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Limitation of Perpetual Points for Confirming Conservation in Dynamical Systems

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Perpetual Points (PPs) have been introduced as an interesting new topic in nonlinear dynamics, and there is a hypothesis that these points can determine whether a system is dissipative or not. This paper demonstrates that this hypothesis is not true since there are counterexamples. Furthermore, we explain that it is impossible to determine dissipation of a system based only on the structure of the system and its equations.

Keywords: Perpetual point; dissipation; conservative system; multistability; coexisting attractors.

1. Introduction

Recently, many new chaotic flows have been discovered that are not associated with a saddle point, including ones without any equilibrium points, with only stable equilibria, or with a line containing infinitely many equilibrium points [Jafari & Sprott, 2013; Jafari *et al.*, 2013; Kingni *et al.*, 2014; Lao *et al.*, 2014; Molaie *et al.*, 2013; Pham *et al.*, 2014a; Wang & Chen, 2012, 2013; Wei, 2011; Pham *et al.*, 2014b]. The attractors for such systems have been called hidden attractors [Leonov & Kuznetsov, 2010, 2013a, 2013b, 2013c, 2014; Leonov *et al.*, 2011a; Leonov *et al.*, 2011b, 2012; Leonov *et al.*, 2014; Leonov *et al.*, 2015; Bragin *et al.*, 2011; Kuznetsov *et al.*, 2010; Kuznetsov *et al.*, 2011], and that accounts for the difficulty of discovering them since there is no systematic way to choose initial conditions except by extensive numerical search. Hidden attractors are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or aircraft wing. Another topic that has attracted increasing attention is multistability and coexisting attractors [Angeli *et al.*, 2004; Pisarchik & Feudel, 2014; Blazejczyk-Okolewska & Kapitaniak, 1996, 1998; Kapitaniak, 1985; Maistrenko *et al.*, 1997; Silchenko *et al.*, 1999].

On the other hand, Perpetual Points (PPs) have been introduced as an interesting new topic

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in nonlinear dynamics [Prasad, 2015; Dudkowski *et al.*, 2015]. It has been shown that these points can be used to locate hidden attractors and to find coexisting attractors in multistable systems [Prasad, 2015]. It has also been claimed that PPs can be used to determine whether a system is dissipative or not. Many examples were investigated that supported this conjecture [Prasad, 2015].

In this note we show that this hypothesis is not true for all cases. Moreover, we believe that it is impossible in general to determine dissipation based only on the structure of the system and its equations. In the next part we describe PPs in a simple way (for more details readers can see the original paper [Prasad, 2015]). In Sec. 3, we use examples to prove that this hypothesis is not true for all cases. In Sec. 4, we claim that it is impossible to determine if a system is dissipative based only on the structure of the system and its equations. Finally, Sec. 5 gives the conclusions.

2. Perpetual Points

Consider a general dynamical system

$$\begin{aligned} v_1 &= \dot{x}_1 = f_1(x_1, x_2, \ldots, x_n) \\ v_2 &= \dot{x}_2 = f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ v_n &= \dot{x}_n = f_n(x_1, x_2, \ldots, x_n) \end{aligned} \tag{1}$$

where $x_1, x_2, \ldots, x_n$ are dynamical variables (states), $v_1, v_2, \ldots, v_n$ are the time derivatives of the states (velocities) and $f_1(X), f_2(X), \ldots, f_n(X)$ are the evolution equations (velocity vectors). It is well known that the fixed points (FPs) of the above system are points $(x_1^*, x_2^*, \ldots, x_n^*)$ at which the velocities of all states are zero. We know that analysis of the FPs plays an essential role in dynamical systems [Prasad, 2015; Ott, 2002; Strogatz, 2014]. Since acceleration is the time derivative of velocity, we obtain

$$\begin{aligned} a_1 &= \ddot{x}_1 = \dot{x}_1 \frac{\partial f_1}{\partial x_1} + \dot{x}_2 \frac{\partial f_1}{\partial x_2} + \cdots + \dot{x}_n \frac{\partial f_1}{\partial x_n} \\ a_2 &= \ddot{x}_2 = \dot{x}_1 \frac{\partial f_2}{\partial x_1} + \dot{x}_2 \frac{\partial f_2}{\partial x_2} + \cdots + \dot{x}_n \frac{\partial f_2}{\partial x_n} \\ &\vdots \\ a_n &= \ddot{x}_n = \dot{x}_1 \frac{\partial f_n}{\partial x_n} + \dot{x}_2 \frac{\partial f_n}{\partial x_n} + \cdots + \dot{x}_n \frac{\partial f_n}{\partial x_n} \end{aligned} \tag{2}$$

