International Journal of Bifurcation and Chaos, Vol. 24, No. 1 (2014) 1450009 (9 pages)

DOI: 10.1142/S0218127414500096

Simplest Chaotic Flows with Involutional Symmetries

J. C. Sprott
Department of Physics, University of Wisconsin, Madison, Wisconsin 53706, USA
sprott@physics.wisc.edu

Received August 9, 2013

A chaotic flow has an involutional symmetry if the form of the dynamical equations remains unchanged when one or more of the variables changes sign. Such systems are of theoretical and practical importance because they can exhibit symmetry breaking in which a symmetric pair of attractors coexist and merge into one symmetric attractor through an attractor-merging bifurcation. This paper describes the simplest chaotic examples of such systems in three dimensions, including several cases not previously known, and illustrates the attractor-merging process.

Keywords: Chaos; symmetry; attractor; merging.

1. Introduction

There has been continued interest in finding new simple examples of systems of autonomous ordinary differential equations whose solutions exhibit chaos and that satisfy constraints on their dimensionality, number and types of nonlinearities, Lyapunov spectrum, symmetries, number of wings or scrolls, number and types of equilibria, bifurcations, and routes to chaos. Although an entire book has been devoted to symmetries in chaos [Gilmore & Letellier, 2007], little systematic work has been done to identify the simplest such examples that are invariant with respect to a change in sign of one or more of the variables. Such examples are of interest because they can exhibit symmetry breaking and offer the possibility that a symmetric pair of attractors will exhibit an attractor-merging crisis as some bifurcation parameter is changed. Especially rare are situations in which the system is chaotic before and after the merging. It is therefore of interest to find the simplest examples for which such behavior occurs.

An involution is a mathematical function f that is its own inverse, the simplest nontrivial example of which is $f(x) = -x$ since $f(f(x)) = x$ for all x in the domain of f. We focus here on three-dimensional dynamical systems governed by three autonomous first-order ordinary differential equations in the variables x, y, and z since such systems describe the simplest continuous flows that can exhibit chaos. In such a case, the elementary involutional symmetries are inversion, rotation, and reflection, corresponding to invariance of the equations with respect to changing the sign of three, two, and one variable, respectively. The trivial identity transformation in which none of the variables are changed is also a symmetry, but it is not interesting and will be ignored. Nonlinear differential equations that are invariant with respect to such transformations often lead to a symmetric pair of coexisting attractors in addition to the symmetric ones that are typically observed and studied. This is an example of spontaneous symmetry breaking, and it is common in systems throughout nature. In the interest of simplicity, we will restrict the discussion to cases in which the governing equations are polynomial functions and seek the simplest such polynomials that give the desired behavior.

2. Inversion Invariant Systems

Systems that are invariant with respect to changes in all three variables (also called parity invariant) can only contain terms that are odd powers of the variables. Since linear systems cannot exhibit chaos, the simplest examples of such systems are those that contain linear and cubic terms.

One of the oldest such system is the Moore–Spiegel [1986] oscillator used to model the irregular variability in the luminosity of stars and given in simplified form by

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= z\\ \dot{z} &= -z + ay - x^2y - bx\end{aligned} \tag{1}$$

whose attractor is shown in Fig. 1 for $a = 9$ and $b = 5$. The system can be written more compactly as a jerk equation: $\dddot{x} = -\ddot{x} + (a - x^2)\dot{x} - bx$. The same equation was subsequently studied by Auvergne and Baglin [1985] as a model of the ionization zone in a star. The two-fold symmetry is evident from the figure in which there are two loops despite the fact that the only equilibrium is the one at the origin. However, this system appears not to have coexisting attractors for any choice of the parameters a and b.

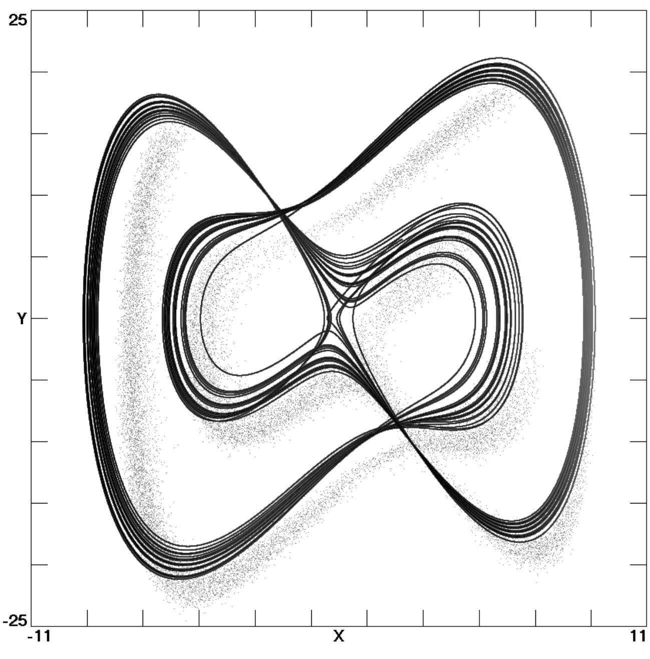

Fig. 1. Moore–Spiegel attractor from Eq. (1) with $a = 9$ and $b = 5$.

