International Journal of Bifurcation and Chaos, Vol. 24, No. 1 (2014) 1450010 (11 pages)

DOI: 10.1142/S0218127414500102

Cost Function Based on Gaussian Mixture Model for Parameter Estimation of a Chaotic Circuit with a Hidden Attractor

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Received August 10, 2013

In this paper, we introduce a new chaotic system and its corresponding circuit. This system has a special property of having a hidden attractor. Systems with hidden attractors are newly introduced and barely investigated. Conventional methods for parameter estimation in models of these systems have some limitations caused by sensitivity to initial conditions. We use a geometry-based cost function to overcome those limitations by building a statistical model on the distribution of the real system attractor in state space. This cost function is defined by the use of a likelihood score in a Gaussian Mixture Model (GMM) which is fitted to the observed attractor generated by the real system in state space. Using that learned GMM, a similarity score can be defined by the computed likelihood score of the model time series. The results show the adequacy of the proposed cost function.

Keywords: Parameter estimation; chaotic circuits; Gaussian mixture model; cost function; state space; stable equilibrium; hidden attractors.

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1. Introduction

Recently there has been increasing attention on some unusual chaotic systems as those having no equilibrium, stable equilibria, or coexisting attractors [Jafari *et al.*, 2013b; Molaie *et al.*, 2013; Uyaroğlu & Pehlivan, 2010; Wang & Chen, 2012, 2013; Wang *et al.*, 2012a; Wang *et al.*, 2012b; Wei, 2011a, 2011b; Wei & Yang, 2010, 2011, 2012]. Recent research has involved categorizing periodic and chaotic attractors as either self-excited or hidden [Bragin *et al.*, 2011; Kiseleva *et al.*, 2012; Kuznetsov *et al.*, 2010; Kuznetsov *et al.*, 2011a; Kuznetsov *et al.*, 2011b; Kuznetsov *et al.*, 2013; Leonov *et al.*, 2010; Leonov *et al.*, 2011a; Leonov *et al.*, 2011b; Leonov *et al.*, 2012; Leonov *et al.*, 2013; Leonov & Kuznetsov, 2011a, 2011b; Leonov & Kuznetsov, 2012, 2013a, 2013b, 2013c]. A self-excited attractor has a basin of attraction that is associated with an unstable equilibrium, whereas a hidden attractor has a basin of attraction that does not intersect with small neighborhoods of any equilibrium points. Thus any dissipative chaotic flow with no equilibrium or with only stable equilibria must have a hidden strange attractor. Only a few such examples have been reported in the literature [Jafari *et al.*, 2013b; Molaie *et al.*, 2013; Uyaroğlu & Pehlivan, 2010; Wang & Chen, 2012, 2013; Wang *et al.*, 2012a; Wang *et al.*, 2012b; Wei, 2011a, 2011b; Wei & Yang, 2010, 2011, 2012]. Hidden attractors are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or an aeroplane wing. In this paper, we introduce a new chaotic system with a hidden attractor and its corresponding electronic circuit. Then we apply a new parameter estimation technique to this system.

A widely used method for parameter estimation of chaotic systems is optimization-based parameter estimation [Chang *et al.*, 2008; Gao *et al.*, 2009; Li *et al.*, 2012; Li & Yin, 2012; Modares *et al.*, 2010; Mukhopadhyay & Banerjee, 2012; Tang & Guan, 2009; Tang *et al.*, 2012; Tao *et al.*, 2007; Tien & Li, 2012; Yuan & Yang, 2012]. In this method, the problem of parameter estimation is formulated as a cost function which should be minimized. Although there are many optimization approaches available for this problem (e.g. genetic algorithm [Tao *et al.*, 2007], particle swarm optimization [Gao *et al.*, 2009; Modares *et al.*, 2010], evolutionary programming [Chang *et al.*, 2008]), they have one common feature: they define a cost function based on similarity between the time series obtained from the real system and ones obtained from the model. They use time correlation between two chaotic time series as the similarity indicator. However, this indicator has limitations. It is well known that chaotic systems are sensitive to initial conditions [Hilborn, 2001]. Thus there can be two completely identical (both in structure and parameters) chaotic systems that produce time series with no correlation due to a small difference in initial conditions [Jafari *et al.*, 2012; Jafari *et al.*, 2013a]. One way to overcome this problem is using near term correlation and to reinitializing the system frequently (i.e. not letting significant divergence of the trajectories occur). However, this approach also has limitations. In many systems, we do not have access to a time series for all of the system variables, and thus the model cannot be reinitialized. Hence, we prefer a new kind of similarity indicator and corresponding cost function.