where $a_1, a_2, \ldots, a_n$ are the second derivatives of the states (accelerations). It is obvious that for all the FPs, all the accelerations are zero, since $v_1, v_2, \ldots, v_n$ are zero. By definition [Prasad, 2015], Perpetual Points (PPs) are points like $(x_1^*, x_2^*, \ldots, x_n^*)$ at which all the accelerations are zero but the velocities are not. For more details, readers can see the original paper [Prasad, 2015].

3. Perpetual Points Cannot Determine Whether a System is Dissipative or Not

There is a hypothesis (although without mathematical proof) in [Prasad, 2015], which suggests that if a system has a PP, then it is dissipative, and if not, it is conservative. We investigate this hypothesis using two examples:

3.1. *Example one*

Consider the Sprott Case A system [Silchenko *et al.*, 1999], which is one of the oldest examples of a chaotic flow with no equilibria:

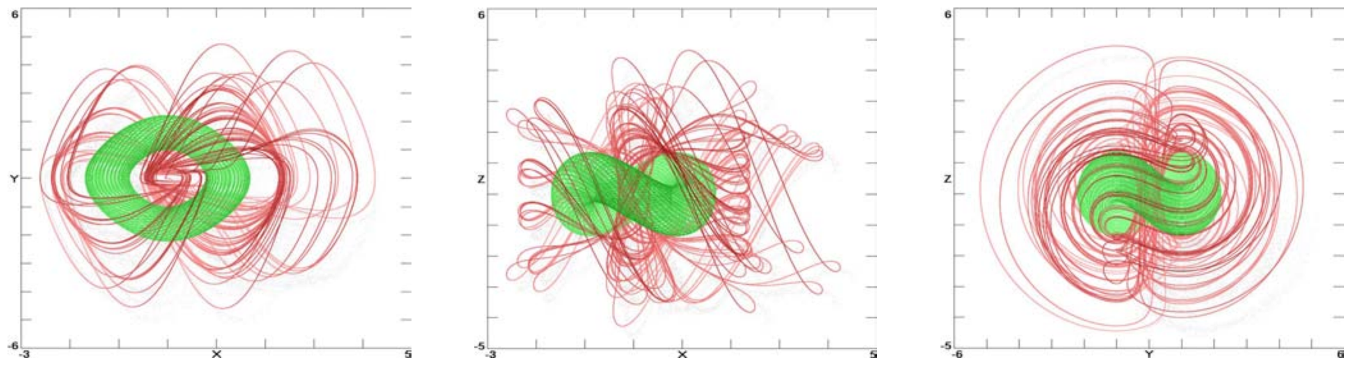
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= 1 - y^2. \end{aligned} \tag{3}$$

Lyapunov exponents of this system are (0.0139, $0, -0.0139$) and the Kaplan–Yorke dimension is 3.0. This system is a special case of the Nose–Hoover oscillator [Hoover, 1995] and describes many natural phenomena [Posch *et al.*, 1986]. This is a conservative system, and thus it does not have attractors, but there is a chaotic sea coexisting with a set of nested tori as shown in Fig. 1.

This system has no FPs. The PPs for this system can be obtained by solving the following equations:

$$\begin{aligned} \ddot{x} &= \dot{y} = 0 \to -x + yz = 0 \\ \ddot{y} &= -\dot{x} + \dot{y}z + y\dot{z} \\ &= 0 \to -y^3 - xz + yz^2 \\ &= -y^3 + z(-x + yz) = 0 \\ \ddot{z} &= -2y\dot{y} = 0 \to -2y(-x + yz) = 0. \end{aligned} \tag{4}$$

The solution is $(0, 0, z)$, which means there is an infinite line of PPs along the z-axis. Thus the hypothesis would predict that it is dissipative, but in fact it is conservative as can be shown by numerically averaging the trace of the Jacobian matrix



(a) (b) (c)

Fig. 1. The system in Eq. (3) with initial conditions $(0, 5, 0)$ gives a chaotic sea (Red) and $(0, 1, 0)$ gives a conservative torus (Green).

along the orbit ($TR(J) = \langle z \rangle = 0$) to verify that it is accurately zero.

