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Elementary quadratic chaotic flows with a single non-hyperbolic equilibrium

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ABSTRACT

This paper describes a class of third-order explicit autonomous differential equations, called jerk equations, with quadratic nonlinearities that can generate a catalog of nine elementary dissipative chaotic flows with the unusual feature of having a single non-hyperbolic equilibrium. They represent an interesting sub-class of dynamical systems that can exhibit many major features of regular and chaotic motion. The proposed systems are investigated through numerical simulations and theoretical analysis. For these jerk dynamical systems, a certain amount of nonlinearity is sufficient to produce chaos through a sequence of period-doubling bifurcations.

1. Introduction

In the investigation of chaos and its applications, it is important to generate new chaotic systems or to enhance complex dynamics and topological structure based on existing chaotic attractors. For a generic three-dimensional smooth quadratic autonomous system, Sprott found by exhaustive computer search nineteen simple chaotic flows with no more than three equilibria [1].

In the continuous case, some questions about periodic homoclinic and heteroclinic orbits and classification of chaos are related to questions about the dynamics of some chaotic systems. It is concerned with the classification and determination of the type of chaos observed experimentally, proved analytically, or tested numerically in theory and practice. One of the commonly agreed-upon analytic criteria for proving chaos in autonomous systems is the existence of Smale horseshoes and the Shilnikov condition [2]. Therefore, there exist four kinds of chaos: homoclinic chaos, heteroclinic chaos, a combination of homoclinic and heteroclinic chaos, and chaos without homoclinic orbits or heteroclinic orbits [3].

Recently, Leonov et al. [4–6] have proposed another type of attractor, called a "hidden attractor" whose basin of attraction does not contain neighborhoods of any equilibria and thus cannot be computed with the help of the standard procedure for anterior types. Therefore, there has been increasing interest in some unusual examples of three-dimensional autonomous quadratic systems such as those having no equilibria [7,8], stable equilibria [9–12], or coexisting attractors [13], and in four-dimensional autonomous quadratic systems with no equilibria [14–17].

However, there is little knowledge about the characteristics of chaotic flows with a single non-hyperbolic equilibrium. Such systems can have neither homoclinic nor heteroclinic orbits, and thus the Shilnikov method cannot be used to verify the chaos. A non-hyperbolic equilibrium point has one or more eigenvalues with a zero real part. There are eleven such types in three-dimensional flows. Six of these have all eigenvalues real and are of the form $(0,-,-)$, $(+,0,-)$, $(+,+,0)$, $(0,0,-)$, $(+,0,0)$, and $(0,0,0)$. Five have one real and a complex conjugate pair of eigenvalues, only two of which have nonzero real eigenvalues. The stability of those systems that do not have an eigenvalue with a positive real part cannot be determined from the eigenvalues and requires a nonlinear analysis.

Relatively few such examples have been previously reported. The oldest and best-known is the Sprott E system [1], which has a single equilibrium point with one real negative eigenvalue and a complex conjugate pair with zero real parts. Recently, three

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other dissipative examples have been reported. One is the Chen–Zhou system [18], which has a single equilibrium point with one zero eigenvalue and a complex conjugate pair with positive real parts. Another is the system proposed by Yang et al. [19], which has a single equilibrium point with one negative real eigenvalue and a complex conjugate pair with zero real parts. Very recently, Sprott [20] reported a similar example as well as one with one positive real eigenvalue and a complex conjugate pair with zero real parts. Here we identify the most elementary functional forms of the other eight types of three-dimensional dynamical systems with a single non-hyperbolic equilibrium that have a chaotic attractor thus demonstrating that chaos can exist in three-dimensional systems with all eleven types of non-hyperbolic equilibria.

2. The main results

In the investigation of chaos and its applications, it is useful to generate new chaotic systems with only one equilibrium point. Thus it is natural to ask whether chaotic attractors can exist in systems with only one non-hyperbolic equilibrium for each of the eleven types that can occur in such systems where the traditional Shilnikov theorem is not applicable.

The two cases of systems that have a complex conjugate pair of eigenvalues of the form $\pm i\omega$ and a real nonzero eigenvalue (negative or positive) are given in [20]. Therefore, we only consider the remaining nine types where at least one eigenvalue is real and zero. A system that admits a wide variety of chaotic solutions is the jerk system

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = f(x, y, z), \end{cases} \tag{1}$$

where $f(x, y, z) = a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz + a$.

