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When Two Dual Chaotic Systems Shake Hands

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This letter reports an interesting finding that the parametric Lorenz system and the parametric Chen system "shake hands" at a particular point of their common parameter space, as the time variable $t \to +\infty$ in the Lorenz system while $t \to -\infty$ in the Chen system. This helps better clarify and understand the relationship between these two closely related but topologically nonequivalent chaotic systems.

Keywords: Chaos; Lorenz system; Chen system; topological equivalence.

1. Introduction

The celebrated parametric Lorenz system [Lorenz, 1963] is described by

$$\begin{aligned} \frac{dX}{dt} &= \sigma(Y - X) \\ \frac{dY}{dt} &= \rho X - Y - XZ \\ \frac{dZ}{dt} &= -\beta Z + XY \end{aligned} \tag{1}$$

where σ, ρ and β are real parameters.

The term *chaotic Lorenz system* (or *Lorenz system, Lorenz equation*) in the literature typically refers to its specific setting

$$\begin{aligned} \frac{dX}{dt} &= 10(Y - X) \\ \frac{dY}{dt} &= 28X - Y - XZ \\ \frac{dZ}{dt} &= -\frac{8}{3}Z + XY \end{aligned} \tag{2}$$

which is chaotic on a particular parameter set, $(\sigma, \rho, \beta) = (10, 28, 8/3)$[1] in the real parameter space of the general system (1); but for other parameter sets, system (1) may not be chaotic. The chaotic Lorenz system (2) has received sustained research interest in the literature since its discovery. For example, in [Tucker, 1999; Stewart, 2000], a rigorous proof confirms the existence of the chaotic attractor in system (2); in [Franceschini *et al.*, 1993], the chaotic attractor of system (2) is characterized in terms of unstable periodic orbits; in [Gilmore *et al.*, 2003, 2007; Mindlin *et al.*, 1990;

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[1]Due to the structural stability of the Lorenz system, this parameter set can be somewhat enlarged, but using this set does not affect the discussions throughout. The same remark applies to system (4).

Tsankov & Gilmore, 2004, 2003; Gilmore & Lefranc, 2002, 2008], various topological properties of the chaotic system (2) are studied.

The above-mentioned research were all carried out based on a common observation that, with different parameter sets other than $(\sigma, \rho, \beta) = (10, 28, 8/3)$, the parametric Lorenz system (1) may be simply convergent, divergent, periodic, or have other types of complex dynamics irrelevant to or very different from the chaotic attractor of system (2).

In some sense [Čelikovský & Chen, 2002] being a *dual system* to the Lorenz system (1), the so-called parametric Chen system [Chen & Ueta, 1999] is described by

$$\begin{aligned} \frac{dx}{dt} &= a(y - x) \\ \frac{dy}{dt} &= (c - a)x + cy - xz \\ \frac{dz}{dt} &= -bz + xy \end{aligned} \tag{3}$$

where a, b and c are real parameters.

It was found that this parametric system is chaotic for the parameter set $(a, b, c) = (35, 3, 28)$, giving

$$\begin{aligned} \frac{dx}{dt} &= 35(y - x) \\ \frac{dy}{dt} &= -7x + 28y - xz \\ \frac{dz}{dt} &= -3z + xy. \end{aligned} \tag{4}$$

Likewise, for other parameter sets of (a, b, c), system (3) may not be chaotic.

2. When the Two Systems Shake Their Hands

Recently, we found that, with the parameter set $(\sigma, \rho, \beta) = (0.4, -1.4, -0.4)$, system (1) becomes

$$\begin{aligned} \frac{dX}{dt} &= 0.4(Y - X) \\ \frac{dY}{dt} &= -1.4X - Y - XZ \\ \frac{dZ}{dt} &= 0.4Z + XY \end{aligned} \tag{5}$$