Although chaotic systems have random-like behavior in the time domain, they are ordered in state space and usually have a specific topology called a strange attractor. In this work, we propose a similarity indicator between these attractors as an objective function for parameter estimation. To do this, we model the attractor of the real system by a statistical and parametric model. In [Povinelli *et al.*, 2004; Johnson *et al.*, 2005; Shekofteh & Almasgani, 2013b] a Gaussian Mixture Model (GMM) was proposed as a parametric model of a phoneme attractor in the state space. Their results of isolated phoneme classification have shown that the GMM is a useful model to capture structure and topology of the phoneme attractors in the state space. Thus we propose to use the GMM as a parametric model of the strange attractor obtained from a real system. Based on the learned GMM, a similarity indicator can be achieved by matching the time series obtained from the model of a real system with different sets of parameters to evaluate the properness of each set. Therefore our proposed cost function will consist of two steps; first, a training stage which includes fitting a GMM to the attractor of the real system in the state space, and second, an evaluation step to compute the similarity between the learned GMM and the attractors of the model with estimated parameters.

The rest of the paper is organized as follows. Section 2 introduces the new chaotic model and its corresponding electronic circuit. Section 3 details the proposed GMM-based cost function. In Sec. 4, our experimental results are introduced and discussed. In Sec. 5 we show that using Takens' theorem, this new method can be applied when we do not have access to all the time series from all the state space variables. This is in fact the main benefit of the proposed method. Finally, we draw conclusions in the last section.

2. New Chaotic System and Its Corresponding Circuit

Consider the following chaotic system:

$$\begin{cases} \dot{x} = -z \\ \dot{y} = -x - z \\ \dot{z} = 2x - 1.3y - 2z + x^2 + z^2 - xz. \end{cases} \tag{1}$$

The parameters in this system have been set in such a way that their values be deemed "elegant" [Sprott, 2010]. Typical initial conditions which are in the basin of attraction of strange attractor are $(-0.1, 3.4, -1.7)$. There is only one equilibrium at the origin $E(0,0,0)$, and it is stable. Thus a point attractor coexists with the strange attractor. Figure 1 shows a cross-section in the xy-plane at $z = 0$ of the basin of attraction for the two attractors. Note that the cross-section of the strange attractor nearly touches its basin boundary as is typical of low-dimensional chaotic flows. It is interesting that besides the area around the origin, there is another separate area that converges to the stable equilibrium.

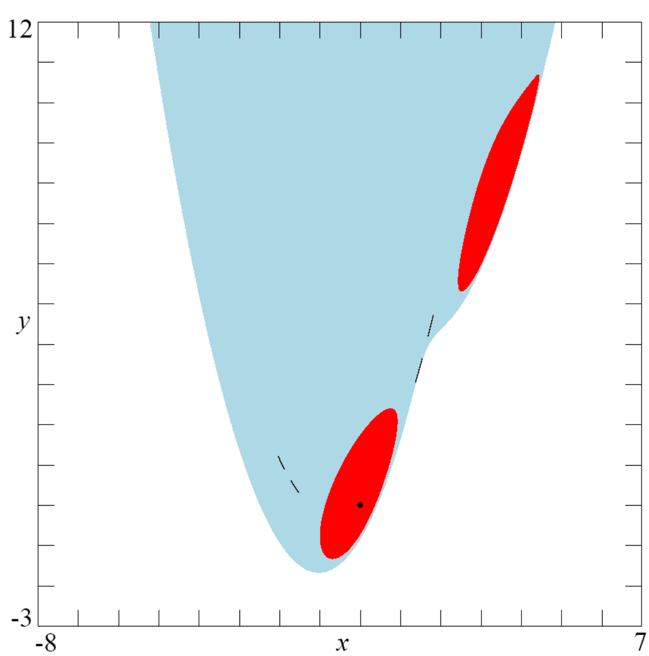


Fig. 1. Cross-section of the basins of attraction of the two attractors in the xy-plane at $z = 0$. Initial conditions in the white region lead to unbounded orbits, those in the red region lead to the point attractor shown as a black dot, and those in the light blue region lead to the strange attractor shown in cross-section as a pair of black lines.

The eigenvalues that correspond to $E(0,0,0)$ are

$$\begin{aligned} \lambda_1 &= -1.9783, \\ \lambda_2 &= -0.0108 + 0.8106i, \\ \lambda_3 &= -0.0108 - 0.8106i. \end{aligned} \tag{2}$$

Since the characteristic equation has a negative real root and two imaginary roots with negative real parts, $E(0,0,0)$ is a stable focus. On the other hand, the Lyapunov spectrum for the strange attractor was estimated as $LE_1 = 0.018, LE_2 = 0, LE_3 = -2.018$. Thus system (1) is believed to be chaotic and has only a single stable equilibrium, though positive Lyapunov exponent is not always an indication of chaos [Kuznetsov & Leonov, 2005; Leonov & Kuznetsov, 2007].