One can see that when

$$a_4 = 0,$$

or

$$a_4 \neq 0, a = \frac{a_1^2}{4a_4},$$

the system (1) has only one equilibrium. The case with $a_4 = 0$ has been examined by Molai et al. [12]. Here, we consider the case $a_4 \neq 0, a = \frac{a_1^2}{4a_4}$ and obtain some novel results, which have not been previously reported.

Under the linear transformation $x \to x - \frac{a_1}{2a_4}$, the lone equilibrium is moved to the origin $O(0, 0, 0)$, and system (1) becomes

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = \frac{a_2 - a_1a_7}{2a_4}y + \frac{a_3 - a_1a_8}{2a_4}z + a_4x^2 + a_5y^2 + a_6z^2 \\ \qquad + a_7xy + a_8xz + a_9yz. \end{cases} \tag{2}$$

Rearranging $b_2 = \frac{a_2 - a_1a_7}{2a_4}$, $b_3 = \frac{a_3 - a_1a_8}{2a_4}$, and $b_i = a_i$ $(i = 4, 5, 6, 7, 8, 9)$ gives

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = b_2y + b_3z + b_4x^2 + a_6z^2 + b_5y^2 + b_7xy + b_8xz + b_9yz. \end{cases} \tag{3}$$

In the interest of simplicity, we set $a_6 = a_9 = 0$. The goal of this paper is to give chaotic examples of the equilibrium types that have not been previously reported using the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = b_2y + b_3z + b_4x^2 + b_5y^2 + b_7xy + b_8xz, \end{cases} \tag{4}$$

where $b_2, b_3, b_4, b_5, b_7, b_8 \in R$.

A non-hyperbolic equilibrium at the origin $O(0, 0, 0)$ has eigenvalues λ that satisfy

$$\lambda(\lambda^2 - b_3\lambda - b_2) = 0 \tag{5}$$

whose solutions are $\lambda = 0$ and $\lambda = \left(b_3 \pm \sqrt{b_3^2 + 4b_2}\right)/2$. Adjusting the parameters b_2 and b_3 gives the following four cases:

Case 1: Condition: $b_3^2 + 4b_2 < 0$

In this case, eigenvalues at the one equilibrium $O(0, 0, 0)$ are $(0, \sigma \pm i\omega)$, where $\sigma \in R, \omega \neq 0$. There are three candidates for these types of equilibria: $\sigma > 0$, $\sigma < 0$, or $\sigma = 0$.

Case 2: Condition: $b_3^2 + 4b_2 \geq 0$

In this case, eigenvalues at the one equilibrium $O(0, 0, 0)$ are $(0, \mu, \nu)$, where $\mu\nu \neq 0$. There are three candidates for these types of equilibria: $\mu > 0, \nu > 0$, $\mu < 0, \nu < 0$ or $\mu\nu < 0$.

Case 3: Condition: $b_3 \neq 0, b_2 = 0$

In this case, eigenvalues at the one equilibrium $O(0, 0, 0)$ are $(0, 0, \gamma)$, where $\gamma \neq 0$. There are two candidates for these types of equilibria: $\gamma < 0$ or $\gamma > 0$.

Case 4: Condition: $b_3 = 0, b_2 = 0$

In this case, eigenvalues at the one equilibrium $O(0, 0, 0)$ are $(0, 0, 0)$. There is only one such type of equilibrium.

An exhaustive computer search was done considering many thousands of combinations of the coefficients and initial conditions subject to the constraints in Eq. (5), seeking cases for which the largest Lyapunov exponent [21–23] is greater than 0.001. Table 1 shows examples of each of the nine types where at least one eigenvalue is real and zero. The search attempted to identify the simplest example of each case. Thus we believe we have identified elementary forms of chaotic flows with quadratic nonlinearities that have a single non-hyperbolic equilibrium for each of the nine types.

All nine of the resulting attractors are shown projected onto the xy-plane in Fig. 1. The Lyapunov spectra and initial conditions near the attractor are given in Table 1. All the cases appear to approach chaos through a succession of period-doubling limit cycles. For example, Fig. 2 shows the local maximum values of x for Model B as the parameter $b_5 = d \in [1, 1.8]$ is varied with the other parameters fixed at $b_2 = -1, b_3 = -1, b_4 = -4, b_7 = 0, b_8 = -1$. The plot shows a period-doubling Feigenbaum-tree.

