Chaotic Systems and Circuits with Hidden Attractors

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Abstract

Categorizing dynamical systems into systems with hidden attractors and systems with self-excited attractors is new topic in dynamical systems. In this chapter we describe three newly introduced families of chaotic systems with hidden attractors. We design a circuit for one example of each family and discuss some important properties of these kinds of circuits.

1. Introduction

Recent research has involved categorizing periodic and chaotic attractors as either self-excited or hidden [1-10]. A self-excited attractor has a basin of attraction that is associated with an unstable equilibrium, whereas a hidden attractor has a basin of attraction that does not intersect with small neighborhoods of any equilibrium points. The classic attractors of Lorenz, Rössler, Chua, Chen, Sprott (cases B to S), and other widely-known attractors are self-excited with one or more unstable equilibrium points. From a computational standpoint, this allows one to use a numerical method in which a trajectory started from a point on the unstable manifold in the neighborhood of the unstable equilibrium, reaches an attractor and identifies it [7]. Hidden attractors cannot be found by this method and are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or an airplane wing.

It has been shown in [11-13] that the chaotic attractors in dynamical systems without any equilibrium points, with only stable equilibria, or with a line of equilibria are examples of hidden attractors. That may be the reason why such systems are rarely found, and only a few such examples have been reported in the literature [11-23]. These systems are challenging, and studying them may reveal new phenomena in dynamical systems.

On the other hand, the circuit implementation of chaotic systems has attracted much interest in the past decades [24-30]. It has bolstered the confidence that chaos is a real phenomenon and has provided experimental data for many applications in the study of chaos.

In the next section, we briefly introduce the above mentioned three families of hidden attractors. In the third section we show a designed circuit for one example of each family. We conclude with some discussion in the last section where we mention some important properties of chaotic circuits with hidden attractors.

2. Three new families of hidden chaotic attractors

2.1. Chaotic flows with no equilibria

In this section we consider chaotic flows with no equilibria. Such systems have neither homoclinic nor heteroclinic orbits [31], and thus the Shilnikov method [32, 33] cannot be used to verify the chaos. The oldest and best-known example is the conservative Sprott A system [34] listed as NE_1 in Table 1. This is an important system since it is a special case of the Nose-Hoover oscillator [35] which describes many natural phenomena [36], and thus it suggests that such systems may have practical as well as theoretical importance. In [11] we performed a systematic search to find three-dimensional chaotic systems with quadratic nonlinearities and no equilibria. Our search was based on the methods proposed in [37] and used our own custom software. The objective was to find the algebraically simplest cases which cannot be further reduced by the removal of terms without destroying the chaos. The main method for finding these systems was to constrain the search space to cases where we could show algebraically that the equilibrium points are imaginary. For example, any chaotic solution of a parametric system such

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= z \\
\dot{z} &= k_1x + k_2y + k_3z + k_4x^2 + k_5y^2 + k_6z^2 + k_7xy + k_8xz + k_9yz + a \\
k_1^2 &- 4k_4a \le 0
\end{aligned} \tag{1}$$

is a candidate. An exhaustive computer search was done considering many thousands of combinations of the coefficients and initial conditions, seeking cases for which the largest Lyapunov exponent is greater than 0.001.

In addition to the seventeen cases listed in the Table 1, dozens of additional cases were found that are extensions of these cases with additional terms. For each case that was found, the space of coefficients was searched for values that are deemed "elegant" [37], by which we mean that as many coefficients as possible are set to zero with the others set to ±1 if possible or otherwise to a small integer or decimal fraction with the fewest possible digits. Except for NE_1, all these cases are dissipative. The Lyapunov spectra and Kaplan-Yorke dimension are shown in Table 1 along with initial conditions that are close to the attractor.

Table 1
Seventeen simple chaotic systems with no equilibria.

Model	Equations	a	LEs	D_{KY}	(x_0, y_0, z_0)
NE_1	$\dot{x} = y$	1.0	0.0138, 0, -0.0138	3.0000	(0, 5, 0)

	$\dot{y} = -x - zy$ $\dot{z} = y^2 - a$				
NE_2	$\dot{x} = -y$ $\dot{y} = x + z$ $\dot{z} = 2y^2 + xz - a$	0.35	0.0776, 0, -1.5008	2.0517	(0, 0.4, 1)
NE_3	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -y + 0.1x^2 + 1.1xz + a$	1.0	0.0522, 0, -2.6585	2.0196	(1, 1, -1)
NE_4	$\dot{x} = -0.1y + a$ $\dot{y} = x + z$ $\dot{z} = xz - 3y$	1.0	0.0235, 0, -8.480	2.0028	(-8.2, 0, -5)
NE_5	$\dot{x} = 2y$ $\dot{y} = -2x - z$ $\dot{z} = -y^2 + z^2 + a$	2.0	0.0168, 0, -0.3622	2.0465	(0.98, 1.8, -0.7)
NE_6	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -y - xz - yz - a$	0.75	0.0280, 0, -3.4341	2.0082	(0, 3, -0.1)
NE_7	$\dot{x} = y$ $\dot{y} = -x + z$ $\dot{z} = -0.8x^2 + z^2 + a$	2.0	0.0252, 0, -6.8524	2.0037	(0, 2.3, 0)
NE_8	$\dot{x} = y$ $\dot{y} = -x - yz$	1.3	0.0314, 0, -10.2108	2.0031	(0, 0.1, 0)

	$\dot{z} = xy + 0.5x^2 - a$				
NE_9	$\dot{x} = y$ $\dot{y} = -x - yz$ $\dot{z} = -xz + 7x^2 - a$	0.55	0.0504, 0, -0.3264	2.1544	(0.5, 0, 0)
NE_{10}	$\dot{x} = z$ $\dot{y} = z - y$ $\dot{z} = -0.9y - xy + xz + a$	0.6	0.0061, 0, -1.3002	2.0047	(1, 0.7, 0.8)
NE_{11}	$\dot{x} = y$ $\dot{y} = -x + z$ $\dot{z} = z - 2xy - 1.8xz - a$	1.0	0.0706, 0, -0.6456	2.1094	(0, 1.6, 3)
NE_{12}	$\dot{x} = z$ $\dot{y} = x - y$ $\dot{z} = -4x^2 + 8xy + yz + a$	0.1	0.0654, 0, -2.0398	2.0321	(0.5, 0, -1)
NE_{13}	$\dot{x} = -y$ $\dot{y} = x + z$ $\dot{z} = xy + xz + 0.2yz - a$	0.4	0.1028, 0, -2.1282	2.0483	(2.5, 0, 0)
NE_{14}	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = x^2 - y^2 + 2xz + yz + a$	1.0	0.0532, 0, -11.8580	2.0045	(1, 0, -4)
NE_{15}	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = x^2 - y^2 + xy + 0.4xz + a$	1.0	0.1101, 0, -1.3879	2.0793	(0, 1, -4.9)