3.2. *Example two*

Consider the following 2-D system:

$$\dot{x} = f(y), \quad \dot{y} = g(x). \tag{5}$$

This system is obviously conservative since the trace of the Jacobian matrix is zero. This system has a Hamiltonian of the following form:

$$H = F(y) - G(x) = \text{const} \tag{6}$$

where $F(y) = \int f(y)dy$ and $G(x) = \int g(x)dy$.

Proof. $\frac{\partial H}{\partial t} = \dot{H} = \dot{F} - \dot{G} = \dot{y}\frac{\partial F}{\partial y} - \dot{x}\frac{\partial F}{\partial x} = g(x) \times f(y) - f(y)g(x) = 0.$ ■

It is easy to make system (5) have PPs:

$$\ddot{x} = \dot{y}\frac{\partial f}{\partial y} = g(x)\frac{\partial f}{\partial y}, \quad \ddot{y} = \dot{x}\frac{\partial g}{\partial x} = f(y)\frac{\partial g}{\partial x}. \tag{7}$$

For system (5), FPs are the points for which $f(y) = g(x) = 0$. Equations (7) will be zero for these points. However, the equations are also satisfied and thus have PPs provided

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 0 \quad \text{and} \quad g(x) \neq 0 \quad \text{and} \quad f(y) \neq 0. \tag{8}$$

For example, consider the system

$$\begin{aligned} \dot{x} &= y^2 - 1 \\ \dot{y} &= x^2 - 1 \\ \text{FPs} &= (\pm 1, \pm 1) \end{aligned} \tag{9}$$

$$\begin{aligned} \ddot{x} &= 2y\dot{y} = 2y(x^2 - 1) \\ \ddot{y} &= 2x\dot{x} = 2x(y^2 - 1) \\ \text{PP} &= (0, 0). \end{aligned} \tag{10}$$

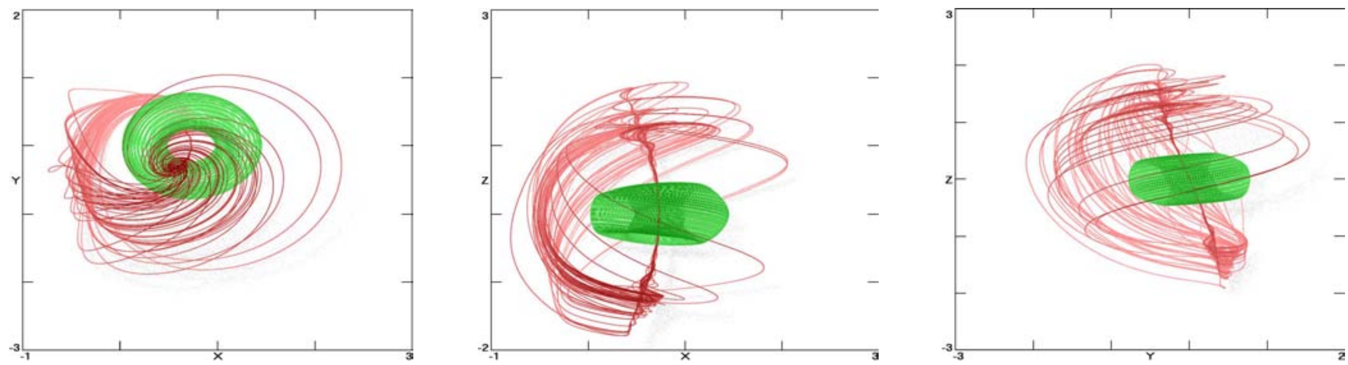
This means that a conservative system has PPs, which contradicts the hypothesis in [Prasad, 2015].

4. Can We Determine Whether a System is Dissipative or Not?

There are some systems of ODEs that have conservative solutions for some initial conditions and dissipative solutions for others even for all the parameters fixed. Thus, in general, it is impossible to determine whether a system is dissipative or not based only on the structure of the system and its equations, since there are cases [Sprott & Li, 2014] in which the initial conditions play an important role in the dynamics. To clarify the issue, we show some examples of systems with coexisting behaviors. Recently, Sprott, in a numerical search for chaotic systems that have no equilibrium points, discovered a simple three-dimensional time-reversible system of ODEs with quadratic nonlinearities and the property that it exhibits conservative behavior for some initial conditions and dissipative behavior for others [Sprott, 2014; Sprott *et al.*, 2014] as given by

