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Review

Strange attractors with various equilibrium types

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Abstract. Of the eight types of hyperbolic equilibrium points in three-dimensional flows, one is overwhelmingly dominant in dissipative chaotic systems. This paper shows examples of chaotic systems for each of the eight types as well as one without any equilibrium and two that are nonhyperbolic. The systems are a generalized form of the Nosé–Hoover oscillator with a single equilibrium point. Six of the eleven cases have hidden attractors, and six of them exhibit multistability for the chosen parameters.

1 Introduction

Nearly all the common examples of strange attractors in three-dimensional autonomous systems of ordinary differential equations occur in systems for which there are one or more spiral saddle points, where the Shilnikov condition is satisfied [1]. Such attractors have been called "self-excited" since they can be found by choosing initial conditions on the unstable manifold in the vicinity of one of these equilibria [2,3].

It is natural to ask whether strange attractors can exist in systems with only one equilibrium of the other eighteen types that can occur in such systems. A system that admits a wide variety of chaotic solutions is a generalized version of the Nosé–Hoover oscillator [4,5] given by

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x + yz\\ \dot{z} &= f(x,y,z)\end{aligned} \tag{1}$$

where

$$f(x,y,z) = a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6xy + a_7xz + a_8yz + a_9 \tag{2}$$

is the most general function $f(x,y,z)$ with quadratic nonlinearities that is guaranteed to provide a single equilibrium point at $(x,y,z) = (0,0,-a_9/a_3)$ provided $a_3 \neq 0$ or $a_9 = 0$. An equilibrium point, sometimes called a fixed point or singular point, is a value of (x,y,z) where $\dot{x} = \dot{y} = \dot{z} = 0$. Note that it is also possible to include a z^2 term in Eq. (2) and adjust the constant a_9 so that the two roots of $f(x,y,z) = 0$ coincide, but that turns out to be unnecessary for the present purpose.

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The simplest chaotic system of this form has $f = 1 - y^2$ and has been much studied, but it is a conservative system without any equilibrium points. However, there is a weakly dissipative form of System (1) with $f = 1 + \epsilon \tanh x - y^2$ that has a strange attractor but with no equilibrium points since $a_3 = 0$ and $a_9 \neq 0$ [6]. Note that all systems in the form of Eq. (1) have a nonlinear damping whose magnitude and sign depend on the time average of z along the orbit because of the yz term.

System (1) with various forms of $f(x, y, z)$ has previously been shown to have strange attractors for two of the equilibrium types [7,8]. The goal of this paper is to give simple examples of the remaining types, which have not been previously reported. This involves putting appropriate constraints on the eigenvalues λ given by

$$\lambda^3 + \left(\frac{a_9}{a_3} - a_3\right)\lambda^2 + (1 - a_9)\lambda - a_3 = 0 \tag{3}$$

and searching the nine-dimensional parameter space and three-dimensional space of initial conditions for solutions that are bounded and dissipative with a positive Lyapunov exponent and then simplifying the resulting systems to the extent possible by removing unnecessary terms and setting parameters to unity.

2 System with no equilibrium

Shown first is a simple example of a dissipative chaotic system in the form of Eq. (1) with no equilibrium given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= x^2 - 4y^2 + 1. \end{aligned} \tag{ET0}$$

For initial conditions of $(0, 2, 0)$, it has a strange attractor with Lyapunov exponents of $(0.0131, 0, -0.0155)$ and a relatively large Kaplan–Yorke dimension of 2.8455 as shown in Fig. 1. This and the following examples are labeled with "ET" for "equilibrium type" followed by a number that denotes the type and that allows it to be easily referenced in future publications. This strange attractor is "hidden" in the sense that it cannot be found by using an initial condition in the vicinity of an equilibrium point since no such point exists [2,3].

The system is time-reversal invariant under the transformation $(x, y, z, t) \rightarrow (x, -y, -z, -t)$. The strange attractor is slightly asymmetric and is accompanied by a strange repellor that is symmetric with it under a 180° rotation about the x-axis and intertwined with it. The attractor and repellor exchange roles when time is reversed. The asymmetry in z is the source of the weak nonlinear damping, since its value time-averaged along the chaotic orbit is $\langle z \rangle \approx -0.0024$.