When d exceeds a critical value of about 1.210, the attractor of Model B undergoes a period-doubling bifurcation which converts a period-1 limit cycle to a period-2 limit cycle. As d is further increased, a second bifurcation converts the period-2 attractor to a period-4 attractor when $d = 1.474$. The third bifurcation converts the period-4 to attractor to a period-8 attractor when $a = 1.535$. Finally, if we look carefully, we can see a hint of a period-8 to period-16 bifurcation, just before the start of the solid red chaotic region.

From the observed bifurcations, a scaling law is obtained in the form:

$$\delta = \frac{1.474 - 1.210}{1.535 - 1.474} = 4.328.$$

Therefore, δ is a mere 7.3% lower than Feigenbaum's constant. Although Model B in this parameter region has a single non-hyperbolic equilibrium, the existence of a universal ratio characterizes the transition to chaos via period-doubling bifurcations, and this behavior is typical of the other systems.

In addition to the nine cases listed in the table, dozens of additional cases were found that are extensions of these cases with

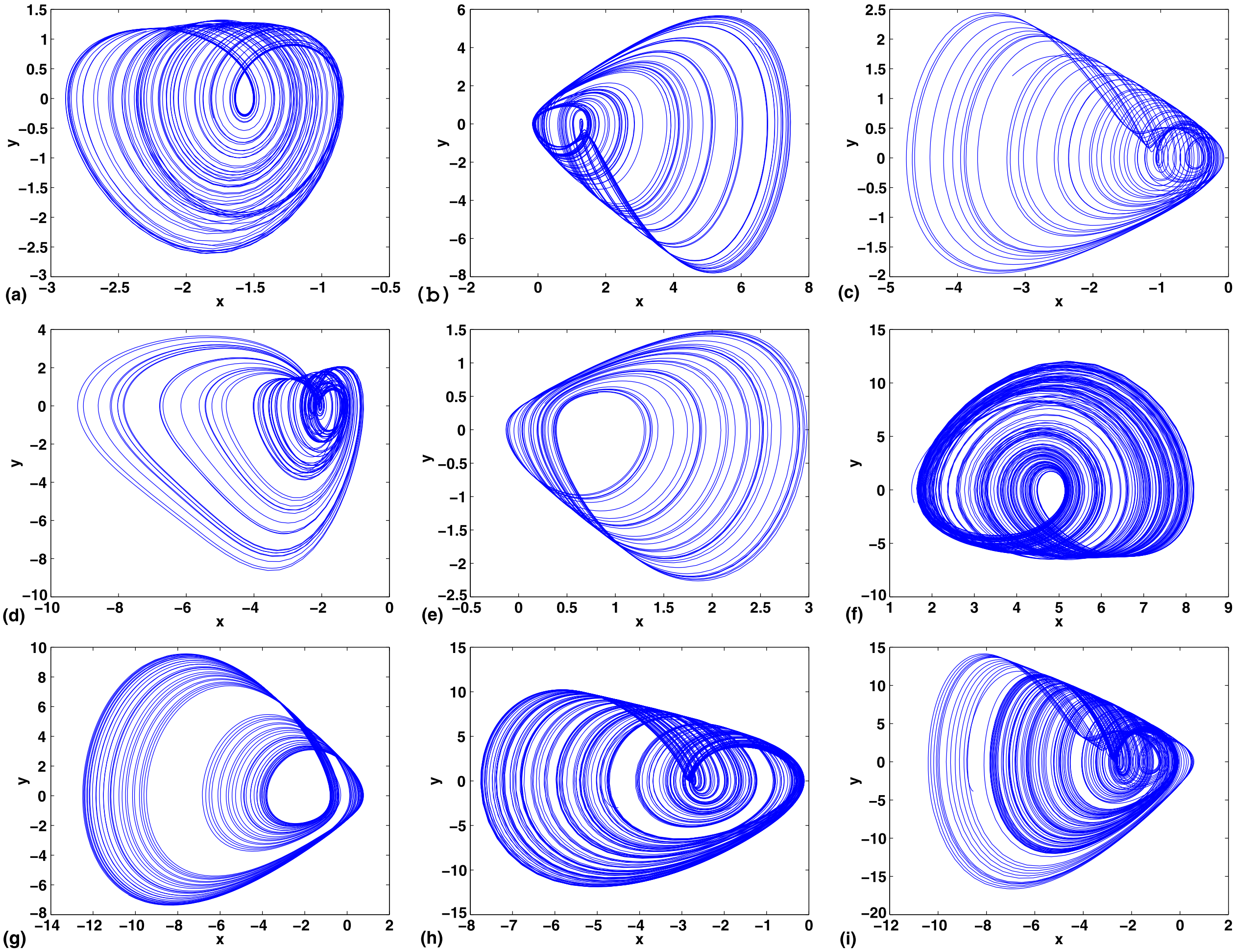

Fig. 1. State space diagram of the cases in Table 1 projected onto the *xy*-plane.

Table 1
Nine simple chaotic systems with a single non-hyperbolic equilibrium at (0, 0, 0).

Model	System	Eigenvalues	Lyapunov exponents	(x_0, y_0, z_0)
A	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -7y + 4.5z + x^2 + 3xz$	$\lambda_1 = 0$ $\lambda_{2,3} = 2.25 \pm 1.3919i$	$L_1 = 0.1127$ $L_2 = 0.0000$ $L_3 = -0.7174$	$(-4.5, -3, -2.5)$