NE_{16}	$\dot{x} = -0.8x - 0.5y^2 + xz + a$ $\dot{y} = -0.8y - 0.5z^2 + yx + a$ $\dot{z} = -0.8z - 0.5x^2 + zy + a$	1.0	0.0607, 0, -0.1883	2.3224	(0, 1, -1)
NE_{17}	$\dot{x} = -y - z^2 + 2.3xy + a$ $\dot{y} = -z - x^2 + 2.3yz + a$ $\dot{z} = -x - y^2 + 2.3zx + a$	2.0	0.2257, 0, -1.7477	2.1292	(1, -1, 0)

2.2. Chaotic flows with stable equilibria

In [12], the search for chaotic flows with a stable equilibrium first focused on jerk systems. Consider a general equation with quadratic nonlinearities of the form

$$\dot{x} = y$$
$$\dot{y} = z \tag{2}$$
$$\dot{z} = f(x, y, z)$$
$$f = a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz + a_{10}$$

Any equilibrium point of (x^*, y^*, z^*) of system (2) must have $y^* = z^* = 0$ and eigenvalues λ that satisfy

$$\lambda^3 - f_z\lambda^2 - f_y\lambda - f_x \tag{3}$$

in which $f_x = a_1 + 2a_4x^*$, $f_y = a_2 + a_7x^*$, and $f_z = a_3 + a_8x^*$. Using the Routh-Hurwitz stability criterion, we require $f_z < 0$, $f_y f_z + f_x > 0$, and $f_x < 0$ for that equilibrium to be stable.

We can find x^* from $a_1x + a_4x^2 + a_{10} = 0$. For $a_4 \neq 0$, we have $x^*_{1,2} = \left(-a_1 \pm \sqrt{\Delta}\right)/2a_4$ where $\Delta = a_1^2 - 4a_4a_{10}$. To have an equilibrium, Δ should be greater than or equal to zero, in which case, $x_1^* = \left(-a_1 + \sqrt{\Delta}\right)/2a_4$ and $x_2^* = \left(-a_1 - \sqrt{\Delta}\right)/2a_4$. From the stability condition $f_x < 0$ for x_1^*, we have $\sqrt{\Delta} < 0$, which is impossible. Thus a quadratic jerk system cannot have two stable equilibria, and we therefore modify the general case in Eq. (3) to

$$\dot{x} = y$$

$$\dot{y} = z \tag{4}$$

$$\dot{z} = a_1x + a_2y + a_3z + a_4y^2 + a_5z^2 + a_6xy + a_7xz + a_8yz + a_9$$

in which there is no x^2 term in the $\dot{z}$ equation to ensure that one and only one equilibrium exists. This system has a single equilibrium at $(-a_9/a_1, 0, 0)$ whose stability requires

$$a_1 < 0$$

$$\left(a_3 - \frac{a_7a_9}{a_1}\right) < 0 \tag{5}$$

$$\left(a_2 - \frac{a_6a_9}{a_1}\right) < \frac{-a_1}{\left(a_3 - \frac{a_7a_9}{a_1}\right)}$$

Again, an exhaustive computer search was done considering many thousands of combinations of the coefficients a_1 through a_9 and initial conditions subject to the constraints in Eq. (5), seeking cases for which the largest Lyapunov exponent is greater than 0.001. Cases SE_1-SE_6 in Table 2 are six simple cases found in this way. As can be seen, the eigenvalues for the equilibria at the origin have all negative real parts, which means that each equilibrium is stable.

By similar calculations, many other simple structures for chaotic flows were investigated, and 17 additional cases (SE_7-SE_{23}) were added to the previous six jerk systems in Table 2. In addition to the cases listed in the table, dozens of additional cases were found, but they were either equivalent to one of the cases listed by some linear transformation of variables, or they were extensions of these cases with additional terms.

The Lyapunov spectra and Kaplan-Yorke dimensions are shown in Table 2 along with initial conditions that are close to the attractor. Since all the cases have a stable equilibrium, a point attractor coexists with a strange attractor for each case.

Table 2. Twenty-three simple chaotic flows with one stable equilibrium.

Model	Equations	Equilibrium	Eigenvalues	LEs	D_{KY}	(x_0, y_0, z_0)
SE_1	$\dot{x} = y$ $\dot{y} = z$ $\dot{z} = -x - 0.6y - 2z + z^2 - 0.4xy$	0 0 0	−1.9548 −0.0226 ±0.7149i	0.0377 0 −2.0377	2.0185	4 −2 0

