Physics Letters A 378 (2014) 1361–1363

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

A dynamical system with a strange attractor and invariant tori

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A R T I C L E I N F O

Article history:
Received 15 January 2014
Received in revised form 13 March 2014
Accepted 17 March 2014
Available online 21 March 2014
Communicated by C.R. Doering

Keywords:
Dynamical system
Strange attractor
Invariant torus

A B S T R A C T

This paper describes a simple three-dimensional time-reversible system of ODEs with quadratic nonlinearities and the unusual property that it is exhibits conservative behavior for some initial conditions and dissipative behavior for others. The conservative regime has quasi-periodic orbits whose amplitude depend on the initial conditions, while the dissipative regime is chaotic. Thus a strange attractor coexists with an infinite set of nested invariant tori in the state space.

1. Introduction

Continuous-time dynamical systems governed by a set of first-order ordinary differential equations are usually categorized as being either conservative or dissipative. Conservative systems maintain the state space volume when time-averaged along the trajectory and are usually characterized by some conserved quantity such as a Hamiltonian function. Depending on the initial conditions, they typically have orbits that are quasi-periodic and lie on surfaces of nested tori surrounded by a chaotic sea whose dimension is equal to the dimension of the state space.

Dissipative systems have a state space volume that decreases on average along the trajectory so that the orbit approaches an attractor of measure zero in the state space. If the dissipative system is chaotic, the attractor is strange with a non-integer dimension and fractal structure. Furthermore, it is possible for the same dissipative system to have coexisting stable equilibria, limit cycles, and strange attractors each with its own dimension and basin of attraction [1,2].

However, if the dissipation is nonlinear and thus dependent on the position in state space, systems can be dissipative for some initial conditions, while other initial conditions lead to solutions for which the dissipation averages exactly to zero along the orbit [3]. The next section describes a simple example of such a system in which a strange attractor with chaotic orbits coexists with an infinite set of nested invariant tori containing quasi-periodic orbits.

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http://dx.doi.org/10.1016/j.physleta.2014.03.028
0375-9601/

2. Example

In a numerical search for chaotic systems that have no equilibrium points and only bounded orbits for all initial conditions, the following unusual system was discovered:

$$\begin{aligned}\dot{x} &= y + 2xy + xz\\ \dot{y} &= 1 - 2x^2 + yz\\ \dot{z} &= x - x^2 - y^2\end{aligned} \tag{1}$$

The numerical evidence of boundedness is that the flow vector is everywhere inward on a sphere of radius R centered on the point $(1, 0, -1)$ except for small holes at $(0, 0, \pm R)$ that occupy a vanishingly small fraction of the sphere as $R \to \infty$. Furthermore, orbits with initial conditions within these holes loop back to the interior of the sphere for R sufficiently large. Since the system is bounded with no equilibrium points (stable or unstable), the only possible solutions are (quasi)-periodic or chaotic.

Because the system is invariant under the transformation $(x, y, z, t) \to (x, -y, -z, -t)$, two types of solutions can occur. The first type is symmetric under a 180° rotation about the x-axis and is time-reversal invariant. Thus it exhibits conservative behavior. The second type has an attractor in forward time and another attractor in reversed time that are symmetric with one another through a 180° rotation about the x-axis. This reversed time attractor is a repellor in forward time. Since the same points in state space cannot be both an attractor and a repellor, any symmetric solutions necessarily conserve state space volume on average (non-attracting). In fact, it turns out that the system displays both behaviors depending on the initial conditions.

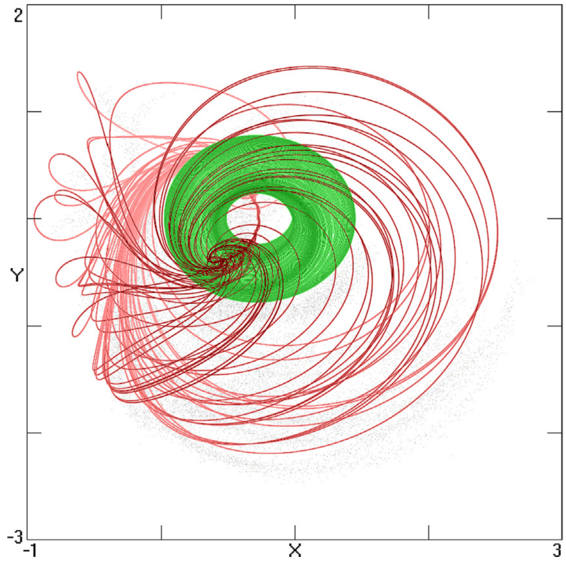

Fig. 1. A strange attractor (red) interlinked with a coexisting invariant torus (green) for System (1). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

