Projective Synchronization Based On Amplitude Control

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Abstract:
A new synchronization system is proposed based on the Lyapunov function scheme by introducing the amplitude parameters for projective synchronization with different ratios. Consequently, the complicated controller is replaced by a signal with scale information from the other side of the synchronization system. Based on amplitude control, different regimes of projective synchronization are obtained which equip different scale factors on partial or total amplitude control and determine the synchronization as same-phase or anti-phase projective synchronization.

Key words: Amplitude control; Projective synchronization; Lyapunov function.

1. Introduction

Chaos synchronization plays an important role in many subjects, especially in secure communication engineering [1-4]. Recently, many synchronization methods have been proposed [5-17], among which, anti-phase [5-7] and projective synchronization [8-10] are especially attractive because of the ready availability of signals from the drive and response systems. However, there is an unavoidable defect in those projective synchronization systems, in that the synchronization controller is very complicated and sensitive to the signals from synchronization systems. Furthermore, once the controller is set up, the scale factor cannot be changed for the demands of rigid feedback, which defines the ration of the variables between the drive system and the response system. In fact, chaotic systems usually have many parameters, some of which control the bifurcations, while others control the amplitude [7, 17-21]. The amplitude control provides a convenient way to realize the flexible projective synchronization.

As shown in Fig. 1, the projective synchronization system can be designed with three parts, a drive system, a response system, and a synchronization controller. Amplitude Parameters [17-21] in the Drive system (APD) and in the Response system (APR) are the inherent scale factors for projective synchronization, which are also embedded in the synchronization controller. Because there are two different types of amplitude parameters, *i.e.*, partial control parameters and total control parameters [18], the regime of projective synchronization consists of several different combinations. The regime can be defined by whether the APD or APR is partial or total. In the following, we illustrate these different regimes with the three-dimensional Lorenz system.

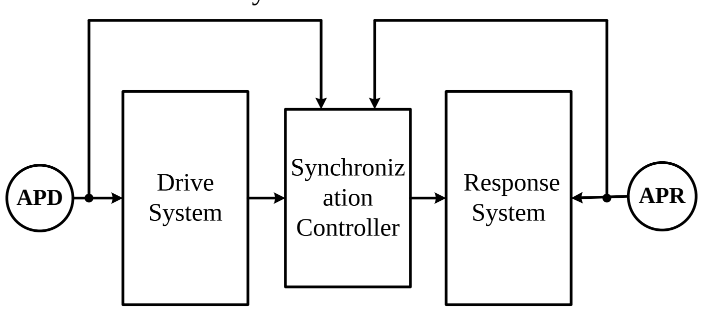

Fig.1. The projective synchronization system model

This paper describes a new flexible method for realizing projective synchronization based on the amplitude parameters in a chaotic system, and hence a series of scale-factor adjustable synchronization systems can be constructed. By adjusting the amplitude parameter in the drive system or response system, as well as in the synchronization controller, the ratio of the signals between the drive and response system can be set to any desired value, which provides a broad outlook for engineering application.

2. Projective synchronization based on single amplitude control

2.1 Projective synchronization based on partial amplitude control

When the APDs and APRs in the synchronization system are partial control parameters, same-phase projective synchronization can be realized. In this condition, the controlled variables in the synchronization system can be used as an adjustable scale factor, while the uncontrolled variables also remain synchronized. Although there are many methods to realize projective synchronization, we consider the active-passive decomposition technology as an example to indicate the function of the amplitude parameters, which provide the simplest controller and rapid synchronization speed.

The drive Lorenz system is,

$$\begin{cases} \dot{x}_1 = -ax_1 + a\sqrt{n/m}\,y_2, \\ \dot{x}_2 = -x_1x_3 + rx_1 - x_2, \\ \dot{x}_3 = mx_1x_2 - bx_3, \end{cases} \tag{1}$$

and the response Lorenz system is,

$$\begin{cases} \dot{y}_1 = a(y_2 - y_1), \\ \dot{y}_2 = -y_1y_3 + ry_1 - y_2, \\ \dot{y}_3 = ny_1y_2 - by_3, \end{cases} \tag{2}$$

where a = 10, b = 8/3, and r = 28. Here, two coefficients m and n are partial amplitude parameters. The parameter m in drive system controls the variables x_1 and x_2 according to $1/\sqrt{m}$, while the parameter n in response system controls the variables y_1 and y_2 according to $1/\sqrt{n}$. The variable x_3 and y_3 remain unchanged. It is clear that m and n should be positive real numbers. We denote the difference $e_1 = \sqrt{m}x_1 - \sqrt{n}y_1$. For $\dot{e}_1 = -ae_1$, the first component converges to zero, and hence the remaining two-dimensional system describing the evolution of the difference, $e_2 = \sqrt{m}x_2 - \sqrt{n}y_2$, $e_3 = x_3 - y_3$ for the limit $t \to \infty$ can be written as

$$\begin{aligned} \dot{e}_2 &= -e_2 - \sqrt{m}x_1e_3, \\ \dot{e}_3 &= \sqrt{m}x_1e_2 - be_3. \end{aligned} \tag{3}$$