	$\dot{x} = y$	0	−0.5103	0.0804		−1
SE_2	$\dot{y} = z$	0	−0.0198 ±0.9896i	0	2.1644	0
	$\dot{z} = -0.5x - y - 0.55z - 1.2z^2 - xz - yz$	0		−0.4889		1
	$\dot{x} = y$	0	−3.9641	0.0711		−2
SE_3	$\dot{y} = z$	0	−0.0179 ±0.9259i	0	2.0175	0
	$\dot{z} = -3.4x - y - 4z + y^2 + xy$	0		−4.0711		2.4
	$\dot{x} = y$	−1	−1.6942	0.0434		0.5
SE_4	$\dot{y} = z$	0	−0.0029 ±0.7683i	0	2.0249	1
	$\dot{z} = -x - 1.7z + y^2 + 0.6xy - 1$	0		−1.7434		0
	$\dot{x} = y$	−2.7	−0.9600	0.0136		−6.1
SE_5	$\dot{y} = z$	0	−0.0200 ±1.0204i	0	2.0134	1
	$\dot{z} = -x - z - z^2 + 0.4xy - 2.7$	0		−1.0136		1
	$\dot{x} = y$	−1	−1.0526	0.0638		−2.2
SE_6	$\dot{y} = z$	0	−0.0237 ±0.9744i	0	2.0600	0.6
	$\dot{z} = -x - 2.9z^2 + xy + 1.1xz - 1$	0		−1.0638		0
	$\dot{x} = y$	0	−2.0000	0.0360		1
SE_7	$\dot{y} = -x + yz$	0	−0.2500 ±0.9682i	0	2.0014	−0.7
	$\dot{z} = -2z - 8xy + xz - 1$	−0.5		−25.6798		0
	$\dot{x} = y$	0	−1.0000	0.1412		0
SE_8	$\dot{y} = -x + yz$	0	−0.0500 ±0.9987i	0	2.1034	0.9
	$\dot{z} = -z - 0.7x^2 + y^2 - 0.1$	−0.1		−1.3649		0
	$\dot{x} = y$	0	−2.0000	0.0203		0
SE_9	$\dot{y} = -x + yz$	0	−0.0750 ±0.9972i	0	2.0082	0.8
	$\dot{z} = 2x - 2z + y^2 - 0.3$	−0.15		−2.4751		−0.2
SE_{10}	$\dot{x} = y$	0	−2.0000	0.0963	2.0061	3.9

	$\dot{y} = -x + yz$ $\dot{z} = x - 0.3y - 2z + xz - 0.1$	0 −0.05	−0.0250 ±0.9997i	0 −15.7010		0 1
SE_{11}	$\dot{x} = y$ $\dot{y} = -x + yz$ $\dot{z} = -y - 12z + x^2 + 9xz - 1$	0 0 −1/12	−12.0000 −0.0417 ±0.9991i	0.0801 0 −14.1917	2.0056	−2 0 0.1
SE_{12}	$\dot{x} = y$ $\dot{y} = -x + yz$ $\dot{z} = -66z + y^2 + 35xz - 1$	0 0 −1/66	−66.0000 −0.0076 ±1.0000i	0.0259 0 −61.6130	2.0004	2 0.6 0
SE_{13}	$\dot{x} = y$ $\dot{y} = -x + yz$ $\dot{z} = -4.9z + 0.4y^2 + xy - 1$	**0** 0 −1/4.9	−4.9000 −0.1020 ±0.9948i	0.0540 0 −4.8228	2.0112	0 −2.2 0
SE_{14}	$\dot{x} = z$ $\dot{y} = x + z$ $\dot{z} = -y - 3z^2 + xy + yz - 0.7$	0 −0.7 0	−0.6082 −0.0459 ±1.2814i	0.0657 0 −1.6407	2.0401	0.5 −1 0
SE_{15}	$\dot{x} = -z$ $\dot{y} = x - z$ $\dot{z} = 0.9y + 0.2x^2 + xz + yz + 1$	**0** −10/9 0	−1.0618 −0.0247 ±0.9203i	0.0414 0 −6.6641	2.0062	3 2 0
SE_{16}	$\dot{x} = -z$ $\dot{y} = -x + z$ $\dot{z} = -7y - 1.4z + x^2 + xz - yz$	0 0 0	−1.0549 −0.1726 ±2.5702i	0.0775 0 −6.7190	2.0115	1 6 −6
SE_{17}	$\dot{x} = z$ $\dot{y} = x - y$ $\dot{z} = -3.1x - 0.3xz + 0.2yz + 0.57$	0.57/3.1 0.57/3.1 0	−1.0000 −0.0092 ±1.7607i	0.0832 0 −0.6549	2.1262	7.5 0 −5
SE_{18}	$\dot{x} = z$ $\dot{y} = -y + z$ $\dot{z} = -2.1x - 0.1z - y^2 + 0.11xz + 0.5yz$	0 0 0	−1.0000 −0.0500 ±1.4483i	0.1469 0 −3.8348	2.0383	− 28 0 0

	$\dot{x} = z$	−0.3	−1.3766	0.0241		0.2
SE_{19}	$\dot{y} = -y + z$	0	−0.0667 ±0.8497i	0	2.0005	6
	$\dot{z} = -x - 2xy + 1.7xz - 0.3$	0		−49.8730		7
	$\dot{x} = z$	0	−0.8543	0.2125		−2.1
SE_{20}	$\dot{y} = -y - z$	0	−0.0728 ±3.5875i	0	2.1753	0
	$\dot{z} = -11x + 2y - 2y^2 - z^2 - yz$	0		−1.2125		5
	$\dot{x} = z$	0	−0.8875	0.0484		0
SE_{21}	$\dot{y} = -y - z$	0	−0.0563 ±2.8279i	0	2.0169	−3
	$\dot{z} = -7.1x + y - 2y^2 + xz - yz$	0		−2.8617		8.2
	$\dot{x} = z$	−0.15	−1.0000	0.0557		−6
SE_{22}	$\dot{y} = -y - z$	0	−0.0750 ±2.4483i	0	2.0194	3.8
	$\dot{z} = -6x - 2y^2 + xz - yz - 0.9$	0		−2.8695		0
	$\dot{x} = -z$	0.5	−0.9060	0.0159		−0.4
SE_{23}	$\dot{y} = -y - z$	0	−0.0470 ±2.1006i	0	2.0156	1
	$\dot{z} = 4x - 0.2z^2 + xy - 2$	0		−1.0159		−9

2.3. Chaotic flows with line equilibria

In [13], the search for chaotic flows with a line equilibrium was inspired by the structure of the conservative Sprott case A system [34],

$$\dot{x} = y$$
$$\dot{y} = -x + yz \tag{6}$$
$$\dot{z} = 1 - y^2$$

We consider a general parametric form of Eq. (6) (without the constant term in $\dot{z}$) with quadratic nonlinearities of the form

$$\dot{x} = y$$
$$\dot{y} = a_1x + a_2yz \tag{7}$$

$\dot{z} = a_3x + a_4y + a_5x^2 + a_6y^2 + a_7xy + a_8xz + a_9yz$

As can be seen, this system has a line equilibrium at (0, 0, z) with no other equilibria (in other words the z-axis is an infinite line equilibrium of equilibrium).