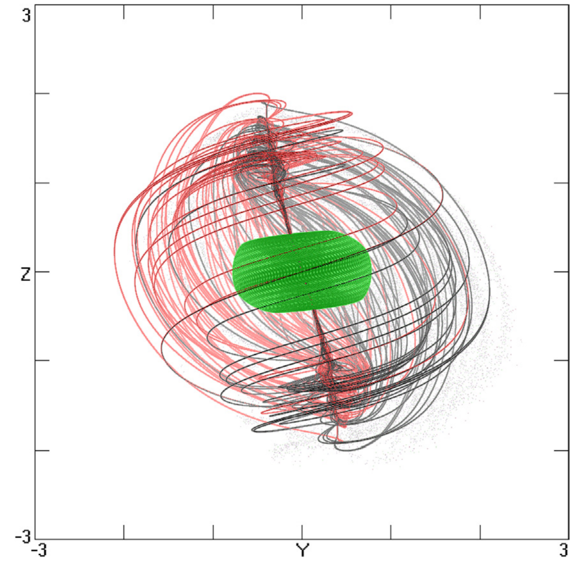

Fig. 2. A different view of the strange attractor (red) intertwined with a symmetric repellor (black) and both interlinked with a coexisting invariant torus (green) for System (1). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The dissipation is given by the trace of the Jacobian matrix, $\mathrm{Tr}(J) = 2(y+z)$ whose value depends on y and z and thus should be time-averaged along the orbit, and whose average value is the sum of the Lyapunov exponents. The surprise was that the time average of $\langle y+z \rangle$ is negative for some initial conditions such as $(x_0, y_0, z_0) = (2, 0, 0)$ and zero for others such as $(1, 0, 0)$. The boundedness of the system ensures that $\langle y+z \rangle$ cannot be positive, and the absence of equilibrium points ensures that all orbits are time-dependent. The first initial condition gives a strange attractor with Lyapunov exponents $(0.0540, 0, -0.1575)$ and a Kaplan–Yorke dimension of 2.3429, and the second initial condition gives a torus with Lyapunov exponents $(0, 0, 0)$ and a dimension of 2.0.

The strange attractor is strongly multifractal with a capacity dimension of about 2.92 and a correlation dimension of about 1.49. Since there are no equilibrium points, the strange attractor is "hidden" in the sense described by Leonov and Kuznetsov [4,5], meaning that its basin does not intersect with small neighborhoods of any equilibrium points, and thus it cannot be found by standard computational methods. However, because the system is bounded, any initial condition sufficiently far from the origin will go to the attractor ($R = 5$ is sufficient).

The two solutions are shown in Fig. 1 projected onto the xy-plane. The strange attractor and torus are interlinked, with the attractor resembling the "cord" attractor described by Letellier and Aguirre [6]. A different projection edge-on to the torus and also showing the intertwined repellor is in Fig. 2. The attractor and repellor are tightly twisted like the strands of a rope where they pass through the hole in the torus. The repellor is identified by simply reversing the sign of time in the equations and showing a portion of the resulting orbit after discarding the initial transient. Additional confirmation that the torus is non-attracting comes from a very long calculation (to a time of $t = 1 \times 10^{10}$) using a fourth-order Runge–Kutta integrator with adaptive step size, which gives an upper bound of $|\langle y+z \rangle| < 1 \times 10^{-10}$.