This system is unusual because the strange attractor is intertwined with a set of nested conservative tori, which are symmetric under rotation about the x-axis and have $\langle z \rangle = 0$. Figure 1 shows one such torus for initial conditions of $(0, 1.2, 0)$ for which the Lyapunov exponents are $(0, 0, 0)$. Figure 2 shows a cross section in the $z = 0$ plane of the nested tori surrounded by what looks like a chaotic sea but is actually a weakly dissipative strange attractor. Sixty-three initial conditions were spaced uniformly over the range $-2.9625 \leq x \leq -0.6522$ with $y = z = 0$. The blue background shows the initial conditions that give conservative orbits (tori), and the yellow background is the basin of attraction for the strange attractor. It appears that there are additional thin tori toward the outer edge of the strange attractor. The basin of attraction of the strange attractor is the whole of state space except for the region of finite volume occupied by tori. Only a few other such examples are known [6,9,10].

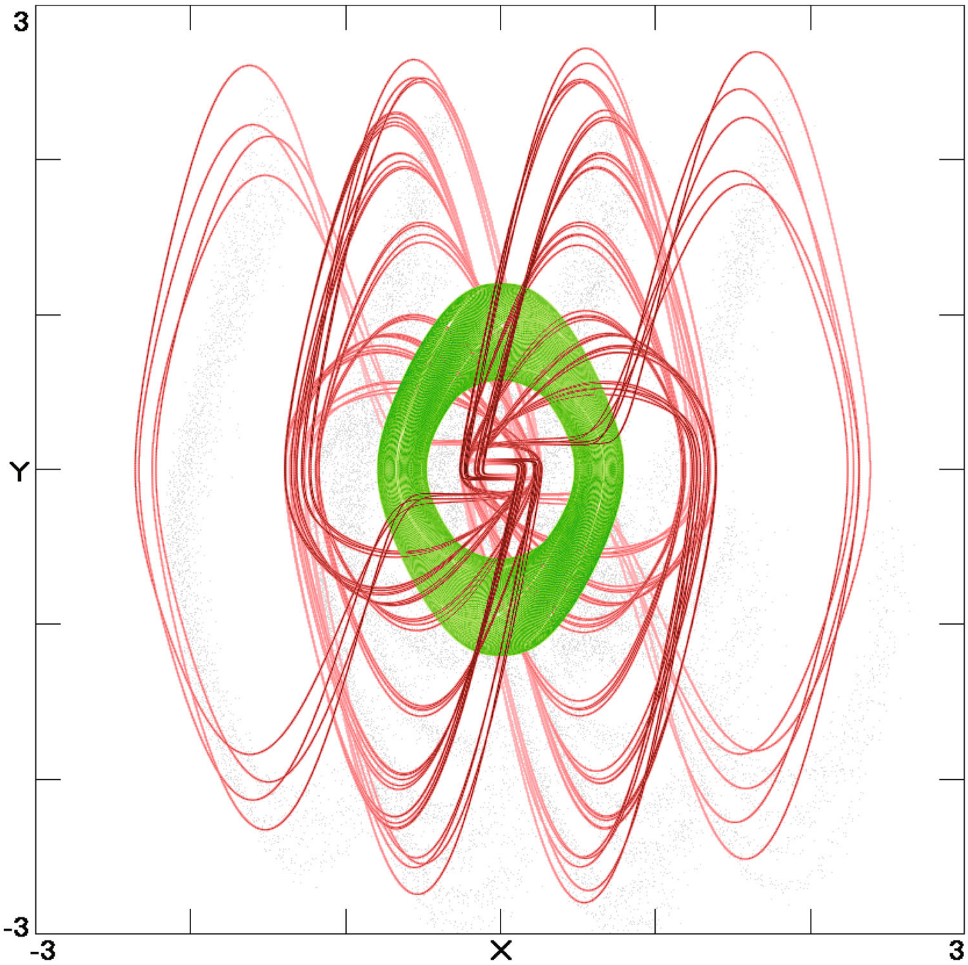

Fig. 1. A strange attractor (in red) for System ET0 coexisting with a conservative invariant torus (in green) projected onto the xy-plane.