As before, an exhaustive computer search was done considering millions of combinations of the coefficients a_1 through a_9 and initial conditions, seeking dissipative cases for which the largest Lyapunov exponent is greater than 0.001. Cases LE_1-LE_6 in Table 3 are six simple cases found in this way with only six terms. With a similar procedure, three other similar cases LE_7-LE_9 were found and included in Table 3.

The equilibria, eigenvalues, Lyapunov exponent spectra, and Kaplan-Yorke dimensions are shown in Table 3 along with initial conditions that are close to the attractor. A discussion of why these systems belong to the category of hidden attractors is in [13].

Table 3
Six simple chaotic flows with a line equilibrium.

Case	Equations	(a,b)	Equilibrium	Eigenvalues	LEs	D_{KY}	(x_0, y_0, z_0)
LE_1	$\dot{x} = y$ $\dot{y} = -x + yz$ $\dot{z} = -x - axy - bxz$	$a = 15$ $b = 1$	0 0 z	$\frac{z \pm \sqrt{z^2 - 4}}{2}$ 0	0.0717 0 −0.5232	2.1371	0 0.5 0.5
LE_2	$\dot{x} = y$ $\dot{y} = -x + yz$ $\dot{z} = -y - axy - bxz$	$a = 17$ $b = 1$	0 0 z	$\frac{z \pm \sqrt{z^2 - 4}}{2}$ 0	0.0564 0 −0.2927	2.1927	0 0.4 0
LE_3	$\dot{x} = y$ $\dot{y} = -x + yz$ $\dot{z} = x^2 - axy - bxz$	$a = 18$ $b = 1$	0 0 z	$\frac{z \pm \sqrt{z^2 - 4}}{2}$ 0	0.0556 0 −0.3245	2.1714	0 −0.4 0.5
LE_4	$\dot{x} = y$ $\dot{y} = -x + yz$ $\dot{z} = -axy - bxz - yz$	$a = 4$ $b = 0.6$	0 0 z	$\frac{z \pm \sqrt{z^2 - 4}}{2}$ 0	0.0539 0 −0.3147	2.1712	0.2 0.7 0
LE_5	$\dot{x} = y$ $\dot{y} = -ax + yz$ $\dot{z} = -x^2 + y^2 - bxy$	$a = 1.5$ $b = 5$	0 0 z	$\frac{z \pm \sqrt{z^2 - 4a}}{2}$ 0	0.1386 0 −1.3764	2.1007	0.7 1 0

	$\dot{x}=y$	$a=0.04$	0	$\frac{z\pm\sqrt{z^2-4}}{2}$	0.0543		1
LE_6	$\dot{y}=-x+yz$	$b=0.1$	0		0	2.0860	2
	$\dot{z}=ay^2-xy-bxz$		z	0	−0.6314		0
	$\dot{x}=z$	$a=1.85$	0	$\frac{-0.3y\pm\sqrt{0.09y^2-4y}}{2}$	0.1144		5.1
LE_7	$\dot{y}=x+yz$	$b=0.3$	y		0	2.0140	7
	$\dot{z}=ax^2-xy-byz$		0	0	−1.0270		0
	$\dot{x}=z$	$a=3$	0	$\pm\sqrt{y}$	0.0521		0
LE_8	$\dot{y}=x-yz$	$b=1$	y		0	2.0647	−0. 3
	$\dot{z}=ax^2+xy+bxz$		0	0	−0.8053		−1
	$\dot{x}=z$		x	$\frac{-0.62\pm\sqrt{6.8644-4x^2}}{2}$	0.0642		0
LE_9	$\dot{y}=-ay-xz$	$a=1.62$	0		0	2.0939	1
	$\dot{z}=z-bz^2+xy$	$b=0.2$	0	0	– 0. 6842		0.8

3. Circuit realization of three new families of hidden chaotic attractors

3.1. Circuit realization of one chaotic flow with no equilibria

The electronic chaos generator of the chaotic flow with no equilibrium NE_6 was designed (Fig. 1). The circuit consists of common electronic components such as resistors, operational amplifiers, capacitors and multipliers. The variables of the systems *x, y, z* correspond to the voltages of capacitors C_1, C_2, and C_3, respectively. Hence the dynamics of the proposed circuit can be expressed as three differential equations in the corresponding voltage variables v_{C_1}, v_{C_2}, v_{C_3} as

$$\begin{cases} \dfrac{dv_{C_1}}{dt}=\dfrac{1}{R_1C_1}\dfrac{R_8}{R_7}v_{C_2}, \\ \dfrac{dv_{C_2}}{dt}=\dfrac{1}{R_2C_2}\dfrac{R_{10}}{R_9}v_{C_3}, \\ \dfrac{dv_{C_3}}{dt}=-\dfrac{1}{R_3C_3}v_{C_2}-\dfrac{1}{10R_4C_3}v_{C_1}v_{C_3}-\dfrac{1}{10R_5C_3}v_{C_2}v_{C_3}-\dfrac{1}{R_6C_3}V_a. \end{cases} \tag{8}$$

The values of the components are chosen as follows: $R_1 = R_2 = R_3 = R_6 = R_7 = R_8 = R_9 = R_{10} = 10k\Omega$, $R_4 = R_5 = 1k\Omega$, $C_1 = C_2 = C_3 = 20nF$, and $V_a = 0.75V_{DC}$ .The designed circuit was simulated using the electronic simulation package Multisim for the initial conditions $\left(v_{C_1}(0), v_{C_2}(0), v_{C_2}(0)\right) = (0V, 3V, -0.1V)$. The resulting phase portraits shown in Fig. 2 are similar to the theoretical ones in [11].