Applying the Lyapunov function $L = \frac{1}{2}(e_2{}^2 + e_3{}^2)$ shows that $\dot{L} = -(e_2{}^2 + be_3{}^2) < 0$, which means the synchronization is globally stable and occurs for all types of drive signal.

Here the partial amplitude control parameters m and n provide the scale factors $k_{1,2} = x_1/y_1 = x_2/y_2 = \sqrt{n}/\sqrt{m}$, while the scale factor of the variables without amplitude control remain $k_3 = x_3/y_3 = 1$. The physical meaning of the scale factors (k_1, k_2, k_3) is the relationship of the signals between the drive system and the response system. A value greater than one (k_i >1) (i =1, 2, 3) means a signal attenuation in response system, while a value less than one indicates a signal amplification for drive system. Using initial conditions in the drive system and response system of (1, 1, 1) and (-3, -2, -

1), respectively, the output signals in the synchronization system and the time evolution of the error vectors are shown in Fig. 2. When m = 1 and n = 4, the scale factors are $k_1 = k_2 = 2$ and $k_3 = 1$. The amplitude parameters m and n are both positive, and the scale factors are also positive, hence the projective synchronization is same-phase synchronization. The additional coefficient in the controller indicates the scale factor between two subsystems which are controlled by amplitude parameters.

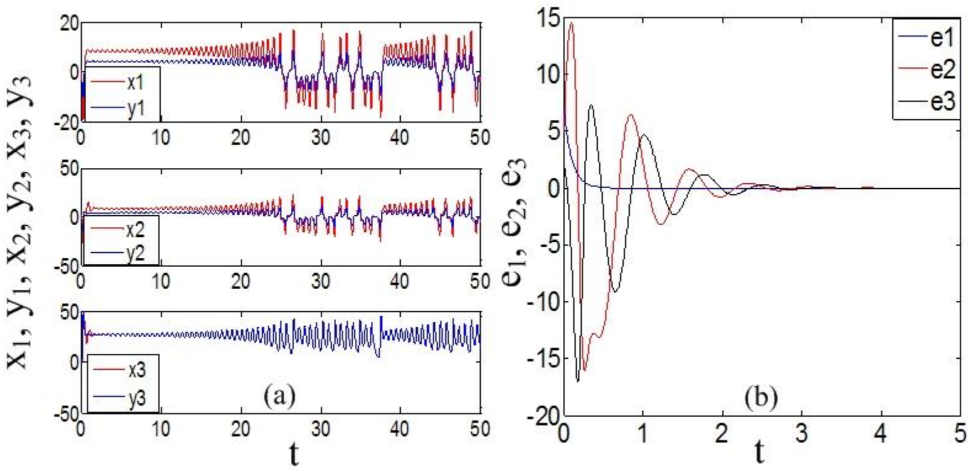

Fig.2. The time evolution of (a) signals (red line for drive system and blue line for response system) and (b) the error vectors with m = 1, n = 4.

2.2 Projective synchronization based on total amplitude control

When the APDs and APRs in the synchronization system are both total control parameters, the synchronization control is flexible and can achieve same-phase or anti-phase synchronization. In this case, all variables can be set at a unified scale factor. A drive system with total amplitude control parameters is

$$\begin{cases} \dot{x}_1 = -ax_1 + a\dfrac{n}{m}y_2, \\ \dot{x}_2 = -mx_1x_3 + rx_1 - x_2, \\ \dot{x}_3 = mx_1x_2 - bx_3, \end{cases} \tag{4}$$

and a corresponding response system is

$$\begin{cases} \dot{y}_1 = a(y_2 - y_1), \\ \dot{y}_2 = -ny_1y_3 + ry_1 - y_2, \\ \dot{y}_3 = ny_1y_2 - by_3, \end{cases} \tag{5}$$

We define the difference $e_1 = mx_1 - ny_1$, $e_2 = mx_2 - ny_2$, and $e_3 = mx_3 - ny_3$. Because of $\dot{e}_1 = -ae_1$, which means the first component converges to zero, the remaining two-dimensional system describing the evolution of the difference can for in the limit $t \to \infty$ be written as

$$\begin{aligned} \dot{e}_2 &= -e_2 - mx_1e_3, \\ \dot{e}_3 &= mx_1e_2 - be_3. \end{aligned} \tag{6}$$

Selecting the same Lyapunov function and using the same reasoning, it is found that the synchronization is globally stable and occurs for all types of drive signal.