3 Hyperbolic examples

There are eight types of hyperbolic equilibrium points in three-dimensional flows as shown in Fig. 3. A hyperbolic equilibrium is one in which all eigenvalues have a nonzero real part. Type 7 is overwhelmingly the most common in dissipative chaotic systems, but examples also occur for the other seven cases as shown below.

3.1 Equilibrium Type 1 (index-0 node)

This system has a single equilibrium point with three real eigenvalues, all negative. Hence it is a stable node with index 0, where the index is the number of eigenvalues with a positive real part, or, equivalently, the dimension of the unstable manifold. A system of this type is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= -z - 8xy + 0.3xz - 3. \end{aligned} \tag{ET1}$$

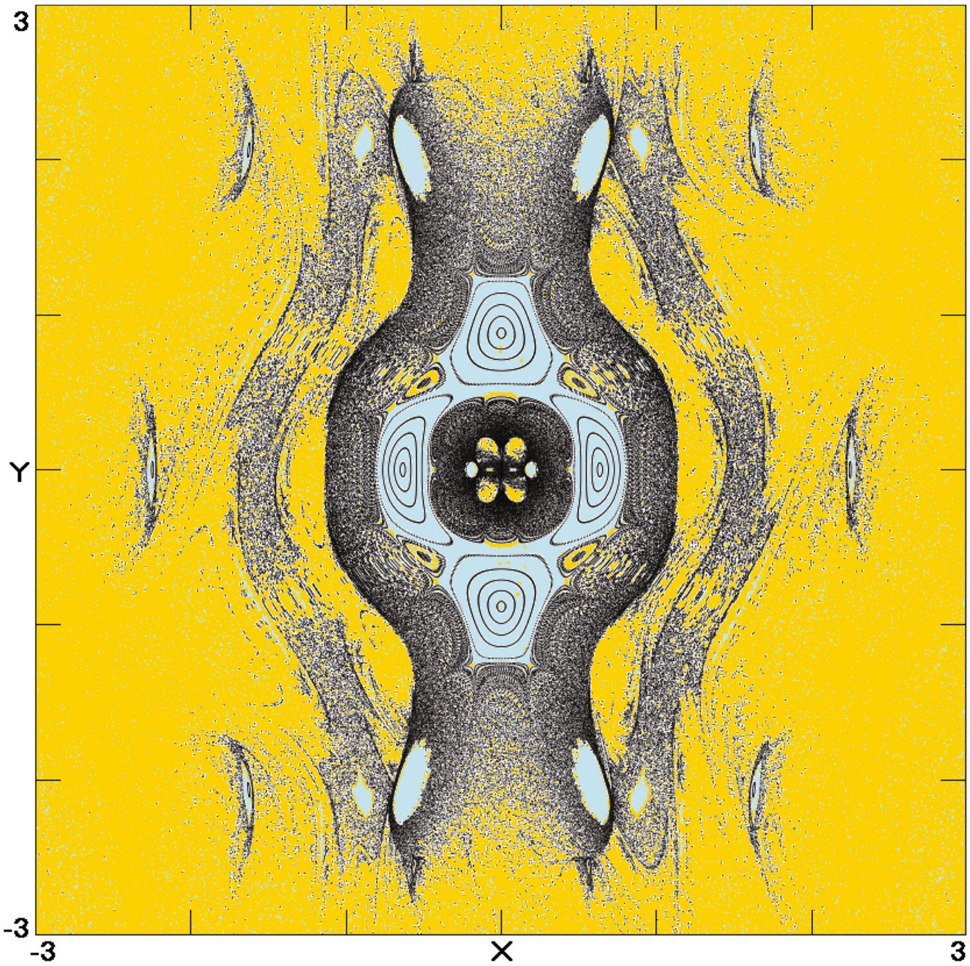

Fig. 2. Cross section in the $z = 0$ plane of the nested tori surrounded by a strange attractor for System ET0. The blue background shows the initial conditions that give conservative orbits (tori), and the yellow background is the basin of attraction for the strange attractor.