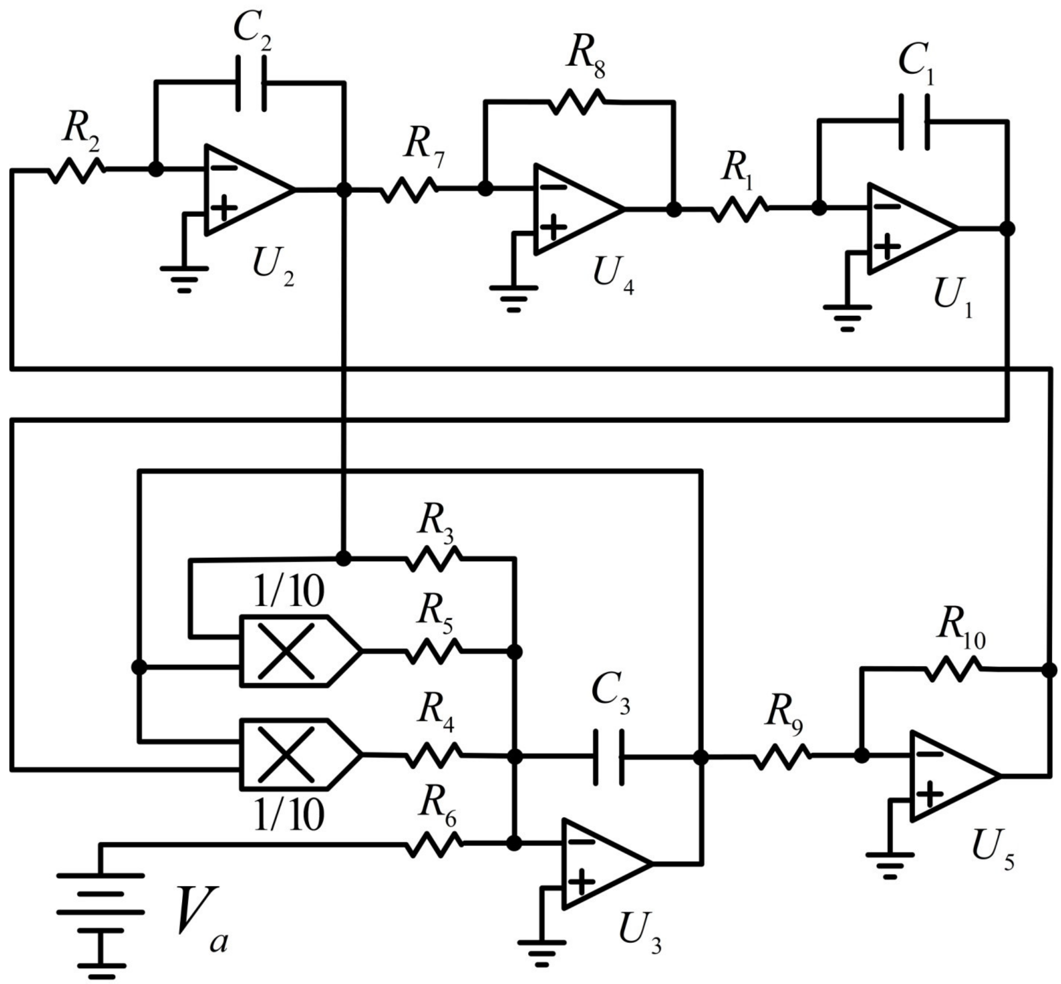

Fig.1. Circuit realization of the chaotic system with no equilibrium NE_6.

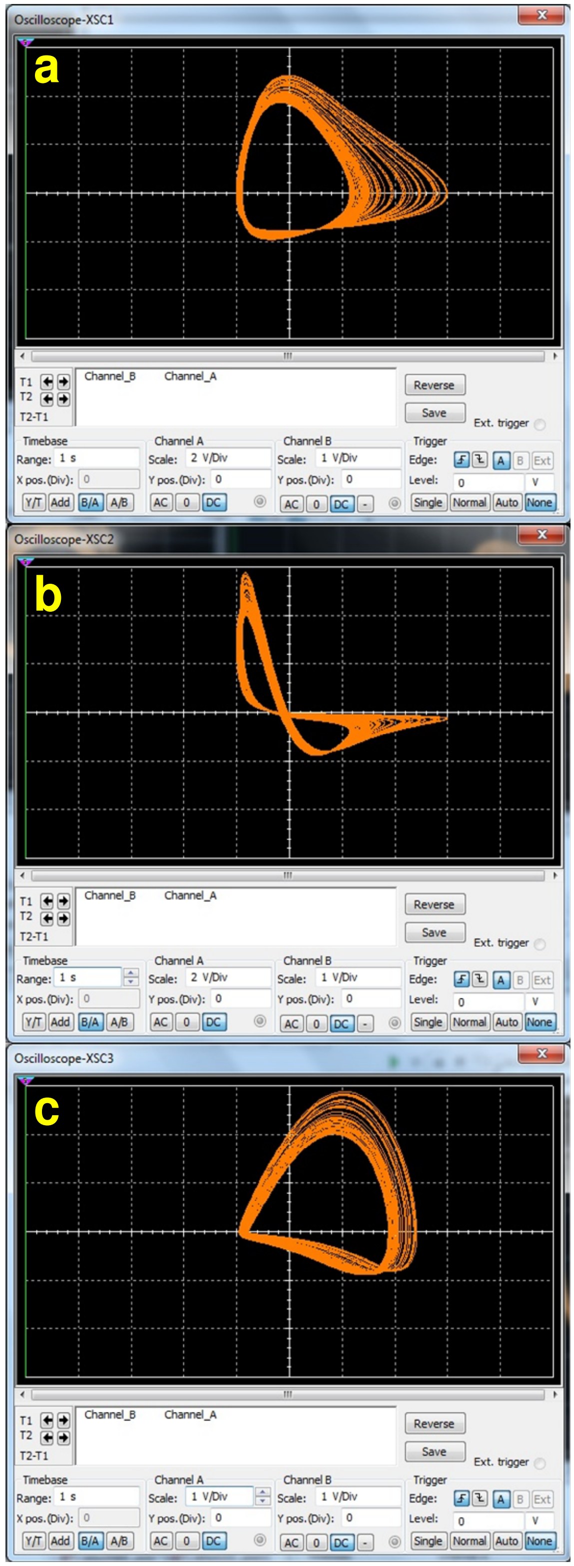

Fig. 2. Simulation results of the designed circuit for the NE$_6$ system using Multisim software: a) $v_{C_1} - v_{C_2}$ phase portrait b) $v_{C_1} - v_{C_3}$ phase portrait c) $v_{C_2} - v_{C_3}$ phase portrait