B	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -y - z - 4x^2 + 3y^2 - xz$	$\lambda_1 = 0$ $\lambda_{2,3} = -0.5 \pm 0.86603i$	$L_1 = 0.1476$ $L_2 = 0.0000$ $L_3 = -2.7904$	$(6.38, 4.22, -3.15)$
C	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -y + x^2 - 3y^2 + xz$	$\lambda_1 = 0$ $\lambda_{2,3} = \pm i$	$L_1 = 0.1028$ $L_2 = 0.0000$ $L_3 = -1.3193$	$(-3.19, 1.38, 1.43)$
D	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -9y + 6z + x^2 + 3xz$	$\lambda_1 = 0$ $\lambda_{2,3} = 3$	$L_1 = 0.1117$ $L_2 = 0.0000$ $L_3 = -2.5959$	$(-1.74, 1.59, -7.07)$
E	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -0.2y - z - 2x^2 + 3y^2 - xz$	$\lambda_1 = -0.7236$ $\lambda_2 = -0.2764$ $\lambda_3 = 0$	$L_1 = 0.0684$ $L_2 = 0.0000$ $L_3 = -2.1072$	$(3.90, -2.16, -12.42)$
F	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = y + 4z - x^2 - 3xy - xz$	$\lambda_1 = -0.2361$ $\lambda_2 = 0$ $\lambda_3 = 4.2361$	$L_1 = 0.1629$ $L_2 = 0.0000$ $L_3 = -0.7119$	$(1.58, -1.22, 10.35)$
G	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -z + x^2 - y^2 + 0.6xz$	$\lambda_{1,2} = 0$ $\lambda_3 = -1$	$L_1 = 0.0389$ $L_2 = 0.0000$ $L_3 = -3.1208$	$(-1.55, 3.41, -5.37)$
H	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = 0.5z + 7x^2 + 4.5xy - 2y^2 + xz$	$\lambda_{1,2} = 0$ $\lambda_3 = 0.5$	$L_1 = 0.1503$ $L_2 = 0.0000$ $L_3 = -2.3917$	$(-4.5, -3, -2.5)$
I	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = 9x^2 + 4.5xy - 2.5y^2 + xz$	$\lambda_{1,2,3} = 0$	$L_1 = 0.1575$ $L_2 = 0.0000$ $L_3 = -2.7936$	$(-8.54, -3.85, 2.32)$