It has an equilibrium with eigenvalues given by $\lambda = (-0.381966, -1, -2.618034)$ and a strange attractor with Lyapunov exponents of $(0.0505, 0, -17.2283)$. The strange attractor is hidden since all initial conditions in the vicinity of the equilibrium are attracted to the equilibrium point. It is also multistable since the strange attractor coexists with a point attractor. Initial conditions close to the attractor are $(-2, 1, 0.7)$, and the basin of attraction is very small.

3.2 Equilibrium Type 2 (index-1 saddle point)

This system has three real eigenvalues, with two negative and one positive. A system of this type is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= 0.5z - y^2 + 5. \end{aligned} \tag{ET2}$$

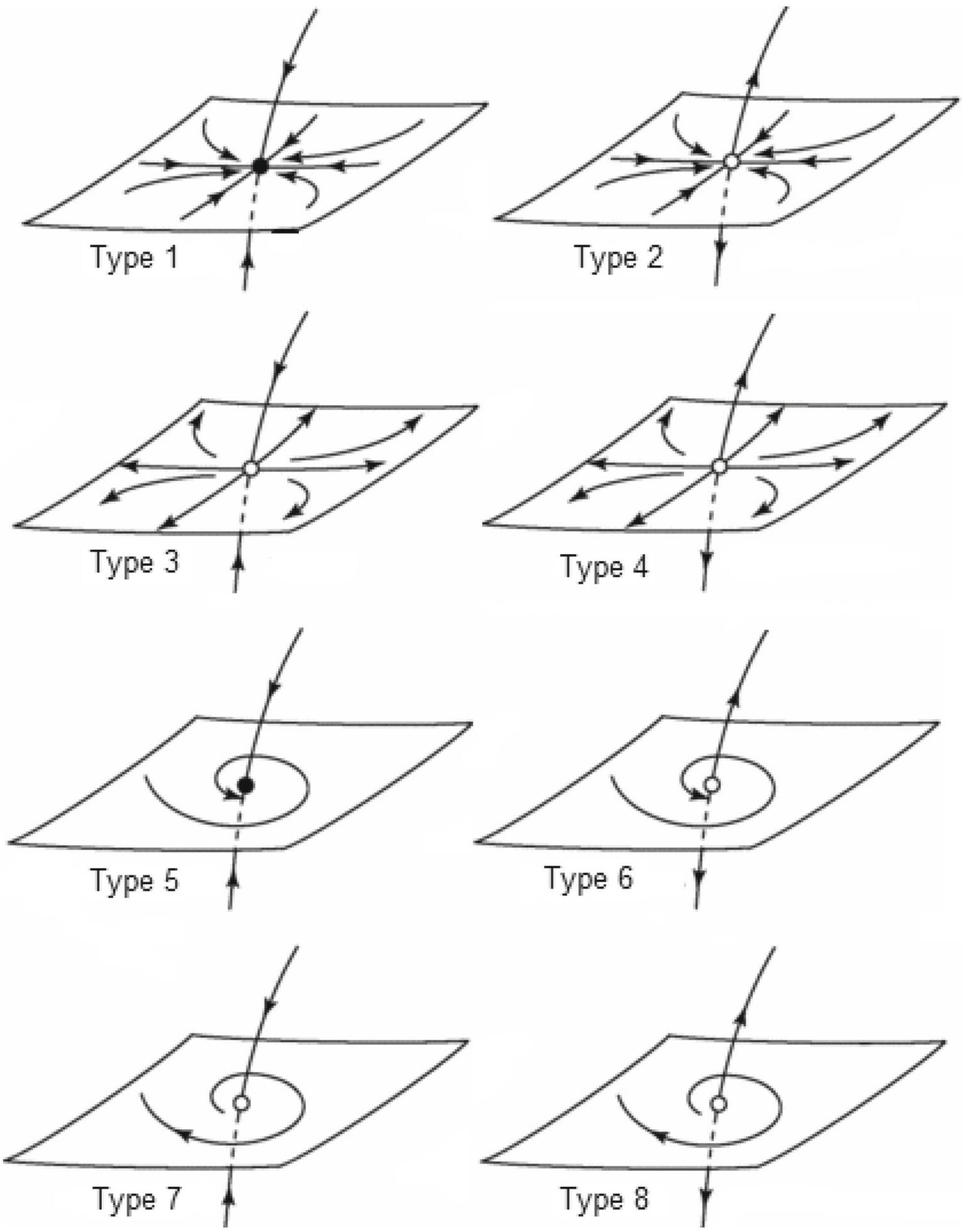

Fig. 3. Types of hyperbolic equilibria in three-dimensional ODEs.