3.2. Circuit realization of one chaotic flow with stable equilibria

An analog circuit emulating the chaotic flow SE_1 with a stable equilibrium is illustrated in Fig. 3. The schematic consists of eleven resistors (from R_1 to R_{11}), three capacitors (C_1, C_2, and C_3), and five operational amplifiers (from U_1 to U_5). By applying Kirchhoff's law to the circuit in Fig. 3, the dynamics of the designed circuit is described by the circuit equations

$$\begin{cases} \dfrac{dv_{C_1}}{dt} = \dfrac{1}{R_1C_1}\dfrac{R_9}{R_8}v_{C_2}, \\ \dfrac{dv_{C_2}}{dt} = \dfrac{1}{R_2C_2}\dfrac{R_{11}}{R_{10}}v_{C_3}, \\ \dfrac{dv_{C_3}}{dt} = -\dfrac{1}{R_3C_3}v_{C_1} - \dfrac{1}{R_4C_3}v_{C_2} - \dfrac{1}{R_5C_3}v_{C_3} + \dfrac{1}{10R_6C_3}v_{C_3}^2 - \dfrac{1}{10R_7C_3}v_{C_1}v_{C_2} \end{cases} \tag{9}$$

where v_{C_1}, v_{C_2}, v_{C_3}, are the voltages across the capacitors C_1, C_2, and C_3, respectively. The values of the electronic components in Fig. 3 are selected to match known parameters of the chaotic flow SE_1 with a stable equilibrium: $R_1 = R_2 = R_3 = 6k\Omega,\ R_5 = 3k\Omega,\ R_6 = 0.6k\Omega,\ R_7 = 1.5k\Omega,\ R_4 = R_8 = R_9 = R_{10} = R_{11} = 10k\Omega,\ C_1 = C_2 = C_3 = 1nF.$

The proposed circuit in Fig. 3 was simulated using the electronic simulation package Multisim with initial conditions of $\left(v_{C_1}(0), v_{C_2}(0), v_{C_2}(0)\right) = (4V, -2V, 0V)$. The resulting chaotic attractors are shown in Fig. 4.

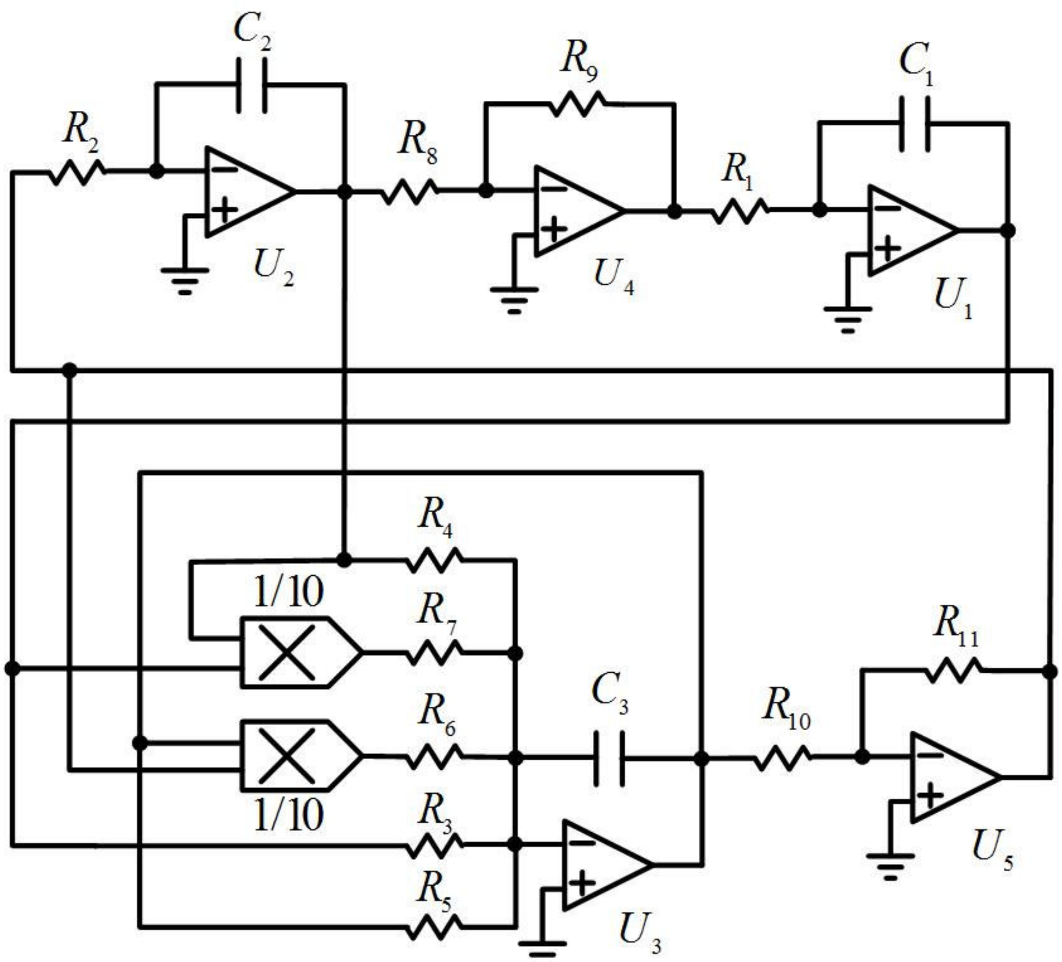

Fig. 3. The proposed circuit which emulates the chaotic flow with a stable equilibrium SE_1.