It is possible to produce electronic signals for the above system. An electronic circuit, as shown in Fig. 2, was designed using operational amplifiers such as summing amplifiers, inverting amplifiers, multipliers, and integrators. System (1) is rewritten in the form of:

$$\begin{cases} \dot{x} = -\dfrac{1}{R1C1} z \\ \dot{y} = -\dfrac{1}{R2C2} x - \dfrac{1}{R3C2} z \\ \dot{z} = \dfrac{1}{R4C3} x - \dfrac{1}{R5C3} y - \dfrac{1}{R6C3} z \\ \quad + \dfrac{1}{10R7C3} x^2 + \dfrac{1}{10R8C3} z^2 - \dfrac{1}{10R9C3} xz. \end{cases} \tag{3}$$

The values of the resistors and capacitors are chosen to be $R1 = R2 = R3 = 1\,M\Omega$, $R4 = 500\,k\Omega$, $R5 = 769.2\,k\Omega$, $R6 = 500\,k\Omega$, $R7 = R8 = R9 = 100\,k\Omega$, $R10 = R11 = R12 = R13 = 1\,M\Omega$, and $C1 = C2 = C3 = 1\,\mu$F. The circuit was implemented in the electronic simulation package *Multisim*®. The initial condition was selected as $x_0 = -0.1$, $y_0 = 3.4$, and $z_0 = -1.7$. The phase diagram of system is plotted

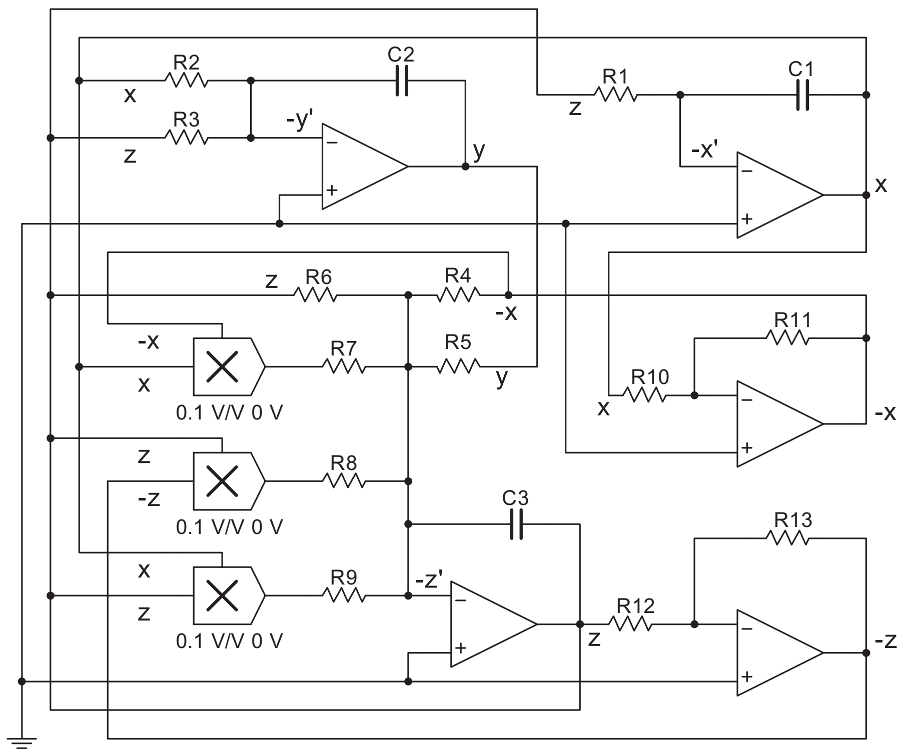


Fig. 2. Electronic circuit of system (3).