It has an equilibrium with eigenvalues given by $\lambda = (0.5, -0.101021, -9.898980)$ and a symmetric pair of tightly intertwined strange attractors with Lyapunov exponents of $(0.0141, 0, -0.3030)$ that coexist with three limit cycles, a symmetric one with Lyapunov exponents of $(0, -0.1239, -0.2336)$ and a symmetric pair with Lyapunov exponents of $(0, -0.0264, -0.0264)$. This is possible because System ET2, like the Lorenz system, has a rotational symmetry about the z-axis as evidenced by its invariance under the transformation $(x, y, z) \to (-x, -y, z)$, and hence any solutions either share that symmetry or there is a symmetric pair of them. All the attractors are hidden since all initial conditions in the vicinity of the equilibrium point produce unbounded orbits. Initial conditions that give the five attractors are $(\pm 0.9, 0, -2)$, $(0.43, 2, 0.18)$, and $(\pm 0.4, \pm 3, 1)$, and the basins of attraction of the strange attractors are relatively small and bounded (finite volume).

3.3 Equilibrium Type 3 (index-2 saddle point)

This system has three real eigenvalues, with two positive and one negative. A system of this type is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= -x - 0.1z - y^2 + 0.3. \end{aligned} \tag{ET3}$$

It has an equilibrium with eigenvalues given by $\lambda = (2.618034, 0.381966, -0.1)$ and a self-excited strange attractor with Lyapunov exponents of $(0.02191, 0, -0.3181)$. Initial conditions close to the attractor are $(0, 0.1, 0)$, and the basin of attraction is relatively large.

3.4 Equilibrium Type 4 (index-3 repellor)

This system has three real eigenvalues, all positive, and hence it is an unstable node, sometimes simply called a "repellor". A system of this type has recently been reported [8] and has the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= z + 8.894x^2 - y^2 - 4. \end{aligned} \tag{ET4}$$

It has an equilibrium with eigenvalues given by $\lambda = (3.732051, 1, 0.267949)$ and a strange attractor with Lyapunov exponents of $(0.1767, 0, -0.9158)$. The equations have a rotational symmetry since they are invariant under the transformation $(x, y, z) \to (-x, -y, z)$, and the system does have a symmetric pair of solutions for some parameters, but not for the ones given above. The symmetric strange attractor for this case is hidden since all initial conditions chosen in the vicinity of the equilibrium lead to unbounded solutions. Initial conditions close to the attractor are $(0, 3.8, 0.7)$, and the basin of attraction is very small.

3.5 Equilibrium Type 5 (index-0 spriral node)

This system has one real negative eigenvalue, and a complex conjugate pair with a negative real part. Twenty-three chaotic examples of this type have recently been reported [7]. One typical case in the form of Eq. (1) is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= 2x - 2z + y^2 - 0.3. \end{aligned} \tag{ET5}$$

It has an equilibrium with eigenvalues given by $\lambda = (-2, -0.075 \pm 0.997184i)$ and a strange attractor with Lyapunov exponents of $(0.0203, 0, -2.4751)$. All chaotic systems of this type are multistable since the strange attractor coexists with a stable equilibrium point, and the strange attractor is hidden since it cannot be found by using initial conditions in the vicinity of the equilibrium. Initial conditions close to the strange attractor are $(0.9, 0, 0.7)$, and the basin of attraction is very large.

3.6 Equilibrium Type 6 (index-1 spiral saddle)

This system has one real positive eigenvalue, and a complex conjugate pair with a negative real part. A system of this type is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= 0.28z - xy + 0.48. \end{aligned} \tag{ET6}$$

It has an equilibrium with eigenvalues given by $\lambda = (0.28, -0.857143 \pm 0.515079i)$ and a symmetric pair of strange attractors with Lyapunov exponents of $(0.0677, 0, -1.5020)$. The equations have a rotational symmetry since they are invariant under the transformation $(x, y, z) \to (-x, -y, z)$. The strange attractors are hidden, and all initial conditions in the vicinity of the equilibrium point lead to unbounded orbits. Initial conditions close to the attractors are $(0, \pm 4, 2)$, and the basins of attraction are very small.