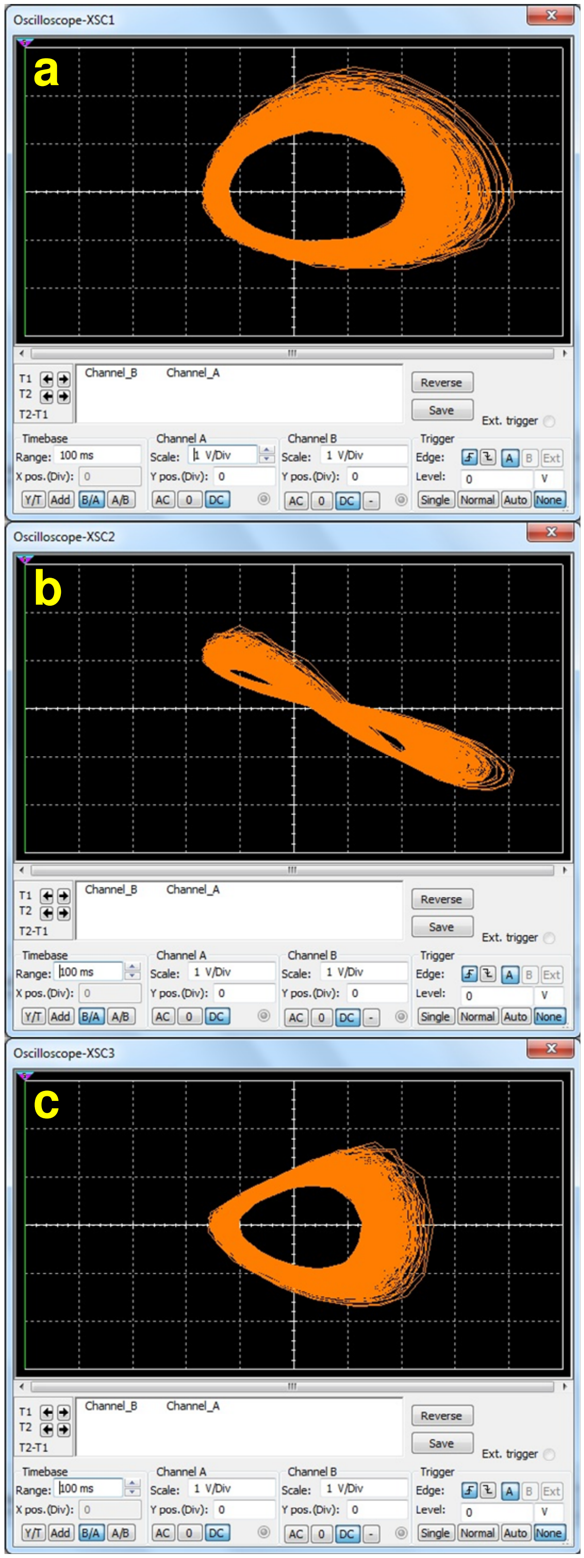

Fig. 4. Chaotic attractors exhibited by the circuit in Fig. 3: a) $v_{C_1} - v_{C_2}$ phase portrait b) $v_{C_1} - v_{C_3}$ phase portrait c) $v_{C_2} - v_{C_3}$ phase portrait

3.3. Circuit realization of one chaotic flow with a line equilibria

An analog circuit was also designed to realize the chaotic flow with a line equilibrium LE_1. The schematic of the proposed circuit is shown in Fig. 5. The circuit equations, which are derived by applying Kirchhoff's law to the circuit in Fig. 5, can be written as

$$\begin{cases} \dfrac{dv_{C_1}}{dt} = \dfrac{1}{R_1 C_1}\dfrac{R_8}{R_7} v_{C_2}, \\ \dfrac{dv_{C_2}}{dt} = -\dfrac{1}{R_2 C_2} v_{C_1} + \dfrac{1}{10 R_3 C_2} v_{C_2} v_{C_3}, \\ \dfrac{dv_{C_3}}{dt} = -\dfrac{1}{R_4 C_3} v_{C_1} - \dfrac{1}{10 R_5 C_3} v_{C_1} v_{C_2} - \dfrac{1}{10 R_6 C_3} v_{C_1} v_{C_3}. \end{cases} \tag{10}$$

Four operational amplifiers, eight resistors, three capacitors, and three multipliers were used. The values of the components are $R_1 = R_2 = R_4 = R_7 = R_8 = 100k\Omega$, $R_3 = R_6 = 10k\Omega$, $R_5 = 0.666k\Omega$, and $C_1 = C_2 = C_3 = 1nF$. Multisim simulations were implemented with the initial conditions $\left(v_{C_1}(0), v_{C_2}(0), v_{C_2}(0)\right) = (0V, 0.5V, 0.5V)$. The resulting phase portraits are shown in Fig. 6.