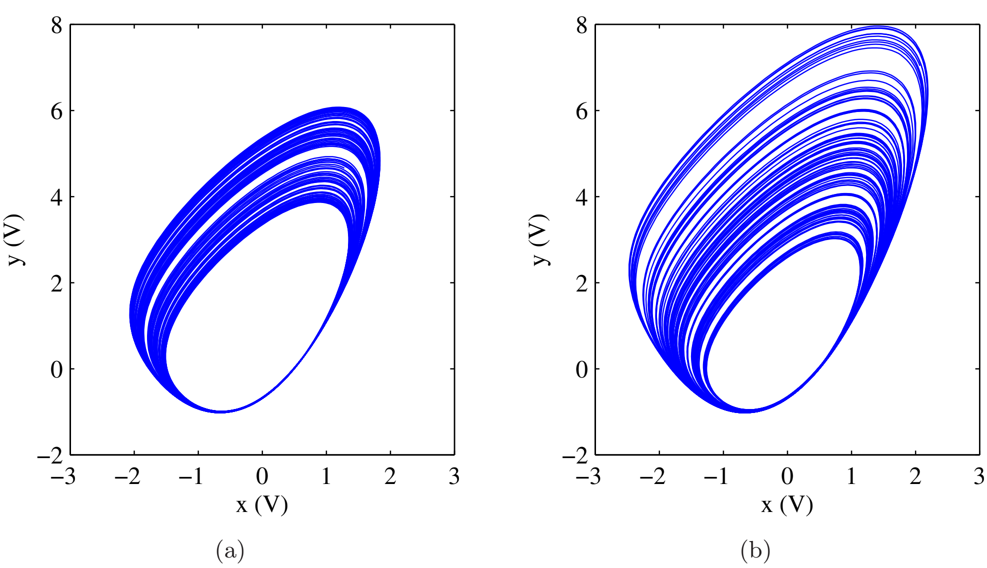


Fig. 3. Phase diagrams of system (3) with (a) $R5 = 769.2\,k\Omega$ and (b) $R5 = 768.6\,k\Omega$.

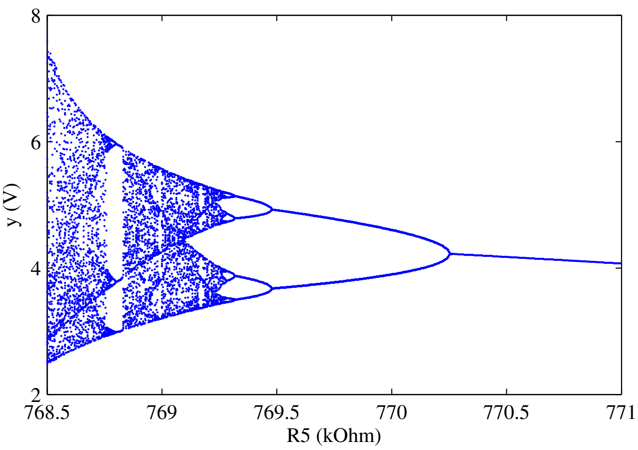


Fig. 4. Bifurcation diagram of output $y_{x=0}$ with varying $R5$ in the electronic circuit.

in Fig. 3(a). In addition, a bifurcation diagram is plotted in Fig. 4 which is produced by varying $R5$. If we change $R5$ to $768.6\,k\Omega$, a wider spread of the trajectories of the system can be observed in the phase diagram depicted in Fig. 3(b). So it is very important to estimate the parameters in an accurate way when modeling a chaotic system.

3. Proposed Cost Function Based on GMM

As mentioned before, the state space is a suitable domain to represent chaotic and nonlinear behaviors of a complex dynamical system. One of the advantages of considering a signal in state space is its time-independent distribution. Based on this characteristic, a statistical distribution of the observed vectors in the state space can capture the attractor geometry and nonlinear system characteristics [Povinelli *et al.*, 2004; Povinelli *et al.*, 2006]. The distribution of points on the attractor is invariant and independent of initial conditions provided the initial conditions are in the basin of attraction and the time series is of infinite length [Kantz & Schreiber, 1997]. Thus we propose a similarity indicator using a conditional likelihood score between a learned statistical model of a real system attractor and a new distribution of an attractor obtained by a specific model of the system. Here, we use a GMM as the statistical model. The GMM is a parametric probability density function represented by a weighted sum of Gaussian component densities [Bishop, 2006]. GMMs were used as a parametric model of the probability distribution of state space vectors in many different systems such as vocal-tract related features in speech recognition system [Shekofteh & Almasgani, 2013b; Jafari & Almasganj, 2012] or EGC signal classification methods [Nejadgholi *et al.*, 2011; Roberts *et al.*, 2001]. One of the powerful characteristics of the GMM is its ability to form smooth approximations of attractors in state space [Nakagawa *et al.*, 2012].

To find the similarity score between the attractor of a real system and the state space points of the model, we calculate likelihood scores which come from GMM computations. Our algorithm consists of two steps; a training stage which includes fitting the GMM to the attractor of the real system, and an evaluation stage to select the best set of parameters in the model to optimize the similarity score in the learned GMM. The following are the steps in detail.