3.7 Equilibrium Type 7 (index-2 spiral saddle)

This system has one real negative eigenvalue, and a complex conjugate pair with a positive real part. This is overwhelmingly the most common type with abundant examples including the familiar Lorenz [11] and Rössler [12] systems, although they have multiple equilibrium points. The simplest such system with a single equilibrium point is the jerk system $\dddot{x} + 2.017\ddot{x} - \dot{x}^2 + x = 0$ [13]. Other simple examples are Sprott Cases I, J, L, N, and R [14]. There are also a number of systems in the form of Eq. (1) that have not been studied including the following

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= -z + xy + 0.39 \end{aligned} \tag{ET7}$$

which is chosen because it is functionally the same as System ET6 with a rotational symmetry but with different parameters. It has an equilibrium with eigenvalues given by $\lambda = (-1, 0.195 \pm 0.980803i)$ and a symmetric pair of strange attractors with Lyapunov exponents of $(0.0820, 0, -0.6920)$. The strange attractors are self-excited, although the equilibrium point lies on their basin boundary, and so which attractor is found depends on exactly where in the vicinity of the equilibrium the initial conditions are chosen. Initial conditions close to the attractors are $(\pm 1.4, \pm 1, 1)$, and the basins of attraction are relatively large.

3.8 Equilibrium Type 8 (index-3 spiral repellor)

This system has one real positive eigenvalue, and a complex conjugate pair with a positive real part. A system of this type is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= 0.2z + 0.1y^2 - xy - 0.08. \end{aligned} \tag{ET8}$$

It has an equilibrium with eigenvalues given by $\lambda = (0.2, 0.2 \pm 0.979796i)$ and a strange attractor with Lyapunov exponents of $(0.1083, 0, -3.2555)$. The equations have a rotational symmetry since they are invariant under the transformation $(x, y, z) \to (-x, -y, z)$, and for the given parameters, the strange attractor is symmetric and self-excited. Initial conditions close to the attractor are $(1, -2, 0.4)$, and the basin of attraction is very large.

4 Nonhyperbolic examples

A nonhyperbolic equilibrium point has one or more eigenvalues with a zero real part. There are eleven such types in three-dimensional flows. Six of these have all eigenvalues real and are of the form $(0, -, -)$, $(+, 0, -)$, $(+, +, 0)$, $(0, 0, -)$, $(+, 0, 0)$, and $(0, 0, 0)$. Five have one real and a complex conjugate pair of eigenvalues, only two of which have nonzero real eigenvalues. The stability of those systems that do not have an eigenvalue with a positive real part cannot be determined from the eigenvalues and requires a nonlinear analysis.

Consider first the nine types where at least one eigenvalue is real and zero. With $\lambda = 0$, Eq. (3) shows that $a_3 = 0$, and thus according to Eq. (2), an equilibrium point is present only if $a_9 = 0$, in which case there is an infinite line of equilibrium points along the z-axis at $(0, 0, z)$. Such cases have been previously studied [15] including ones in the form of Eq. (1), but they fall outside the scope of the present paper which involves chaotic systems with a single equilibrium point. Thus nine of the eleven possible nonhyperbolic isolated equilibrium points cannot occur in Eq. (1), although this does not imply that they cannot exist in other systems.

The remaining two cases have a complex conjugate pair of eigenvalues of the form $\lambda = 0 \pm i\omega$. Substitution into Eq. (3) gives $\omega^2 = 1 - a_9$ and $a_9 = 1 - a_3^2$ or $a_9 = 0$. The remaining real eigenvalue can be negative or positive. Chaotic examples of the two types are given below.