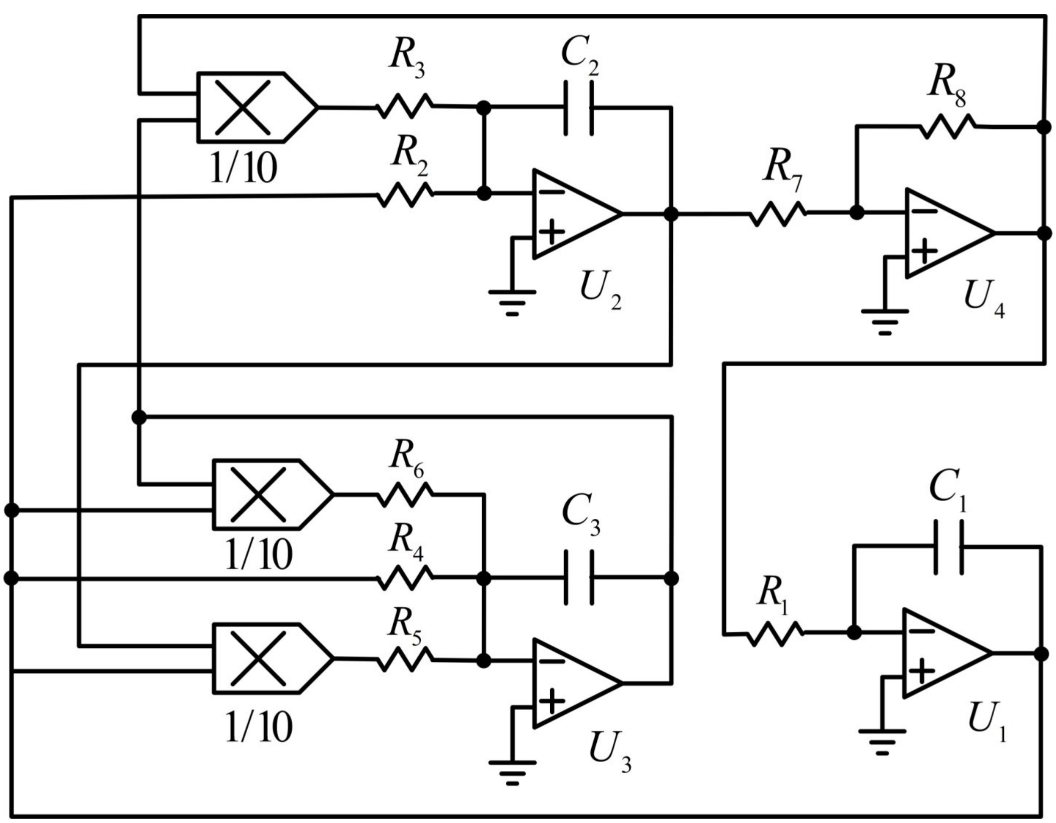

Fig. 5. Schematic of the designed analog circuit for the chaotic system with a line equilibrium LE_1.

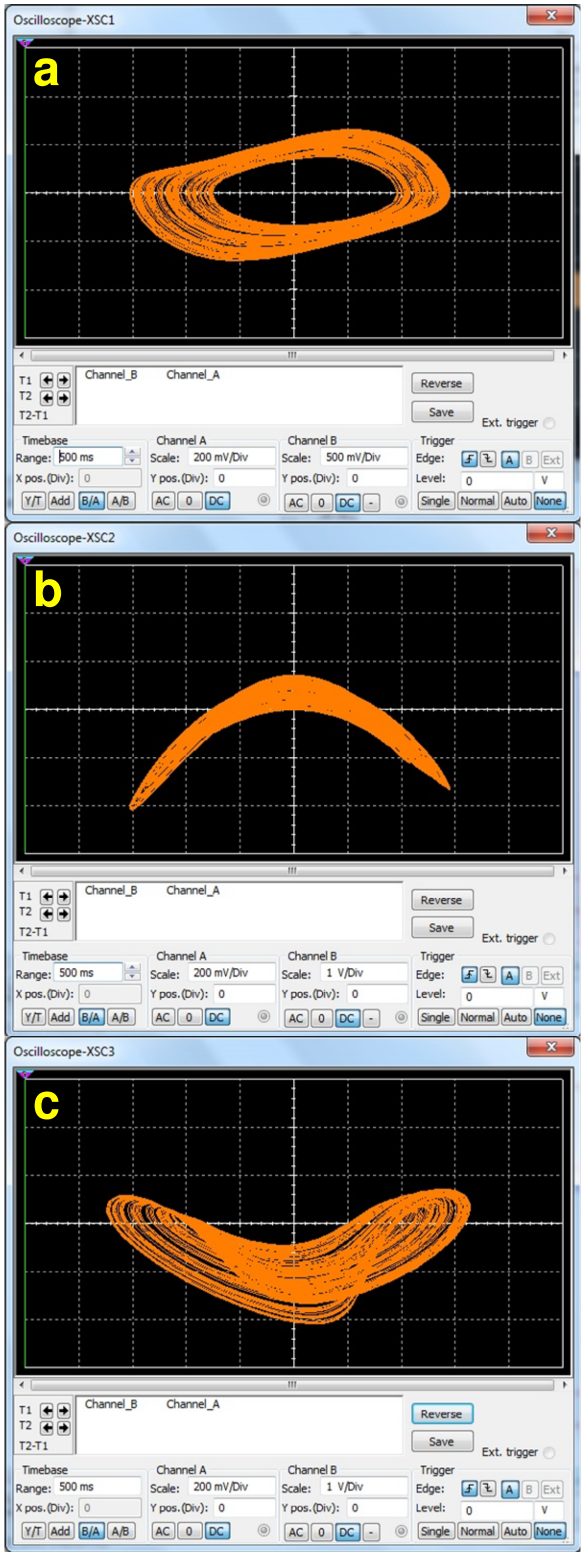

Fig. 6. Simulation results of the designed circuit for the LE_1 system using Multisim software: a) $v_{C_1} - v_{C_2}$ phase portrait b) $v_{C_1} - v_{C_3}$ phase portrait c) $v_{C_2} - v_{C_3}$ phase portrait

3. 4. Discussion

There are some precautions for designing an analog circuit that emulates the dynamics of a chaotic flow, especially when the attractor is hidden:

a) The components of the analog circuit must be selected carefully to match the mathematical model. Choosing correct off-the-shelf discrete components is a practical challenge. For example, it is easy to change the parameters and the eigenvalues of the chaotic system with one stable equilibrium when inappropriate values of the electronic components are used. In such cases, the dynamics of the system can change radically.

b) The limits of operational amplifiers and analog multipliers, such as saturation, power supply voltages, nonlinearities, frequency limitations, and acceptable inputs must be considered. For example in a chaotic system like NE_4, the amplitudes of the variables are much greater than the other cases (see Fig. 1 in [11]).

c) A special sub-circuit generating the initial conditions for the voltages on the capacitors should be implemented to provide the appropriate initial conditions. Setting appropriate initial conditions is essential for realizing a chaotic flow with coexisting attractors since such a circuit will not oscillate if the initial conditions are in the basin of the stable equilibrium. In fact, this is the main difference between chaotic circuits with hidden attractors and those with self-excited attractors. Since the attractor is hidden, it will not be observed when using initial conditions close to an equilibrium.

d) By the abovementioned discussion, it seems that the chaotic flows with one stable equilibrium should not be suggested as student projects in the limit time of a course, since they are more difficult to construct (as we experienced).

4. Conclusion

Categorizing dynamical systems into systems with hidden attractors and systems with self-excited attractors is a new topic in dynamical systems. We described three newly introduced families of chaotic systems with hidden attractors (chaotic attractors in dynamical systems without any equilibrium points, with only stable equilibria, and with a line of equilibria). We designed a circuit for one example of each family. We described the difficulties in implementing circuits with hidden attractors resulting from the necessity of carefully choosing component values and initial conditions.

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