Step 1. The first step of the proposed approach is the learning phase. A GMM learns the probability distribution of the attractor of the real system. This model is a weighted sum of M individual Gaussian densities. It can be represented by a set of system parameters, λ, as follows,

$$\begin{cases} \lambda = \left\{ w_m, \mu_m, \sum_m \right\}, \quad m = 1, \ldots, M \\ p(v \mid \lambda) = \sum_{m=1}^{M} w_m \dfrac{1}{(2\pi)^{D/2}} \dfrac{1}{\left|\sum_m\right|^{1/2}} \\ \qquad \times \exp\left\{ \dfrac{-1}{2}(v - \mu_m)^T \sum_m^{-1} (v - \mu_m) \right\} \end{cases} \tag{4}$$

where M is the number of mixtures (Gaussian components), μ_m is the D-dimensional mean vector of the mth mixture, $\sum_m$ is the $D \times D$ covariance matrix, and $|\cdot|$ denotes the determinant operator. Based on the observation vector v, in this work, $D = 3$ is selected. Also, $p(v \mid \lambda)$ is the likelihood-based similarity score for the observed vector v. This score is obtained by giving v to the learned GMM with its parameters of λ.

Using the prepared training data from the attractor of the real system, the parameters of the GMM are specialized to model the geometry of the attractor. As a popular and well-established method, maximum likelihood (ML) estimation is

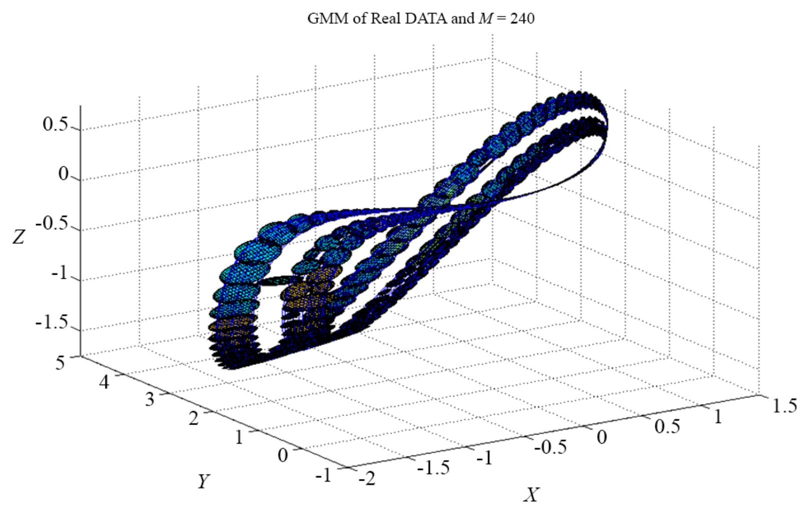


Fig. 5. GMM modeling (with $M = 240$ components) of the strange attractor for the introduced chaotic circuit in a 3-D state space.

applied to identify the GMM parameters [Bishop, 2006; Dempster *et al.*, 1977]. However, there is no analytical solution to determine the optimum number of GMM mixtures needed for a specific problem, which depends on the complexity of the involved data set [Jafari *et al.*, 2010]. Figure 5 shows the strange attractor of the new chaotic circuit in the three-dimensional state space with its GMM modeling using 240 Gaussian component ($M = 240$), where every three-dimensional ellipsoid corresponds to one of the GMM's Gaussian components.

Step 2. The second step of the proposed approach is finding the best model parameters using the learned GMM in Step 1. Here, the search space will be formed from a set of acceptable values of the model parameters. Then, for each set of parameters (here a set of parameters $k = \{a \text{ and } b\}$ which is described in Sec. 4), the model will be simulated, and a new attractor in the state space will be obtained. Finally, the similarity score is computed using an average point-by-point likelihood score obtained from the learned GMM, λ, as follows:

$$p(V^k \mid \lambda) = \frac{1}{N}\sum_{n=1}^{N} \log\{p(v_n^k|\lambda)\} \tag{5}$$

where V^k is a matrix whose rows are composed from the state space vector of the model trajectory with the model's set of parameters k, and N is the number of state space point in the V^k matrix. The model selection is accomplished by computing the conditional likelihoods of the signal under learned GMM and selecting the parameters of a model that gives the best similarity score.

Selection of the best set of parameters, k^*, uses the following criteria. If we use the negative of the similarity score, then the parameter estimation becomes a cost function minimization. Equation (6) shows the final cost function, $J(k)$, based on the negative of its mean log-likelihood score,

$$k^* = \arg\min\{J(k)\} \quad \text{and} \quad J(k) = -p(V^k \mid \lambda) \tag{6}$$

where k is the set of model parameters and λ is the learned GMM of the real system attractor. Our objective is to determine the parameters of the model, k, in such a way that $J(k)$ is minimized.

4. Simulation Results

In this section, we do some simulations to investigate the acceptability of the proposed cost function in estimating parameters of the chaotic circuit. We have used a fourth-order Runge–Kutta method with a step size of 10 ms and a total of 30 000 samples corresponding to a time of 300 s.

