# A rigorous solution of the chaotic Lozi mapping

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**Abstract**

In this letter, a rigorous solution of the chaotic Lozi mapping is given for some regions of the bifurcation parameters. The relevance of this result is that while chaos has been proved for the Lozi mapping, there has not previously been discussion of the possibility of finding a rigorous formula for this solution. This important result is obtained using the Jordan normal form defined for matrices.

*Keywords:* Lozi map, chaotic attractor, rigorous solution.
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## 1 Introduction

Many physical systems are known to be best represented by piecewise maps [di Bernardo, *et al.*,1999; Hassouneh, *et al.*, 2001; Banergee, *et al.*, 1998; Banerjee & Grebogi, 1999; Banerjee & Verghese, 2001; Banerjee, *et al.*, 2004; Rajaraman, *et al.*, 1996; Tse, 2003; Dutta, *et al.*, 1999]. Such maps [Devaney,1984; Lozi, 1978; Aharonov, *et al.*, 1997; Ashwin & Fu, 2002] are interesting for the development of the theory of dynamical systems with practical applications [Scheizer & Hasler, 1996; Abel, *et al.*,1997]. For these maps, the discrete-time state space is divided into several compartments with different functional forms separated by borderlines. The simplest dissipative piecewise time-delayed map with chaotic solutions is the well known Lozi mapping [Lozi, 1978; Misiurewicz, 1980] given by

$$\left\{\begin{array}{r} x_{n+1}=1-a\left|x_n\right|+by_n \\ y_{n+1}=x_n \end{array}\right. \tag{1}$$

where $a$ and $b$ are bifurcation parameters. This map is the subject of many works focused on its various properties. Most of these works are collected with detailed discussions in [Zeraoulia, 2013].

## 2 On a rigorous solution of the chaotic Lozi mapping

Note that the Lozi map (1) can be rewritten as

$$\left\{\begin{array}{l} X_{n+1}=AX_n+C, \text{ if } x_n\geq 0 \\ X_{n+1}=BX_n+C, \text{ if } x_n\leq 0 \end{array}\right. \tag{2}$$

where $X_n=\begin{pmatrix} x_n \\ y_n \end{pmatrix}$, $A=\begin{pmatrix} -a & b \\ 1 & 0 \end{pmatrix}$, $B=\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$, and $C=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

In this section, we give a rigorous formula for the chaotic solution of the Lozi map (1) when $a^2+4b>0$, i.e., $b>\frac{-a^2}{4}$. Indeed, the matrix $A$ has two real and distinct eigenvalues $\lambda_1=\frac{1}{2}\sqrt{a^2+4b}-\frac{1}{2}a$ and $\lambda_2=-\frac{1}{2}a-\frac{1}{2}\sqrt{a^2+4b}$. Hence, if we consider the Jordan normal form for the matrix $A$, i.e., $A=PJP^{-1}$, where $P$ is the matrix whose columns consist of the two eigenvectors $v_1$ and $v_2$ of the matrix $A$, then $J$ is given by $J=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. The formula $A=PJP^{-1}$ implies that $A^n=PJ^nP^{-1}$ for all $n\in\mathbb{N}$, and the eigenvalues of $A^n$ are the same as the eigenvalues of the matrix $J^n$. Here we have $P=\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$ and $P^{-1}=\frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$. On the other hand, the matrix $B$ also has two real and distinct eigenvalues $-\lambda_1$ and $-\lambda_2$. Hence, if we consider the Jordan normal form for the matrix $B$, i.e., $B=QLQ^{-1}$, where $Q$ is the matrix whose columns consist of the two eigenvectors $w_1$ and $w_2$ of the matrix $B$, then $L=-J$. Here we have $Q=\begin{pmatrix} -\lambda_1 & -\lambda_2 \\ 1 & 1 \end{pmatrix}$ and

$Q^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} -1 & -\lambda_2 \\ 1 & \lambda_1 \end{pmatrix}$.

To prove the boundedness of system (1), we assume that it is bounded and then we find its bound. If $x_n \geq 0$, then we have $X_1 = AX_0 + C$ and $X_2 = A^2 X_0 + (A + I_2) C$, and by successive iterations we get

$$X_{n+1} = P \left( J^{n+1} P^{-1} X_0 + \left( J^n + J^{n-1} + \ldots + I_2 \right) P^{-1} C \right)$$

If $x_n \leq 0$, applying the same method gives

$$X_{n+1} = Q \left( (-J)^{n+1} Q^{-1} X_0 + \left( (-J)^n + (-J)^{n-1} + \ldots + I_2 \right) Q^{-1} C \right)$$