Here the united total amplitude parameters m and n control the variables x_1, x_2 and x_3 in drive system according to $1/m$ and the variables y_1, y_2 and y_3 in response system according to $1/n$, correspondingly we make the scale factors $k_{1,2,3} = x_1 / y_1 = x_2 / y_2 = x_3 / y_3 = n / m$. Since the amplitude parameters m and n can be chosen as positive or negative, the projective synchronization can be same-phase or anti-phase. The projective scale factors are flexible and can be defined by the drive or response system. Set (m, n) = (1, 2) and (－2, 1), respectively, and using the initial conditions in drive system and response system as (0.1, 0.1, 0.1) and (-0.3, -0.2, -0.1), respectively, Fig. 3 suggests that the ratio of corresponding coordinates approaches any desired constant even though the initial conditions for the drive and response systems were different and not collinear.

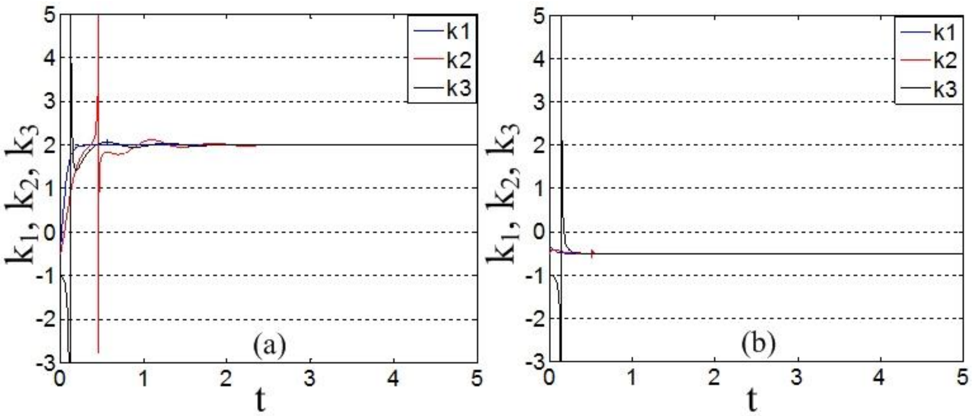

Fig.3. Behavior of the drive and response systems (a) same-phase projective synchronization (scale factor k = 2) (b) anti-phase projective synchronization (scale factor k = −0.5)

3. Projective synchronization based on compound amplitude control

Projective synchronization can be achieved by compound amplitude control, *i.e.*, partial control on one side and total control on the other side, or partial control and total control simultaneously on any side. In this case, same-phase and anti-phase projective synchronization can also be achieved conditionally.

3.1 Partial control in drive system and total control in response system

Suppose the drive system includes a partial amplitude control parameter as

$$\begin{cases} \dot{x}_1 = -ax_1 + \dfrac{an}{\sqrt{m}} y_2, \\ \dot{x}_2 = -x_1x_3 + rx_1 - x_2, \\ \dot{x}_3 = mx_1x_2 - bx_3, \end{cases} \tag{7}$$

and the corresponding response system has a total amplitude control parameter as

$$\begin{cases} \dot{y}_1 = a(y_2 - y_1), \\ \dot{y}_2 = -ny_1y_3 + ry_1 - y_2, \\ \dot{y}_3 = ny_1y_2 - by_3, \end{cases} \tag{8}$$

The difference is now $e_1 = \sqrt{m}x_1 - ny_1$. For $\dot{e}_1 = -ae_1$, the first component converges to zero, and then the remaining two-dimensional system describing the evolution of the difference, $e_2 = \sqrt{m}x_2 - ny_2$, $e_3 = x_3 - ny_3$ can in the limit $t \to \infty$ be written as

$$\begin{aligned} \dot{e}_2 &= -e_2 - \sqrt{m}x_1 e_3, \\ \dot{e}_3 &= \sqrt{m}x_1 e_2 - be_3. \end{aligned} \tag{9}$$

Using the same Lyapunov function and the same reasoning, the synchronization is globally stable and occurs for all types of drive signal.

Under this condition, the amplitude parameters m and n give scale factors $k_{1,2} = x_1 / y_1 = x_2 / y_2 = n / \sqrt{m}, k_3 = x_3 / y_3 = n$. Since the amplitude parameters m can only be positive, same-phase or anti-phase projective synchronization are determined by the united control parameter n from the response system. When $m = 9$ and $n = -2$, the scale factors are $k_1 = k_2 = -2/3$, $k_3 = -2$, as shown in Fig. 4 where the phase projection shows agreement with this prediction. When total control in drive system and partial control in response system is taken, the results are similar to the case discussed above.