Here, we consider a parametric model of Eq. (1):

$$\begin{cases} \dot{x} = -z \\ \dot{y} = -x - z \\ \dot{z} = 2x - 1.3y + az + x^2 + bz^2 - xz \end{cases} \tag{7}$$

where a and b are the parameters of the model which should be estimated by minimization of the

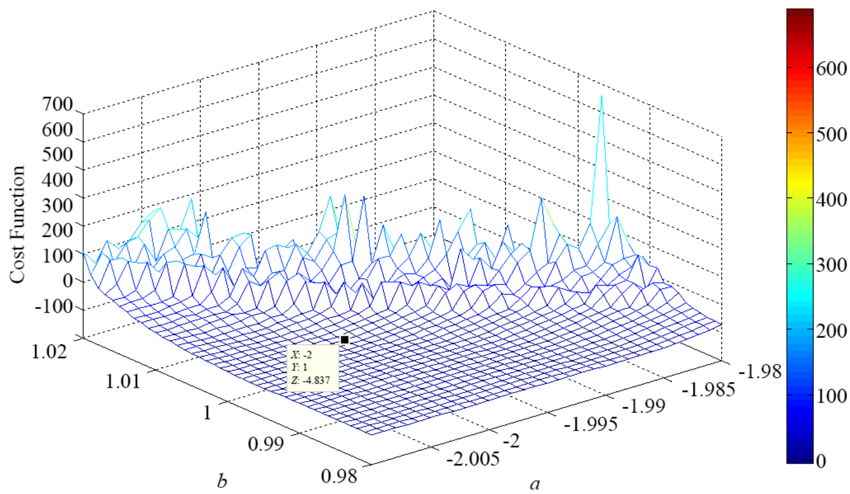
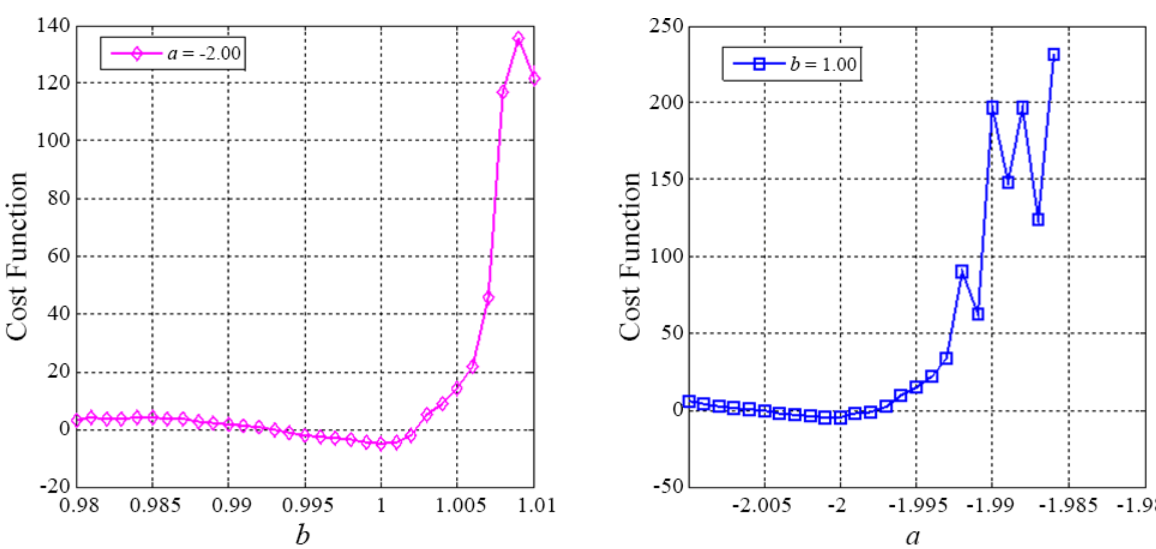


Fig. 6. The surface of the GMM-based cost function of $J(k)$ for the introduced chaotic system of Eq. (7) along with a variation in their parameters, a and b.



(a) (b)

Fig. 7. Cross-section of the surface shown in Fig. 6 for (a) $a = -2.00$ and (b) $b = 1.00$.

proposed cost function. We used $M = 240$ mixtures to model the system attractor in state space.

If we plot the value of $J(k)$, a cost "surface" can be obtained that shows dissimilarity between the real system attractor and each model attractor. In Fig. 6, such a surface is shown for the proposed cost function. The minimum on that surface gives the parameters of the best model. In addition, Fig. 7 shows one-dimensional sections of the surface. The global minimum of the cost function is in the expected place ($a = -2.00$ and $b = 1.00$). Moreover, the surface is almost convex, which makes it a simple case for any optimization approach that moves downhill.