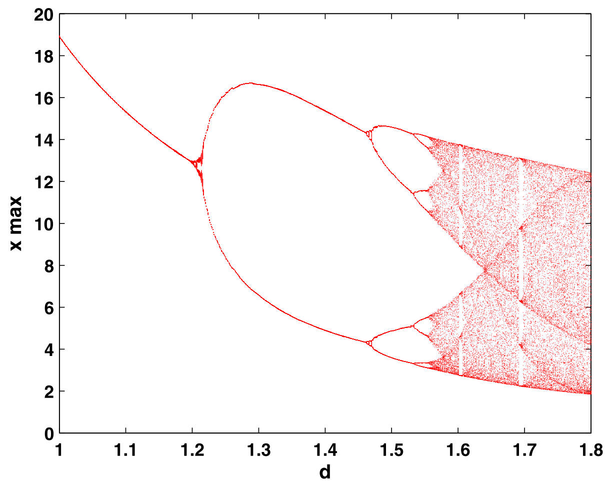

Fig. 2. Bifurcation diagram of model B showing a period-doubling route to chaos.

additional terms. We believe we have identified most if not all of the elementary forms of three-dimensional chaotic systems with quadratic nonlinearities that have a single non-hyperbolic equilibrium.

3. Conclusion

In this paper, systems of autonomous ordinary differential equations of the form of Eq. (1) admit chaotic solutions with chaotic attractors in the presence of a single non-hyperbolic equilibrium point for each of the nine types as shown in Table 1. The results support the idea that any dynamic not explicitly forbidden by some theorem will occur in an appropriately designed dynamical system, and it answers the question whether chaotic attractors can occur in systems with all the types of non-hyperbolic equilibrium points that can occur in three dimensions. There are still abundant and complex dynamical behaviors and topological structures of the new systems that should be completely and thoroughly investigated and exploited, such as finding new criteria for the existence of chaos in systems with no homoclinic or heteroclinic orbits. It is expected that more detailed theoretical analysis and simulation investigations about these systems will be provided in a forthcoming study.

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References

[1] J.C. Sprott, Phys. Rev. E 50 (1994) 647.
[2] C.P. Silva, IEEE Trans. Circuits Syst. I 40 (1993) 657.
[3] T. Zhou, G. Chen, Int. J. Bifurc. Chaos 16 (2006) 2459.
[4] G. Leonov, N. Kuznetsov, Int. J. Bifurc. Chaos 23 (2013) 1330002.
[5] G. Leonov, N. Kuznetsov, V. Vagaitsev, Phys. Lett. A 375 (2011) 2230.
[6] G. Leonov, N. Kuznetsov, V. Vagaitsev, Physica D 241 (2012) 1482.
[7] Z. Wei, Phys. Lett. A 376 (2011) 102.
[8] S. Jafari, J.C. Sprott, S. Golpayegani, Phys. Lett. A 377 (2013) 699.
[9] X. Wang, G. Chen, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 1264.
[10] Z. Wei, Q. Yang, Nonlinear Anal., Real World Appl. 12 (2011) 106.
[11] Z. Wei, Q. Yang, Nonlinear Dyn. 68 (2012) 543.
[12] M. Molaie, S. Jafari, J.C. Sprott, S. Mohammad, Int. J. Bifurc. Chaos 23 (2013) 1350188.
[13] J.C. Sprott, X. Wang, G. Chen, Int. J. Bifurc. Chaos 23 (2013) 1350093.
[14] Z. Wang, S. Cang, E.O. Ochola, Y. Sun, Nonlinear Dyn. 69 (2012) 531.
[15] C. Li, J.C. Sprott, Int. J. Bifurc. Chaos 24 (2014) 1450034.
[16] V.T. Pham, V. Volos, S. Jafari, Z. Wei, X. Wang, Int. J. Bifurc. Chaos 24 (2014) 1450073.
[17] Z. Wei, R. Wang, A. Liu, Math. Comput. Simul. 100 (2014) 13.
[18] B. Chen, T. Zhou, G. Chen, Int. J. Bifurc. Chaos 19 (2009) 1679.
[19] Q. Yang, Z. Wei, G. Chen, Int. J. Bifurc. Chaos 20 (2010) 1061.
[20] J.C. Sprott, Eur. Phys. J. Spec. Top. 224 (2015) 1409.
[21] A. Wolf, J.B. Swift, H.L. Swinney, J.A. Vastano, Physica D 16 (1985) 285.
[22] N.V. Kuznetsov, G.A. Leonov, in: Int. Conf. Physics and Control, 2005, p. 596.
[23] G.A. Leonov, N.V. Kuznetsov, Int. J. Bifurc. Chaos 17 (2007) 1079.