A variant of Eq. (1) given by

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= z\\ \dot{z} &= -z - ay - x^3 + bx\end{aligned} \tag{2}$$

or in jerk form, $\dddot{x} = -\ddot{x} - a\dot{x} - x^3 + bx$, was studied by Coullet *et al.* [1979], and it does admit coexisting strange attractors probably because it has two additional equilibria at $(\pm\sqrt{b}, 0, 0)$ that merge into the one at the origin for $b = 0$. Figure 2 shows that the strange attractors become limit cycles just before and after the merging, after which a new symmetric strange attractor appears. Such behavior is not unusual for systems of this type.

More recently, Malasoma [2000] proposed the simplest dissipative jerk equation that is parity invariant: $\dddot{x} = -a\ddot{x} + x\dot{x}^2 - x$. This system is chaotic over most of the range $2.0278 < a < 2.0840$ except for periodic windows such as a dominant period-3 window near $a = 2.043$, and it does exhibit the merging of two coexisting strange attractors into one at $a = 2.0644\ldots$. However, the basin of attraction is extremely small, and the coexisting attractors are thin and nearly coincident as shown in Fig. 3, making it difficult to observe the merging except by way of Poincaré sections. In an extensive search, no simpler examples of inversion invariant chaotic systems were found.

3. Rotation Invariant Systems

For systems that are invariant with respect to rotation, the possibility exists of cases with only quadratic nonlinearities in which two coexisting strange attractors merge into one. Without loss of generality, we consider quadratic systems that are invariant under the transformation $(x, y, z) \to (-x, -y, z)$, corresponding to a 180° rotation about the z-axis, the most general form of which is

$$\begin{aligned}\dot{x} &= a_1x + a_2y + a_3xz + a_4yz\\ \dot{y} &= b_1x + b_2y + b_3xz + b_4yz\\ \dot{z} &= c_0 + c_1z + c_2x^2 + c_3y^2 + c_4z^2 + c_5xy.\end{aligned} \tag{3}$$