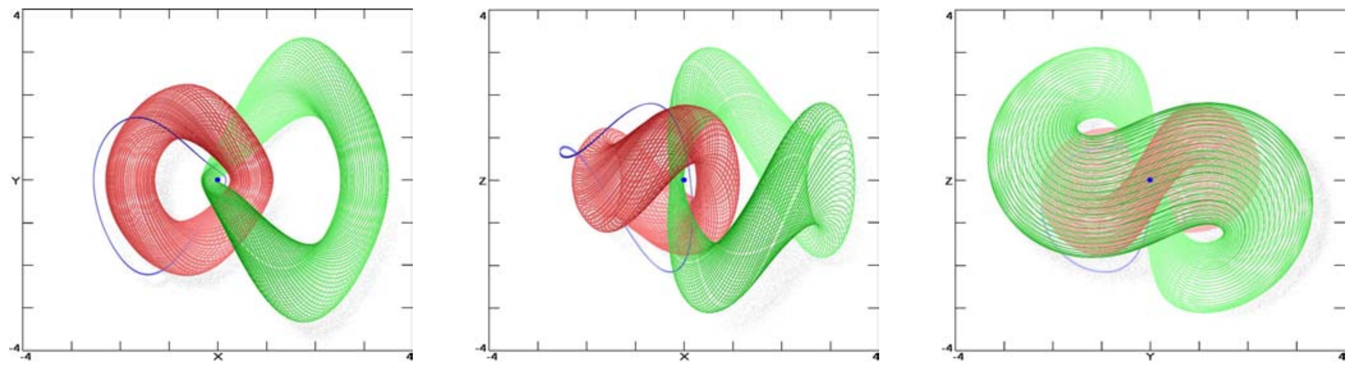
$$\begin{aligned} \dot{x} &= y + 2xy + xz \\ \dot{y} &= 1 - 2x^2 + yz \\ \dot{z} &= x - x^2 - y^2. \end{aligned} \tag{11}$$

The dissipation is given by $\text{Tr}(J) = 2(y + z)$ and the time average of $\langle y + z \rangle$ is negative for some initial conditions such as $(x_0, y_0, z_0) = (2, 0, 0)$



(a) (b) (c)

Fig. 2. The Eq. (11) system with initial conditions $(2, 0, 0)$ gives a dissipative strange attractor (Red) and $(1, 0, 0)$ gives a conservative torus (Green).



(a) (b) (c)

Fig. 3. The Eq. (12) system with initial conditions $(-2.3, 0, 0)$ and $(3.5, 0, 0)$ gives interlinked conservative tori (Red and Green) and $(-2.7, 0, 0)$ gives a dissipative limit cycle (Blue).

and zero for others such as $(x_0, y_0, z_0) = (1, 0, 0)$. The first initial condition gives a strange attractor with Lyapunov exponents $(0.0540, 0, -0.1575)$ and a Kaplan–Yorke dimension of 2.3429, and the second initial condition gives a torus with Lyapunov exponents $(0, 0, 0)$ and a dimension of 2.0. Thus the conservative regime has quasiperiodic orbits, while the dissipative regime is chaotic. These two behaviors are shown in Fig. 2.

Another example of a system with coexisting behaviors is the thermostated harmonic oscillator where the imposed temperature field is a function of the oscillator coordinate x [Sprott, 2014; Sprott *et al.*, 2014] given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - yz \\ \dot{z} &= y^2 - 1 - 0.42\tanh(x). \end{aligned} \tag{12}$$

This system shows interlocked phase-space behaviors, two conservative invariant tori and a dissipative limit cycle as shown in Fig. 3. The tori are produced using the initial conditions $(x_0, y_0, z_0) = (-2.3, 0, 0)$ and $(3.5, 0, 0)$ with Lyapunov exponents $(0, 0, 0)$. The limit cycle is produced using the initial conditions $(-2.7, 0, 0)$ with Lyapunov exponents $(0, -0.0256, -0.0788)$.

As these examples show, one system can have conservative and dissipative behavior for the same parameters and different initial conditions. Thus features based on the structure of the system cannot indicate (in general) whether a system is dissipative or not, although sometimes it can.

5. Conclusion

The hypothesis of the relation between PPs and dissipation of systems (which suggests that if a system has a PP then it is dissipative, and otherwise it is conservative) is not true for all cases. Furthermore, we explain that in general, it is impossible to determine dissipation of systems based only on their structure and equations, and we clarify the issue with examples.

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