5. Reconstruction of True Dynamics Using Takens' Embedding Theorem

One of the interesting topics in dynamical systems theory is the embedding theorem introduced and used by Takens and Sauer [Kantz & Schreiber, 1997]. It is also called the time-delay embedding theorem, and it gives the conditions under which a chaotic system can be reconstructed from a sequence of observations in a reconstructed phase space (RPS). The RPS is a multidimensional space whose coordinates are produced by shift-delay samples of a one-dimensional time series. The chaotic

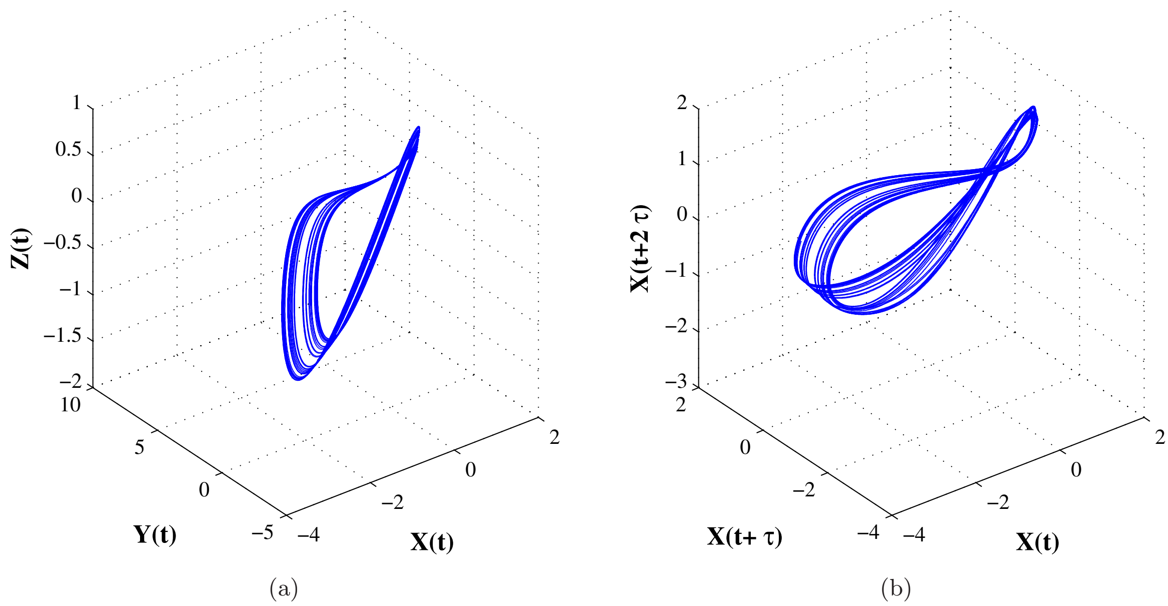


Fig. 8. (a) A segment of the simulated time series in state space and (b) phase space reconstruction using only one variable of $X(t)$ in the three-dimensional RPS ($d = 3$ and $\tau = 45$).

and nonlinear behavior of such a signal is exhibited in the RPS [Kantz & Schreiber, 1997; Shekofteh & Almasganj, 2013a; Kokkinos & Maragos, 2005]. The sequence of embedded points of a signal in the RPS is commonly referred to as a signal trajectory. To construct a signal trajectory, the samples must be embedded in the RPS. A single point of the embedded signal in the RPS is given by

$$S_l = [s_l, s_{l+\tau}, s_{l+2\tau}, \ldots, s_{(d-1)\tau}] \quad \text{where} \quad s = s_1, s_2, s_3, \ldots, s_N \qquad (8)$$

where s_l is the lth sample of an N-point segment of the original one-dimensional signal s, d is the embedding dimension, and τ is the time lag. The concept of embedding dimension and time lag plays an important role in both practical and theoretical aspects of the RPS [Povinelli *et al.*, 2004; Johnson *et al.*, 2005; Shekofteh & Almasgani, 2013b; Kantz & Schreiber, 1997]. The minimum possible embedding dimension can be identified by some heuristic procedures such as false nearest neighbor (FNN). Here, we use $d =$ as a constant of the embedding dimension. Common techniques, including the first minimum of the auto-mutual information function or the first zero crossing of the auto-correlation function, have been used to identify the preferred time lag of the RPS [Hilborn, 2001; Kantz & Schreiber, 1997]. Here we select $\tau = 45$ which is calculated from the first minimum of the auto-mutual information function.

Figure 8 shows a segment of the simulated time series from Eq. (7) and its embedded trajectory in the three-dimensional RPS ($d = 3$) only using one variable of $X(t)$. As can be seen in the figure, the reconstructed trajectory in the RPS has the same geometric structure as the original simulated time series in state space.