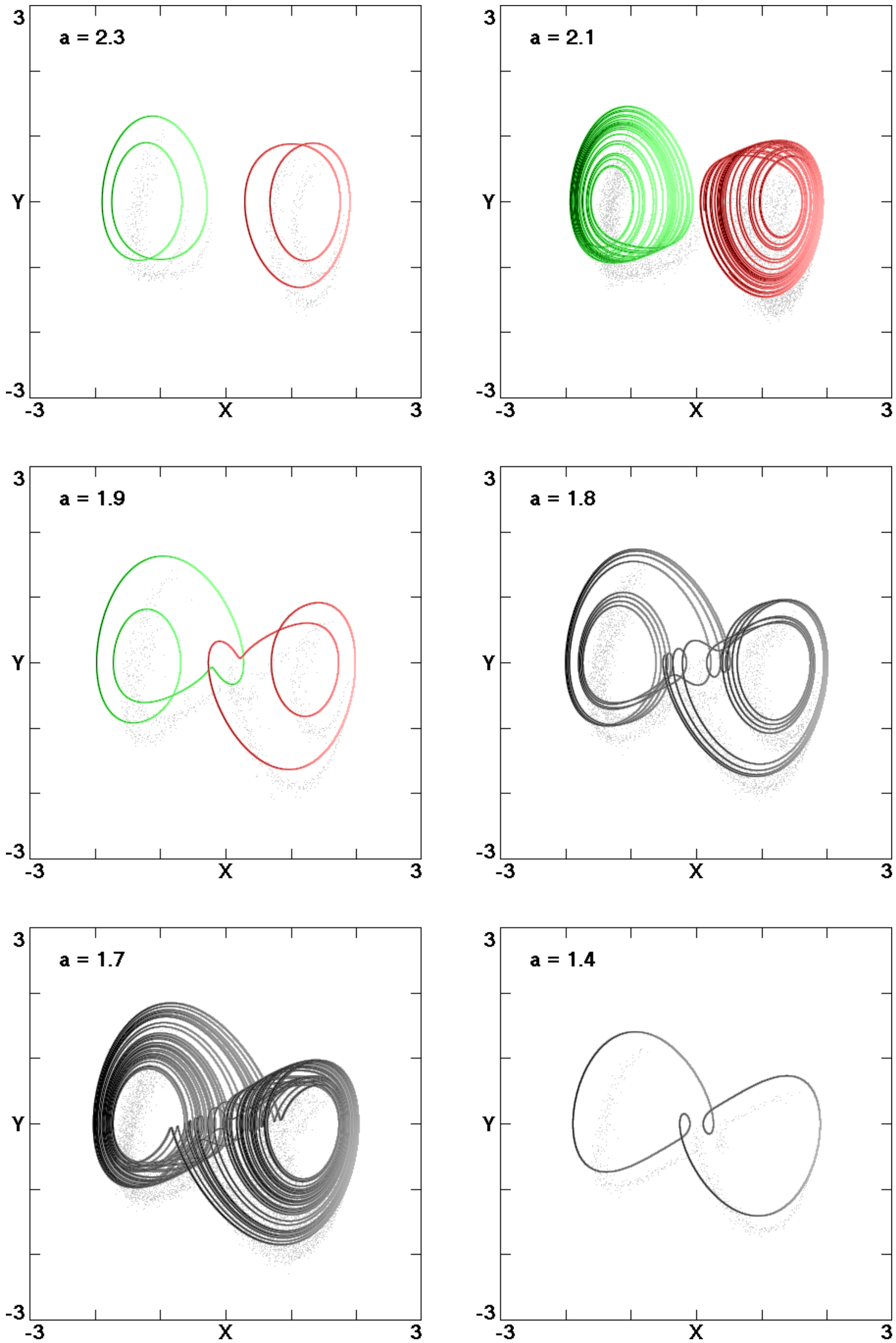

Fig. 2. Attractor merging from Eq. (2) with $b = 2$.

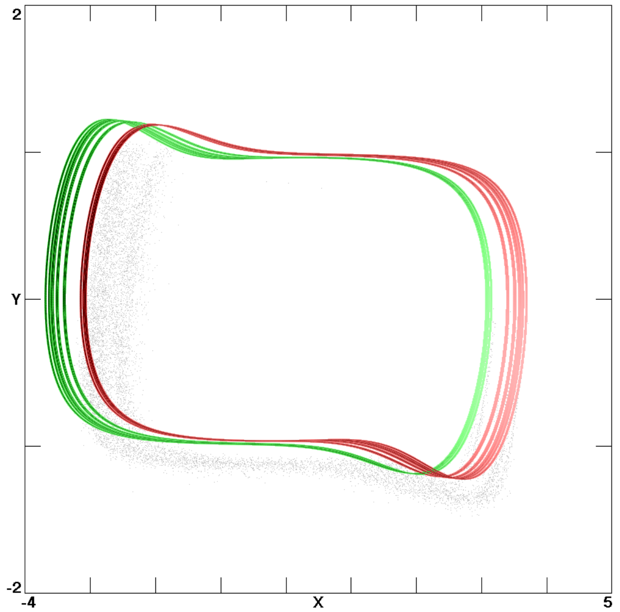

Fig. 3. Coexisting Malasoma strange attractors with $a = 2.08$.

The most familiar example of an equation of this form is the Lorenz [1963] system

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= -xz + rx - y\\ \dot{z} &= xy - bz\end{aligned} \tag{4}$$

which is a special case of Eq. (3) with $a_1 = -\sigma, a_2 = \sigma, a_3 = 0, a_4 = 0, b_1 = r, b_2 = -1, b_3 = -1, b_4 = 0, c_0 = 0, c_1 = -b, c_2 = c_3 = c_4 = 0, c_5 = 1$. For the usual parameters of $\sigma = 10, r = 28$, and $b = 8/3$ the Lorenz system has a single symmetric strange attractor, and that seems to be the case for any positive values of the parameters.

However, in a previously unexplored region of parameter space such as $\sigma = 0.053$, $r = 0$, and $b = -0.3$, the Lorenz system admits a pair of coexisting strange attractors. As σ is increased, there is a broad region where the only solutions are unbounded, until a new larger pair of almost