Thus after some tedious calculations we obtain the next iteration solution $(x_{n+1}, y_{n+1})$ of the Lozi mapping (1) as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{cases} \begin{pmatrix} \xi_1 \lambda_1^{n+2} + \xi_2 \lambda_2^{n+2} + \xi_3 \\ \xi_1 \lambda_1^{n+1} + \xi_2 \lambda_2^{n+1} + \xi_3 \end{pmatrix}, \text{ if } x_n \geq 0 \\ \begin{pmatrix} \xi_4 \lambda_1^{n+2} + \xi_5 \lambda_2^{n+2} + \xi_6 \\ \xi_4 \lambda_1^{n+1} + \xi_5 \lambda_2^{n+1} + \xi_6 \end{pmatrix}, \text{ if } x_n \leq 0 \end{cases} \tag{3}$$

for any initial condition $(x_0, y_0) \in \mathbb{R}^2$ in the basin of attraction of the solution, where $(\xi_i)_{1 \leq i \leq 6}$ are given by

$$\begin{cases} \xi_1 = -\frac{(1-\lambda_1)x_0 + \lambda_2(\lambda_1 - 1)y_0 - 1}{(\lambda_1 - 1)(\lambda_1 - \lambda_2)}, \xi_2 = \frac{(1-\lambda_2)x_0 + \lambda_1(\lambda_2 - 1)y_0 - 1}{(\lambda_2 - 1)(\lambda_1 - \lambda_2)} \\ \xi_3 = \frac{1}{(\lambda_2 - 1)(\lambda_1 - 1)}, \xi_6 = \frac{1}{(\lambda_2 + 1)(\lambda_1 + 1)} \\ \xi_4 = -(-1)^n \frac{(1+\lambda_1)x_0\lambda_2 + \lambda_2(1+\lambda_1)y_0 - 1}{(\lambda_1 + 1)(\lambda_1 - \lambda_2)}, \xi_5 = (-1)^n \frac{(1+\lambda_2)x_0 + \lambda_1(1+\lambda_2)y_0 - 1}{(\lambda_2 + 1)(\lambda_1 - \lambda_2)} \end{cases} \tag{4}$$

with the conditions $\lambda_1 - \lambda_2 \neq 0, \lambda_1 - 1 \neq 0, \lambda_2 - 1 \neq 0$, $\lambda_1 + 1 \neq 0$, and $\lambda_2 + 1 \neq 0$, that is,

$$\mathcal{S} : \begin{cases} b > \frac{-a^2}{4}, b \notin \{a+1, 1-a\} \\ 0 < a < 2 \end{cases} \tag{5}$$

If $x_n$ has a fixed sign for all $n$, then the solution is not chaotic since the corresponding formula is the solution of a linear system given by (3). Hence a chaotic attractor of the Lozi map is possible if $x_n$ has a mixed sign. This claim depends on the location of $a$ and $b$, and thus it is not a general rule for identifying chaos.

The most important results available in the literature for the rigorous proof of chaos in Lozi mappings (1) are given in [Misiurewicz, 1980], with a mathematical proof that the Lozi map (2) has a strange attractor $\Lambda_{a,b}$ for the range of parameters defined by

$$\mathcal{M}:\begin{cases} 0<b<1, a>0, a>b+1 \\ 2a+b<4, b<\frac{a^2-1}{2a+1}, a\sqrt{2}>b+2 \end{cases} \tag{6}$$

and that the basin $B(\Lambda_{a,b})$ contains a neighborhood of $\Lambda_{a,b}$. The method of the proof is based essentially on finding a trapping region for the Lozi map (2) and then proving that this map has a hyperbolic structure.

Thus the rigorous formula given by (3) for the chaotic Lozi mapping is possible when $\mathcal{S}\cap\mathcal{M}$ is not empty, that is

$$\mathcal{S}\cap\mathcal{M}:\begin{cases} 0<b<\min\left(1,\frac{a^2-1}{2a+1}\right), b\notin\{a+1,1-a\} \\ \max\left(b+1,\frac{b+2}{\sqrt{2}}\right)<a<\min\left(2,\frac{1}{2}(4-b)\right) \end{cases} \tag{7}$$

To understood the mechanism for the distribution of points $(x_n, y_n)$ in the Lozi map (1), assume that $x_0>0$. Then the next point is given by the first equation of (3). In this case, $x_1=\xi_1\lambda_1^2+\xi_2\lambda_2^2+\xi_3$. If $x_1>0$, we continue with the first equation of (3) to get the value of $x_2$. If $x_1<0$, then we use the second equation of (3) to calculate $x_2$ and so on...

# 3 Conclusion

In this letter, a rigorous solution of the chaotic Lozi mapping is given for some regions of the bifurcation parameters. The result is determined by using the Jordan normal form defined for matrices. This example shows that it is possible to find rigorous solutions of other piecewise chaotic mappings by using the same method. This method opens an interesting direction for studying piecewise chaotic systems.

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