4.1 Equilibrium Type 9

This system has a single equilibrium point with one real negative eigenvalue and a complex conjugate pair with zero real parts. Chaotic systems of this type have been reported such as Sprott Case E [14]. A system of this type in the form of Eq. (1) is

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= -x + yz \\
\dot{z} &= -z - 4xy + xz.
\end{aligned} \tag{ET9}$$

It has an equilibrium with eigenvalues given by $\lambda = (-1, 0 \pm i)$ and a strange attractor with Lyapunov exponents of $(0.0394, 0, -1.4067)$. The equilibrium at the origin is nonlinearly unstable, and the strange attractor is self-excited. Initial conditions close to the attractor are $(0, 1, 0.4)$, and the basin of attraction is relatively small.

4.2 Equilibrium Type 10

This system has a single equilibrium point with one real positive eigenvalue and a complex conjugate pair with zero real parts. A system of this type is

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= -x + yz \\
\dot{z} &= 0.23z + y^2 - 10xy.
\end{aligned} \tag{ET10}$$

It has an equilibrium with eigenvalues given by $\lambda = (0.23, 0 \pm i)$ and a strange attractor with Lyapunov exponents of $(0.1241, 0, -2.4424)$. The system is invariant under the transformation $(x, y, z) \to (-x, -y, z)$, and the strange attractor is symmetric and