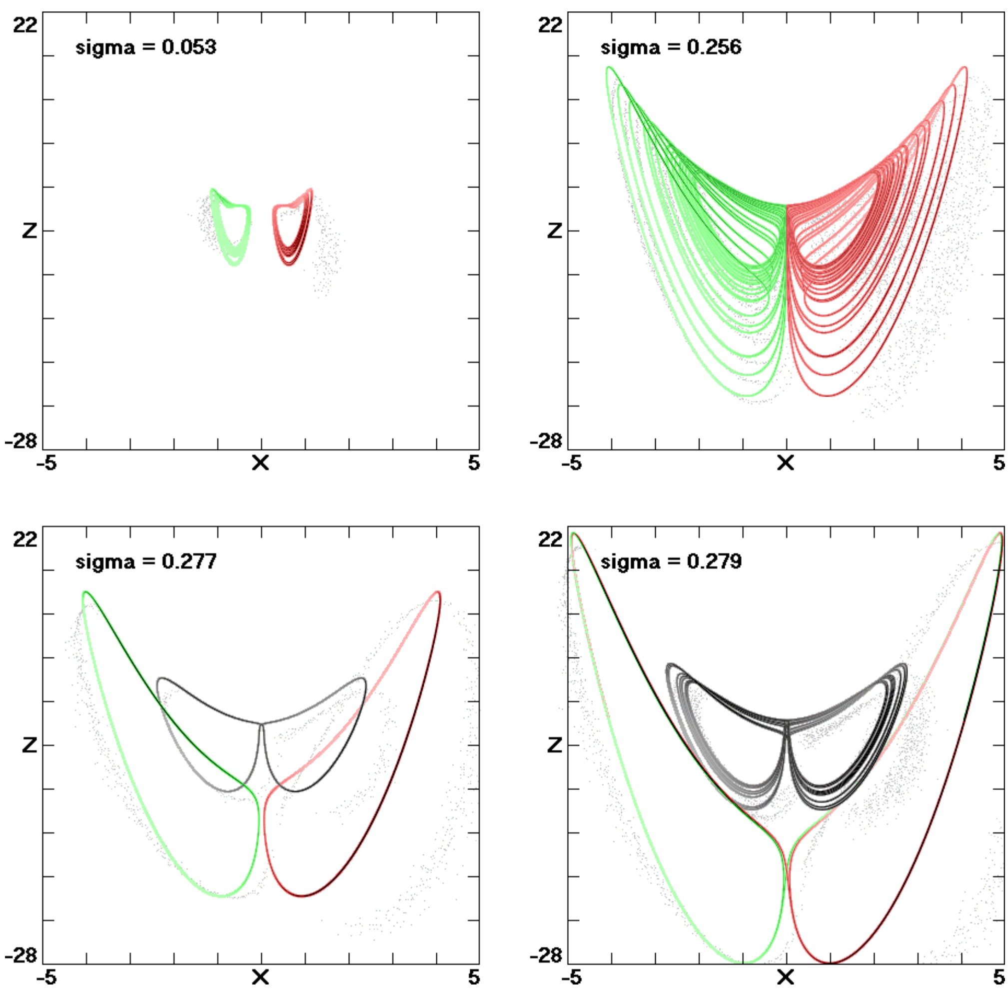

Fig. 4. Coexisting strange attractors in the Lorenz system with $r = 0$ and $b = -0.3$.