Like all strange attractors, this one is dense in unstable periodic orbits. The orbit with the shortest period lies close to the surface of the outermost torus and makes five loops projected onto the xy-plane while linking the torus once. It has a period of approximately 12.58, a net dissipation of $2\langle y+z \rangle \simeq -0.0121$, and can be observed by using the initial conditions $(0.5002, 0.0023, -0.0791)$.

Fig. 3 shows a cross section of the flow in the $z = 0$ plane for 80 initial conditions uniformly distributed over the range $-2 < x_0 < 3$

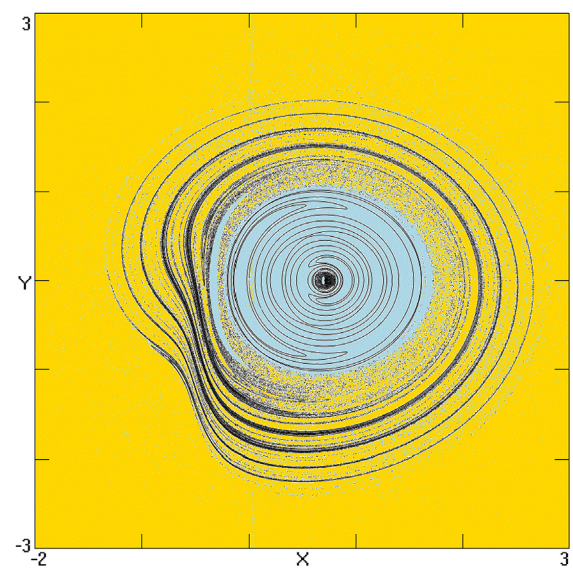

Fig. 3. Cross section in the $z = 0$ plane showing nested invariant tori surrounded by a multifractal strange attractor for System (1). Initial conditions in the conservative region are shown in blue, and initial conditions in the basin of attraction of the strange attractor are shown in yellow.(For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

with $y_0 = z_0 = 0$. This plot is called a "cross section" rather than a "Poincaré section" because the crossings of the $z = 0$ plane are plotted in both directions rather than a single direction. The nested invariant tori are surrounded by a strange attractor. The basin boundary of the strange attractor appears to be coincident with the outermost toroidal surface, and its basin extends to infinity in all directions. The background colors in the figure show the regions of conservative and dissipative behavior, respectively.

The boundary between the conservative and dissipative region in the $z = 0$ plane is near a circle given by $(x - 1/\sqrt{2})^2 + y^2 = 1$, a result that begs for an explanation. Initial conditions near that boundary such as $(0, 0.707, 0)$ or near one of the separatrices within the toroidal region such as $(0.4999, 0.01, 0)$ can appear to be conservative and chaotic, but either the largest Lyapunov exponent eventually converges to zero or the orbit eventually reaches the strongly dissipative strange attractor. There is no convincing evidence for a conservative chaotic sea.

3. Conclusions

This paper describes an unusual example of a chaotic dynamical system that exhibits conservative behavior for some initial conditions and dissipative behavior for others so that an infinite set of nested invariant tori coexist with and are linked by a strange attractor. For initial conditions sufficiently close to the origin, the quasi-periodic orbits are such that the time average of $\langle y + z \rangle$ is identically zero to within numerical precision, while orbits starting farther from the origin have a strongly negative value of that quantity with a sharp, nearly circular toroidal boundary separating the two regions. It would be good to have a better understanding of this phenomenon, a theory for when it will occur, a prediction for the location of the boundary, and details of the bifurcations that occur in the five-dimensional parameter space, but those challenges will be left for the future.

Shortly after discovering this system, another similar example was found in a nonlinearly dissipative Nosé–Hoover oscillator with a hyperbolic tangent nonlinearity and four equilibrium points [7]. In that case, an attracting limit cycle coexists with two sets of invariant tori for some values of the parameters, while a space-filling multifractal strange attractor with a Kaplan–Yorke dimension of 2.945 coexists with the tori for other values of the parameters. This result suggests that the behavior described here may be common in time-reversible dynamical systems with nonlinear damping but apparently not widely known or appreciated.

Acknowledgements

I am grateful to Zeraoulia Elhadj for suggesting the search that uncovered this example and to Bill Hoover for extensive discussions.

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