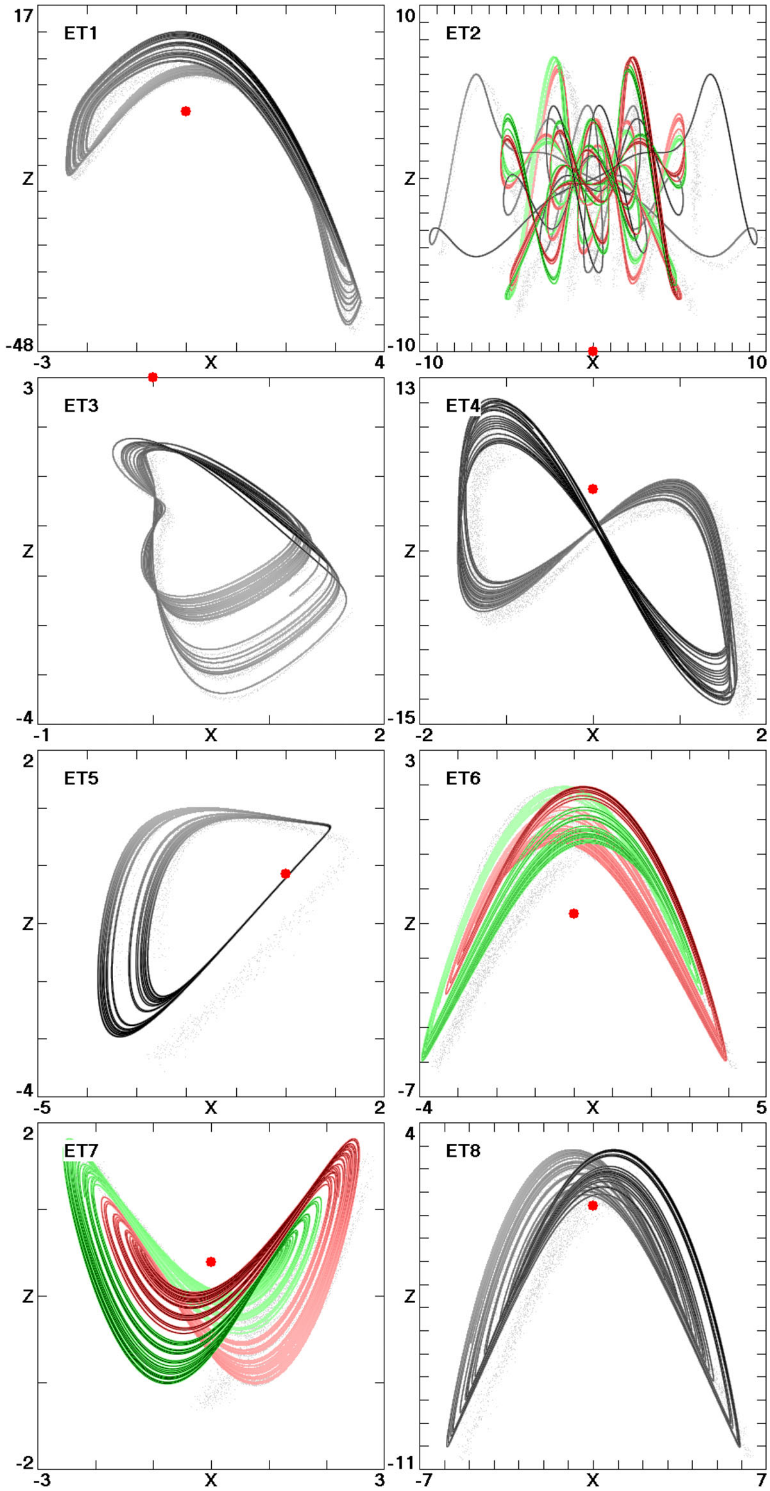

Fig. 4. Attractors for systems with a single hyperbolic equilibrium point for each of the eight types projected onto the xz-plane. The equilibrium points are indicated by red dots and lie in the $y = 0$ plane.

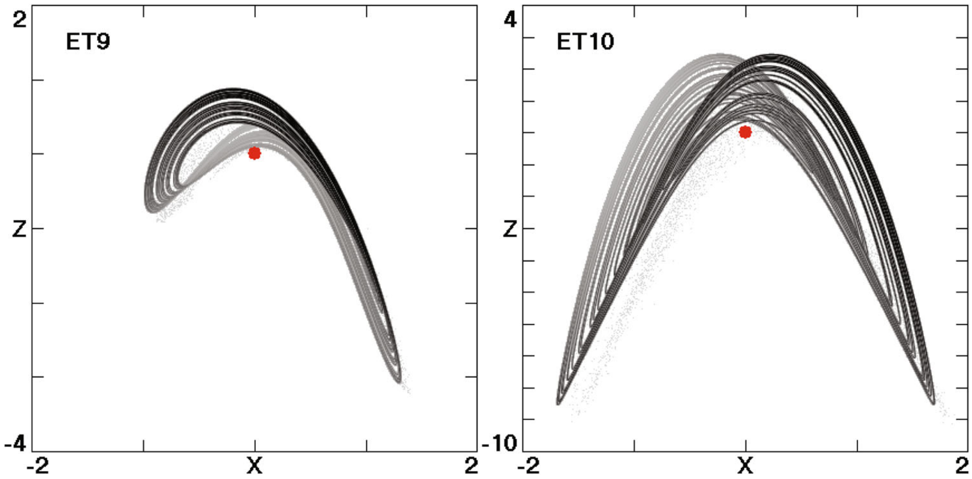

Fig. 5. Attractors for systems with a single nonhyperbolic equilibrium point for two of the eleven types projected onto the xz-plane. The equilibrium points are indicated by red dots and lie in the $y = 0$ plane.

self-excited. Initial conditions close to the attractor are $(0, 1, 1)$, and the basin of attraction is very large.

5 Summary and conclusions

Systems of autonomous ordinary differential equations of the form of Eq. (1) admit chaotic solutions with one or more strange attractors in the presence of a single hyperbolic equilibrium point for each of the eight types as shown in Fig. 4 projected onto the xz-plane. Two of the systems have a strange attractor coexisting with a stable equilibrium, three of the systems have a symmetric pair of strange attractors, and one (ET2) has two strange attractors and three coexisting limit cycles for the given parameters. Five of the eight cases have hidden attractors.

There are eleven types of nonhyperbolic equilibrium points that can occur in three-dimensional systems. However, nine of the eleven types cannot occur in isolation for the chosen system. The other two cases admit chaotic solutions with a single self-excited strange attractor as shown in Fig. 5. All of the systems should have interesting basins of attraction [16] and a rich set of bifurcations as the parameters are varied and deserve further study.

Given that systems in the form of Eq. (1) can exhibit chaos in the absence of equilibrium points as in System ET0, it is not surprising that chaos can accompany isolated equilibrium points of all the allowed types, although in most cases the equilibrium point is relatively close to the attractor and thus would be expected to influence its dynamics. The results presented here support the idea that any dynamic not explicitly forbidden by some theorem will occur in an appropriately designed dynamical system. One needs only to look carefully to find suitable examples. An interesting question is whether strange attractors can occur in systems with the fourteen types of hyperbolic equilibrium points that occur in four dimensions, but that answer will be left to a subsequent publication.

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