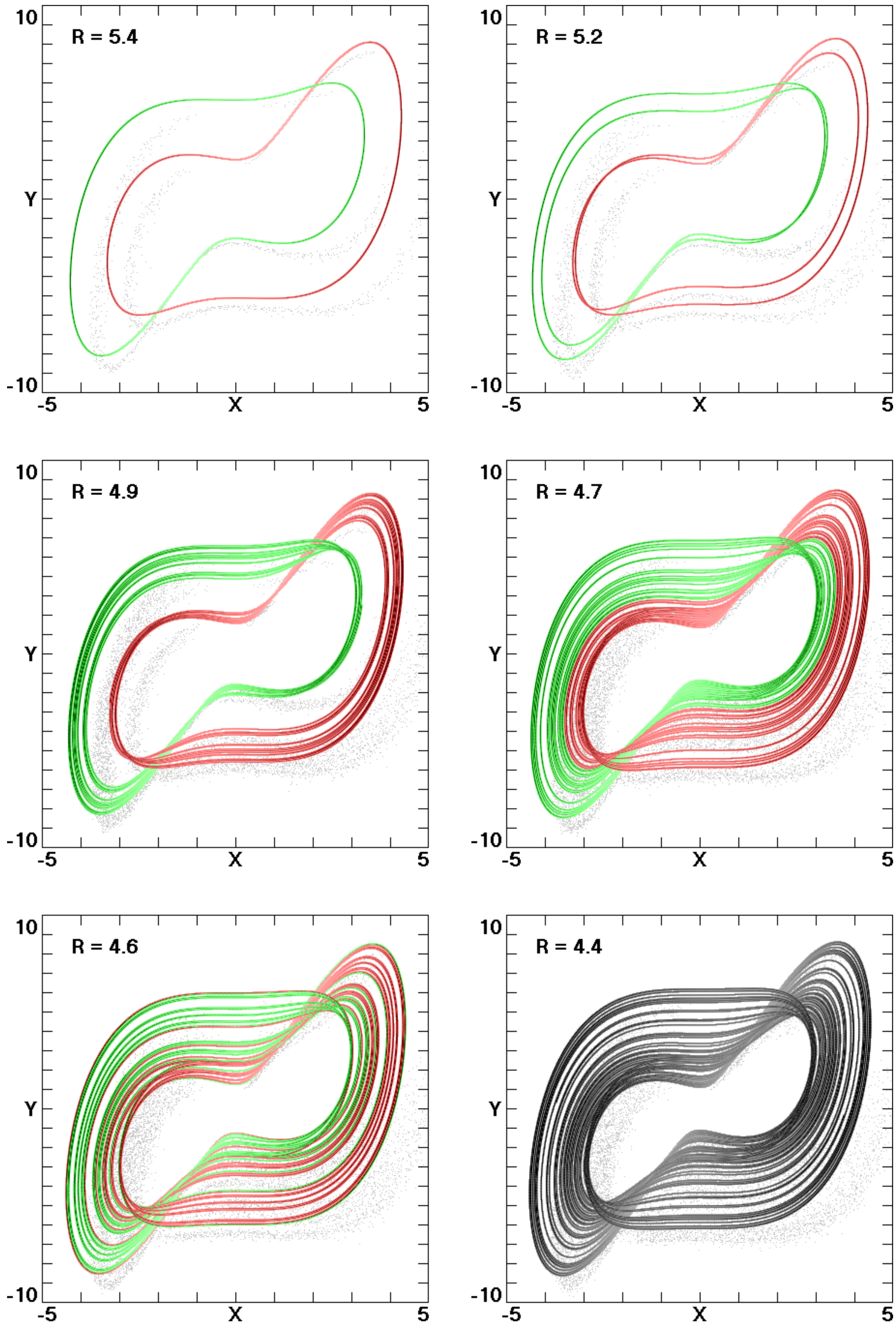

Fig. 5. Attractor merging for Eq. (5).

touching strange attractors are born in a homoclinic bifurcation around $\sigma = 0.256$. These attractors disappear in a sequence of inverse period doublings leading to a pair of limit cycles that coexist with a larger symmetric limit cycle that period-doubles, forming a new coexisting symmetric strange attractor at about $\sigma = 0.279$ as shown in Fig. 4. All three of the attractors are destroyed in a homoclinic bifurcation for σ slightly greater than 0.279. The chaotic regions for the Lorenz attractor with $b < 0$ and $r = 0$ are very narrow with small basins of attraction, but the dynamics are very rich in this new regime and deserving of further study despite being somewhat nonphysical.

Probably the simplest rotationally invariant chaotic flow is the diffusionless Lorenz system [van der Schrier & Maas, 2000; Munmuangsaen & Srisuchinwong, 2009]

$$\begin{aligned} \dot{x} &= y - x \\ \dot{y} &= -xz \\ \dot{z} &= xy - R \end{aligned} \tag{5}$$

derived from Eq. (4) by taking the limit $r, \sigma \to \infty$ but with finite $R = br/\sigma^2$. This system is one of the five cases first discovered in an exhaustive search for three-dimensional chaotic systems with only five terms and two quadratic nonlinearities [Sprott, 1994]. It is chaotic over most of the range $0 < R < 5$ with coexisting strange attractors in the approximate range $4.418 < R < 5$. As shown in Fig. 5, the system has a symmetric pair of linked limit cycles at $R = 5.4$, and these limit cycles undergo a sequence of period doublings as R is decreased, leading to a symmetric pair of linked strange attractors at $R = 4.9$. Such linking is apparently very common in these systems, although it usually goes unnoticed in the usual two-dimensional projections. The strange attractors grow in size, eventually touching along their edges, creating a single symmetric attractor at about $R = 4.418$. At $R = 4.6$ the attractors are strongly entangled despite residing in separate basins of attraction.

This picture is very different from the mental image one might have of two attractors first touching at a single point, but it is apparently common since the Malasoma attractors in Fig. 3, which are also linked, behave similarly. Prior to the merging, the two attractors have an intricate fractal basin boundary whose cross-section at $z = 0$ is shown in Fig. 6 for $R = 4.7$. Fractal basin boundaries