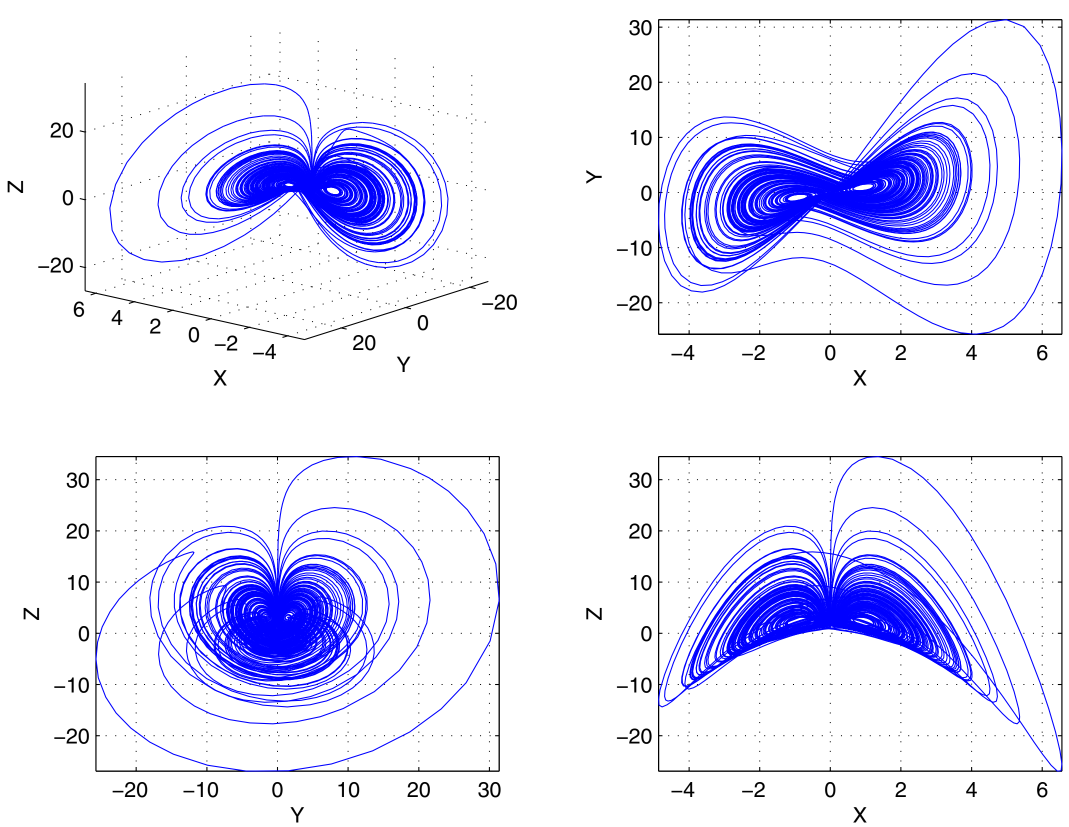

Fig. 1. The same chaotic attractor produced by both — system (5) as $t \to +\infty$ and system (6) as $t \to -\infty$.

and, with the parameter set $(a, b, c) = (-0.4, 0.4, 1)$, system (3) becomes

$$\begin{aligned}\frac{dx}{dt} &= -0.4(y - x)\\ \frac{dy}{dt} &= 1.4x + y - xz\\ \frac{dz}{dt} &= -0.4z + xy.\end{aligned} \tag{6}$$

We observed that these two systems produce the same limiting set — a chaotic attractor if one lets $t \to +\infty$ in system (5) and a chaotic repellor if one lets $t \to -\infty$ in system (6) — the two limiting sets are identical (see Fig. 1).

In fact, a simple transformation $(x, y, z, t) \to (X, Y, -Z, -t)$ can convert system (6) to system (5). Note, however, that according to the conventional definitions of topological equivalence and conjugacy, which preserves the flow orientation (see, e.g. [Wiggins, 2003]), systems (2) and (4) have been proved to be nonequivalent topologically [Hou *et al.*, 2010]. Likewise, systems (5) and (6) are nonequivalent topologically because of time reversal here, and generally, does not preserve flow orientation: a well-known example is the following two linear systems:

$$\frac{dX}{dt} = X \quad \text{and} \quad \frac{dx}{dt} = -x \tag{7}$$

which can be converted from one to another via $t \to -t$, but the first system is unstable and the second one is stable about their zero equilibria, with opposite flow orientations (namely, the time reversion transform $t \to -t$ does not preserve flow orientation).

3. Conclusions

This letter has reported a very interesting finding that the parametric Lorenz system (1) and the parametric Chen system (3) "shake hands" at a single parameter point $(\sigma, \rho, \beta) = (0.4, -1.4, -0.4)$ as $t \to +\infty$ for the former and a single parameter point $(a, b, c) = (-0.4, 0.4, 1)$ as $t \to -\infty$ for the latter. This further reveals more intermittent relationship between the Lorenz system and the Chen system, in addition to their algebraic duality and topological nonequivalence.

A final remark is that, although it has been well known that the parametric Lorenz system (1) is chaotic with the parameter set $(\sigma, \rho, \beta) = (10, 28, 8/3)$, in this letter we found that it is also chaotic with the parameter set $(\sigma, \rho, \beta) = (0.4, -1.4, -0.4)$.

References

Čelikovský, S. & Chen, G. [2002] "On a generalized Lorenz canonical form of chaotic systems," *Int. J. Bifurcation and Chaos* **12**, 1789–1812.

Chen, G. & Ueta, T. [1999] "Yet another chaotic attractor," *Int. J. Bifurcation and Chaos* **9**, 1465–1466.

Franceschini, V., Giberti, C. & Zheng, Z. [1993] "Characterization of the Lorentz attractor by unstable periodic orbits," *Nonlinearity* **6**, 251.

Gilmore, R. & Lefranc, M. [2002] *The Topology of Chaos*: *Alice in Stretch and Squeezeland* (Wiley).

Gilmore, R., Lefranc, M. & Tufillaro, N. B. [2003] "The topology of chaos," *American J. Phys.* **71**, 508.

Gilmore, R., Letellier, C. & Romanazzi, N. [2007] "Global topology from an embedding," *J. Phys. A*: *Math. Theoret.* **40**, 13291.

Gilmore, R. & Lefranc, M. [2008] *The Topology of Chaos* (Wiley).

Hou, Z., Kang, N., Kong, X., Chen, G. & Yan, G. [2010] "On the nonequivalence of Lorenz system and Chen system," *Int. J. Bifurcation and Chaos* **20**, 557–560.

Lorenz, E. N. [1963] "Deterministic nonperiodic flow," *J. Atmosph. Sci.* **20**, 130–141.

Mindlin, G. B., Hou, X.-J., Solari, H. G., Gilmore, R. & Tufillaro, N. B. [1990] "Classification of strange attractors by integers," *Phys. Rev. Lett.* **64**, 2350.

Stewart, I. [2000] "Mathematics: The Lorenz attractor exists," *Nature* **406**, 948–949.

Tsankov, T. D. & Gilmore, R. [2003] "Strange attractors are classified by bounding tori," *Phys. Rev. Lett.* **91**, 134104.

Tsankov, T. D. & Gilmore, R. [2004] "Topological aspects of the structure of chaotic attractors in r^3," *Phys. Rev. E* **69**, 056206.

Tucker, W. [1999] "The Lorenz attractor exists," *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics* **328**, 1197–1202.

Wiggins, S. [2003] *Introduction to Applied Nonlinear Dynamical Systems and Chaos* (Springer).