Now, we do the same simulation as in Sec. 4 to investigate the acceptability of the proposed cost function in estimating parameters of the chaotic circuit only using one variable. Here, we assume that

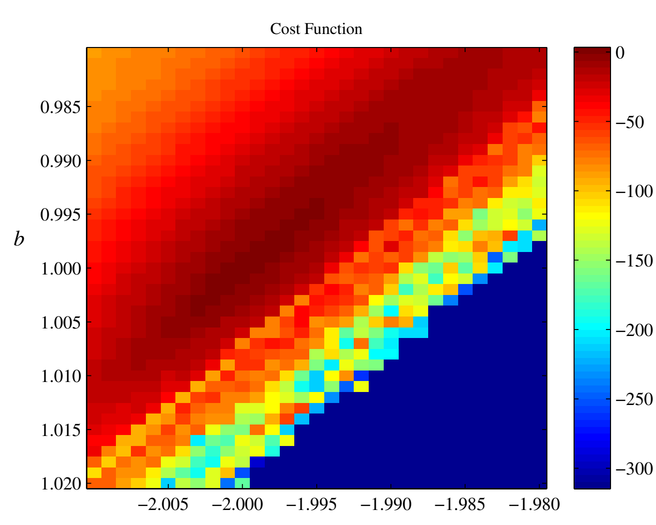
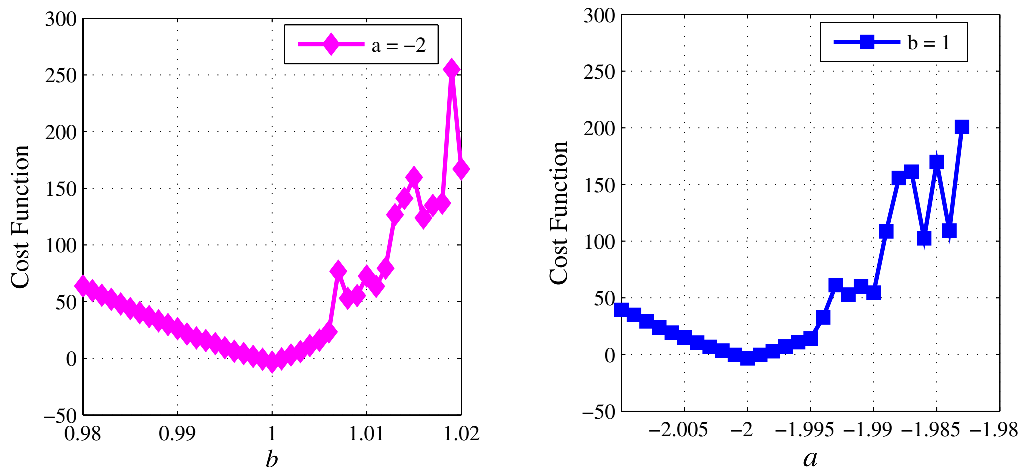


Fig. 9. GMM-based cost function obtained from the reconstruction of $X(t)$ in the RPS with a variation in the model parameters of (7), a and b.



(a) (b)

Fig. 10. Cross-sections of the cost function shown in Fig. 9 for (a) $a = -2.00$ and (b) $b = 1.00$.

the original chaotic system of (7) has the following real value of the parameters $\{a = -2.00$ and $b = 1.00\}$ which should be estimated by minimization of the proposed cost function. Similar to the method proposed in Sec. 4, we use $M = 240$ mixtures to model the attractor of the reconstructed dynamics of the variable X in the RPS. In Fig. 9, the calculated cost function $J(k)$ is shown where the minimum point gives the parameters for the best model. Moreover, Fig. 10 shows its one-dimensional sections. As can be seen in these figures, the proposed cost function can give the true value of the global minimum ($a = -2.00$ and $b = 1.00$).

6. Conclusion

In this paper an appropriate and new cost function has been introduced to be used in parameter estimation of chaotic systems, based on a statistical model of the real system attractor in state space. Since Gaussian Mixture Models (GMMs) are strong tools to be used as parametric models of the probability distribution of state space vectors in many different systems, we have used them in this work as the statistical model. The proposed cost function is the negative of a similarity metric which is formed by averaging some log-likelihood scores. Overall results indicate that the global minimum of the proposed cost function is the true value of the model's parameters. Since this method is based on the topology of strange attractors, one virtue is that the sample time is not critical, and even with short and piecewise time series, the GMM can be trained, although the data do have to adequately cover the attractor. This method has been applied to parameter estimation of chaotic circuits with hidden attractors. To do so, a new chaotic system has been proposed which has only one stable equilibrium and thus a hidden attractor. These kinds of systems are barely investigated and are good cases for further studies.

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