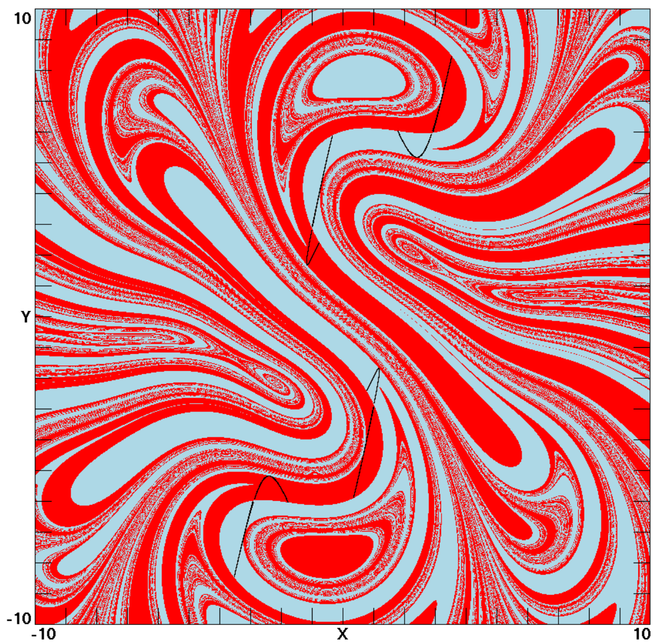

Fig. 6. Cross-section of the fractal basins of attraction in the $z = 0$ plane for Eq. (5) with $R = 4.7$. The black lines are cross-sections of the corresponding strange attractors that nearly touch their basin boundaries.

are a necessary consequence of the merging of two strange attractors along their entire edges. The cross-sections of the attractors are shown in black in the figure, and where the attractors appear to intersect their basin boundary, there is a small gap separating the two attractors, and this gap remains small in other sections such as at $z = \pm 7$ and at $y = 0$ in the xz-plane.

A variant of Eq. (5) in which the xy term is replaced by y^2 also leads to chaotic solutions [Sprott, 1994] over most of the range $0 < R < 2.7$, but it does not appear to admit multiple attractors.

A system almost as simple as Eq. (5) but with one extra term and thus two parameters is obtained by adding a damping term $-bx$ to the Nose–Hoover oscillator [Nose, 1991; Hoover, 1995]

$$\begin{aligned} \dot{x} &= y - bx \\ \dot{y} &= yz - x \\ \dot{z} &= a - y^2 \end{aligned} \tag{6}$$

which for $a = 1$ is chaotic over most of the range $0.67 < b < 1.12$ as well as at $b = 0$ where the system is conservative. For $b > 1.3$ the system has a single symmetric limit cycle that bifurcates into a symmetric pair of linked limit cycles as shown in Fig. 7. The limit cycles undergo a period doubling

