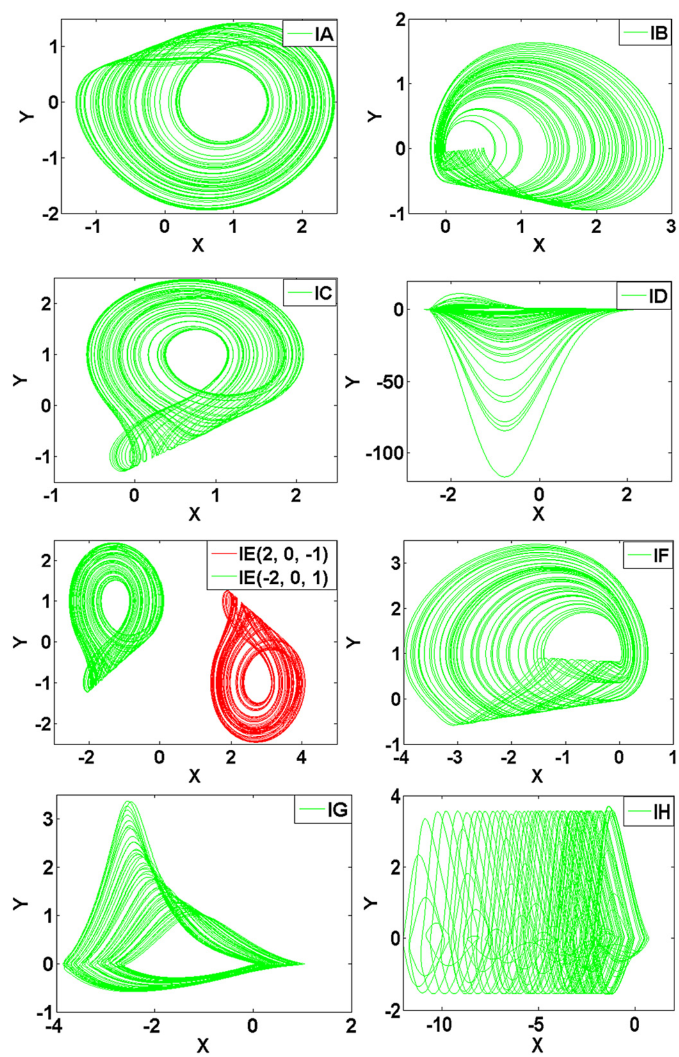

Fig. 8. Phase trajectories in image hypogenetic jerk flows.

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References

[1] Stefan J. Linz, J.C. Sprott, Phys. Lett. A 259 (1999) 240.
[2] J.C. Sprott, Phys. Lett. A 266 (2000) 19.
[3] J.C. Sprott, IEEE Trans. Circuits Syst. II, Express Briefs 58 (2011) 240.
[4] J.C. Sprott, Phys. Lett. A 228 (1997) 271.
[5] J.C. Sprott, Am. J. Phys. 65 (1997) 537.
[6] C. Li, J.C. Sprott, Int. J. Bifurc. Chaos 23 (2013) 1350199.
[7] C. Li, W. Hu, J.C. Sprott, X. Wang, Eur. Phys. J. Spec. Top. 224 (2015) 1493.
[8] C. Li, J.C. Sprott, Phys. Lett. A 378 (2014) 178.
[9] J.C. Sprott, X. Wang, G. Chen, Int. J. Bifurc. Chaos 23 (2013) 1350093.
[10] R. Barrio, F. Blesa, S. Serrano, Physica D 238 (2009) 1087.
[11] G.A. Leonov, V.I. Vagaitsev, N.V. Kuznetsov, Phys. Lett. A 375 (2011) 2230.
[12] G.A. Leonov, V.I. Vagaitsev, N.V. Kuznetsov, Physica D 241 (2012) 1482.
[13] G.A. Leonov, N.V. Kuznetsov, Int. J. Bifurc. Chaos 23 (2013) 1330002.
[14] G.A. Leonov, N.V. Kuznetsov, T. N Mokaev, Eur. Phys. J. Spec. Top. 224 (2015) 1421.
[15] C. Li, J.C. Sprott, Int. J. Bifurc. Chaos 24 (2014) 1450034.
[16] C. Li, J.C. Sprott, Nonlinear Dyn. 73 (2013) 1335.
[17] C. Li, J.C. Sprott, Phys. Lett. A 379 (2015) 888.
[18] C. Li, J.C. Sprott, Z. Yuan, H. Li, Int. J. Bifurc. Chaos 25 (2015) 1530025.
[19] C. Li, J.C. Sprott, W. Thio, H. Zhu, IEEE Trans. Circuits Syst. II, Express Briefs 61 (2014) 977.
[20] B. Munmuangsaen, B. Srisuchinwong, J.C. Sprott, Phys. Lett. A 375 (2011) 1445.
[21] J.C. Sprott, Phys. Rev. E 50 (1994) R647.