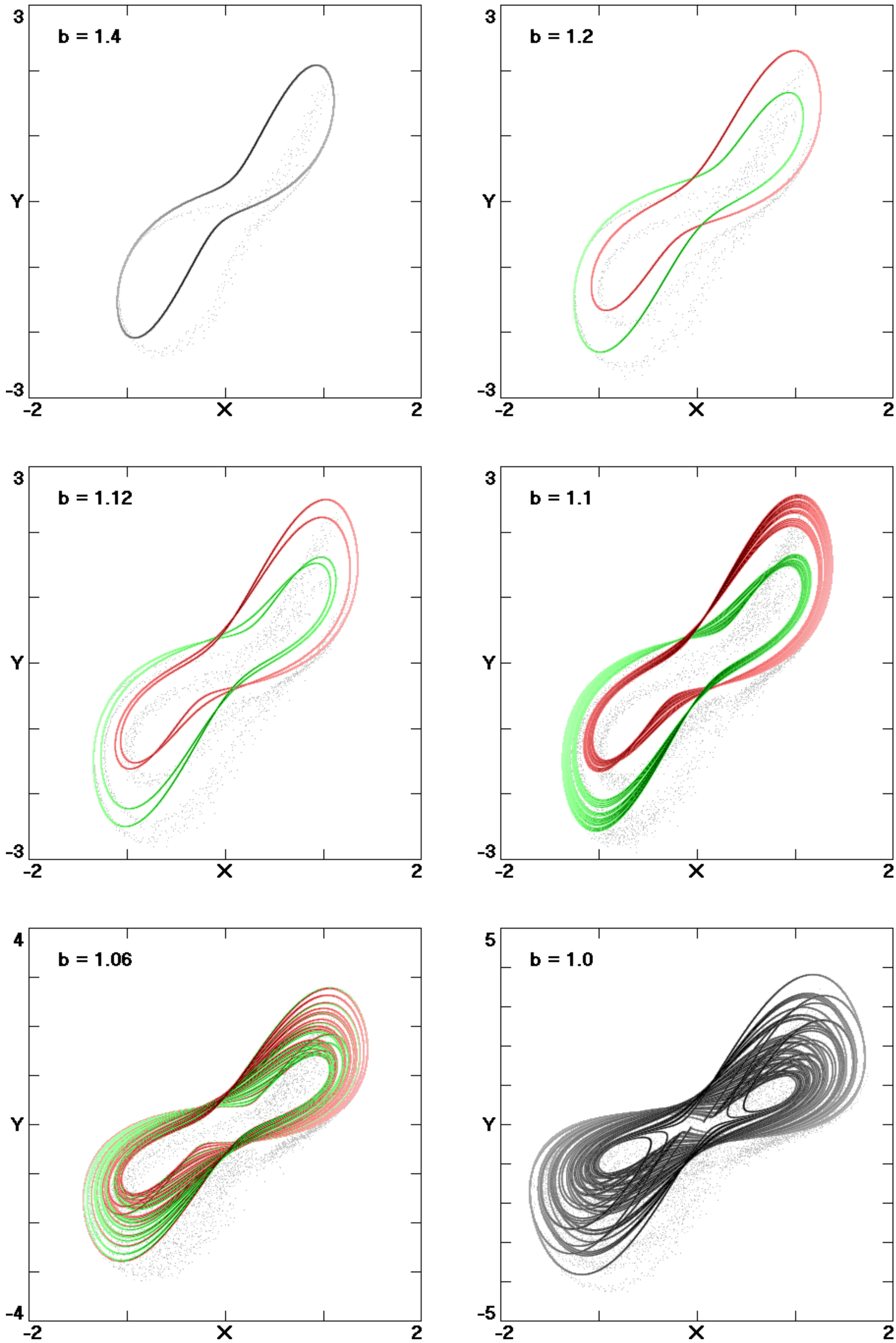

Fig. 7. Attractor merging for Eq. (6) with $a = 1$.

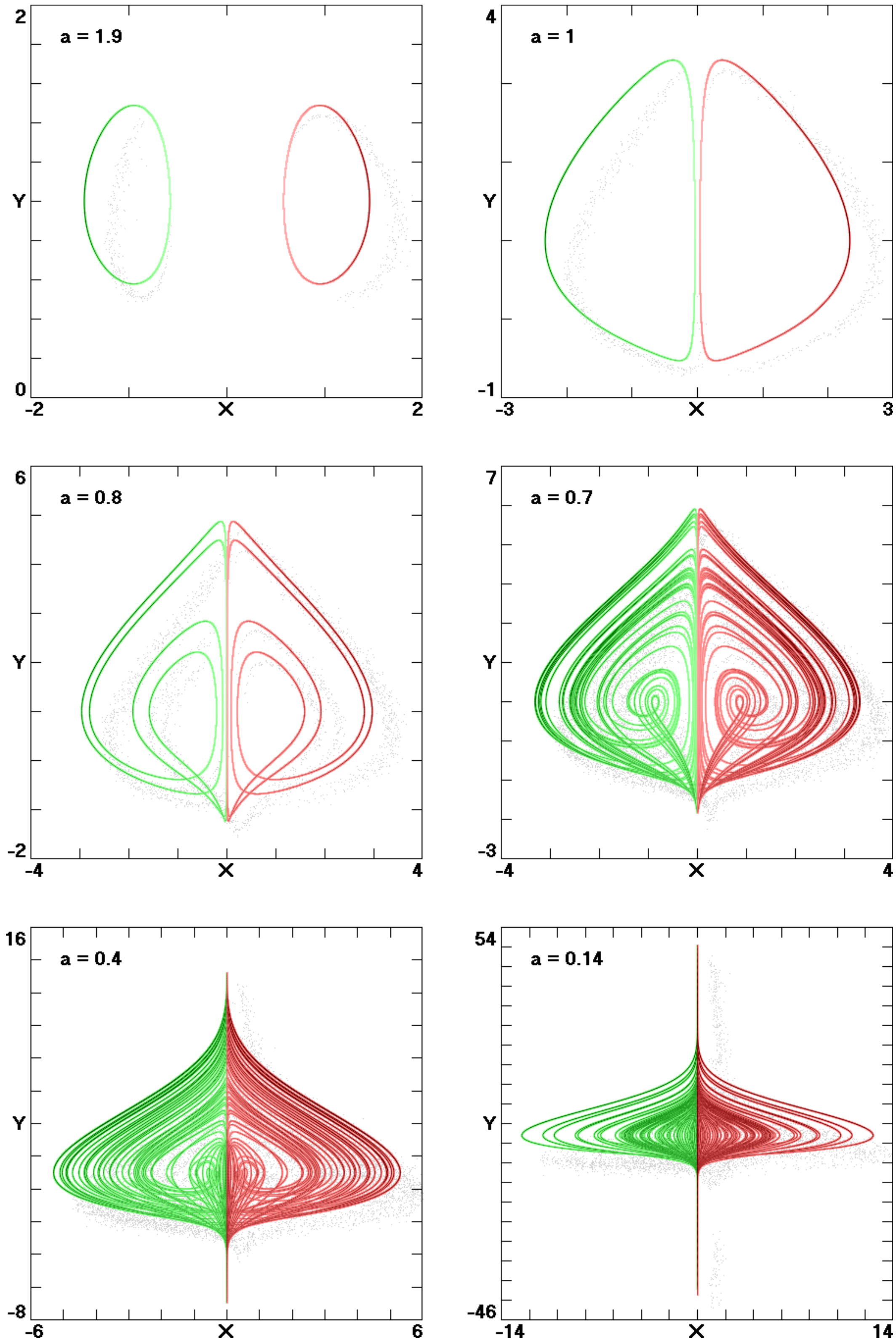

Fig. 8. Attractor "kissing" for Eq. (8).

route to chaos, which onsets around $b = 1.1$ producing a symmetric pair of linked strange attractors that merge into a single symmetric strange attractor around $b = 1.06$ in a manner similar to the case in Fig. 5.

4. Reflection Invariant Systems

Systems that are invariant with respect to reflection also allow multiple strange attractors with only quadratic nonlinearities. In fact, any attractors in such systems necessarily exist as a symmetric pair. Without loss of generality, we consider quadratic systems that are invariant under the transformation $(x, y, z) \to (-x, y, z)$, corresponding to symmetry about the $x = 0$ plane, the most general form of which is

$$\begin{aligned} \dot{x} &= a_1x + a_2xy + a_3xz \\ \dot{y} &= b_0 + b_1y + b_2z + b_3x^2 + b_4y^2 + b_5z^2 + b_6yz \\ \dot{z} &= c_0 + c_1y + c_2z + c_3x^2 + c_4y^2 + c_5z^2 + c_6yz. \end{aligned} \tag{7}$$

The 17-dimensional parameter space is replete with examples of cases with a symmetric pair of strange attractors. However, these attractors can never merge because the trajectory cannot cross the $x = 0$ plane since $\dot{x} = 0$ at $x = 0$, although we might say that they "kiss" since they can come arbitrarily close to one another (and often do).

Perhaps the simplest such system, not previously known, with the fewest number of terms is

$$\begin{aligned} \dot{x} &= x - xy \\ \dot{y} &= z \\ \dot{z} &= -y - az + x^2. \end{aligned} \tag{8}$$

This is actually a two-parameter system, but it is possible to set one of the parameters to unity, in which case the system has a symmetric pair of strange attractors for $a = 0.3$. This system has three equilibria, one at the origin (0, 0, 0) and the others at (± 1, 1, 0). The one at the origin is unstable for all values of a, and the other two are stable for $a > 2$. As a decreases, a supercritical Hopf bifurcation occurs at $a = 2$, whereupon a symmetric pair of stable limit cycles are formed. With further decrease of a, these limit cycles undergo successive period doublings, leading to a symmetric pair of strange attractors at about $a = 0.7263$ that nearly touch one another on either side of the $x = 0$ plane as shown in Fig. 8 in what might be called attractor "kissing".

5. Conclusion

Dynamical systems often possess symmetries in the form of their equations but have solutions for which the symmetry is broken. In many important cases, the result is a symmetric pair of attractors over some range of the parameters. These attractors usually merge to form a single symmetric attractor in an attractor-merging bifurcation. Especially interesting are examples in which the attractor is chaotic before and after the merging. Such behavior seems to be very common and occurs in simple three-dimensional systems of autonomous ODEs with quadratic and cubic nonlinearities. The simplest such examples for the three basic involutional symmetries have now been identified and described.

References

Auvergne, M. & Baglin, A. [1985] "A dynamical instability as a driving mechanism for stellar oscillations," *Astron. Astrophys.* **142**, 388–392.

Coullet, P. H., Tresser, C. & Arnéodo, A. [1979] "Transition to stochasticity for a class of forced oscillators," *Phys. Lett. A* **72**, 268–270.

Gilmore, R. & Letellier, C. [2007] *The Symmetry of Chaos* (Oxford University Press, NY).

Hoover, W. G. [1995] "Remark on 'Some simple chaotic flows'," *Phys. Rev. E* **51**, 759–760.

Lorenz, E. N. [1963] "Deterministic nonperiodic flow," *J. Atmos. Sci.* **20**, 130–141.

Malasoma, J.-M. [2000] "A new class of minimal chaotic flows," *Phys. Lett. A* **264**, 383–389.

Moore, D. W. & Spiegel, E. A. [1986] "A thermally excited non-linear oscillator," *Astrophys. J.* **143**, 871–887.

Munmuangsaen, B. & Srisuchinwong, B. [2009] "A new five-term simple chaotic attractor," *Phys. Lett. A* **373**, 4038–4043.

Nose, S. [1991] "Constant temperature molecular dynamics methods," *Prog. Theor. Phys. Suppl.* **103**, 1–46.

Sprott, J. C. [1994] "Some simple chaotic flows," *Phys. Rev. E* **50**, R647–R650.

van der Schrier, G. & Maas, L. R. M. [2000] "The diffusionless Lorenz equations; Shil'nikov bifurcations and reduction to an explicit map," *Phys. Nonlin. Phenom.* **141**, 19–36.