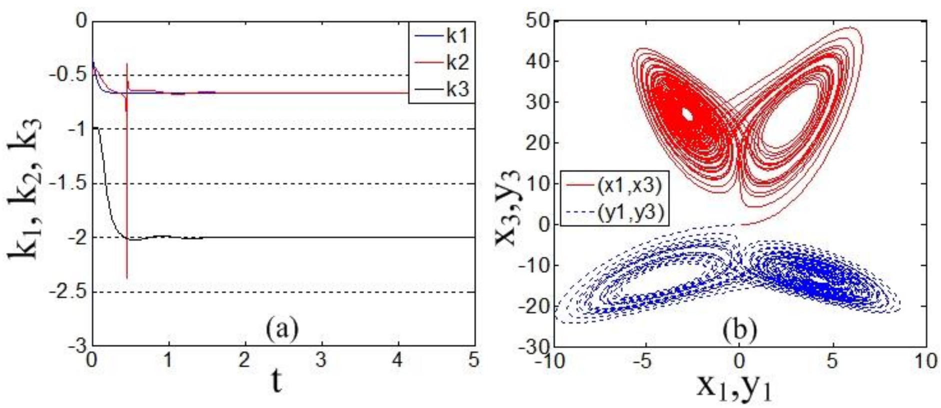

Fig. 4. Behavior of the drive and response systems with initial conditions (0.1, 0.1, 0.1) and (-0.3, -0.2, -0.1), respectively (a) anti-phase projective synchronization (b) projection onto the (x_1-x_3) (the solid line) and (y_1-y_3) (the dash line) plane

3.2 Partial control and total control in drive and response systems

When partial and total control is included simultaneously in any one side, the partial control should be under the regime that is defined by the total control. Thinking of the polarity of the parameters m and n, p and q, the projective synchronization is divided into four modes according to whether the total control parameters m and n are negative or positive.

For the case, $m > 0$, $m + p > 0$, $n > 0$, $n + q > 0$, a drive system with partial and total control parameters is

$$
\begin{cases}
\dot{x}_1 = -ax_1 + \dfrac{a\sqrt{n(n+q)}}{\sqrt{m(m+p)}} y_2, \\
\dot{x}_2 = -mx_1x_3 + rx_1 - x_2, \\
\dot{x}_3 = (m+p)x_1x_2 - bx_3,
\end{cases}
\tag{10}
$$

And the corresponding response system partial and total control parameters is

$$
\begin{cases}
\dot{y}_1 = a(y_2 - y_1), \\
\dot{y}_2 = -ny_1y_3 + ry_1 - y_2, \\
\dot{y}_3 = (n+q)y_1y_2 - by_3,
\end{cases}
\tag{11}
$$

For a difference $e_1 = \sqrt{m(m+p)}x_1 - \sqrt{n(n+q)}y_1$ with $\dot{e}_1 = -ae_1$, the first component converges to zero, and the remaining two-dimensional system describing the evolution of the difference, $e_2 = \sqrt{m(m+p)}x_2 - \sqrt{n(n+q)}y_2$, $e_3 = mx_3 - ny_3$ in the limit $t \to \infty$ is

$$
\begin{aligned}
\dot{e}_2 &= -e_2 - \sqrt{m(m+p)}x_1e_3, \\
\dot{e}_3 &= \sqrt{m(m+p)}x_1e_2 - be_3.
\end{aligned}
\tag{12}
$$

Using the same Lyapunov function and the same reasoning, the synchronization is globally stable and occurs for all types of drive signal.

With this condition, the amplitude parameters *m*, *n*, *p* and *q* give the scale factors $k_{1,2} = x_1 / y_1 = x_2 / y_2 = \dfrac{\sqrt{n(n+q)}}{\sqrt{m(m+p)}}, k_3 = x_3 / y_3 = n / m$. Thus same-phase projective synchronization is determined by the positive parameters *m* and *n*, while the parameters *p* and *q* provide a supplement and fine-tuning for the scale factors k_1 and k_2. For (m, n, p, q) = (5, 4, 1, 26) and (5, 4, 15, －3), respectively, the scale factors are $k_1 = k_2 = 2$, $k_3 = 4/5$, and $k_1 = k_2 = 1/5$, $k_3 = 4/5$, correspondingly, as shown in Fig. 5. For the case, $m < 0$, $m + p < 0$, $n < 0$, and $n + q < 0$, the analysis procedure and the result are the same as this case.

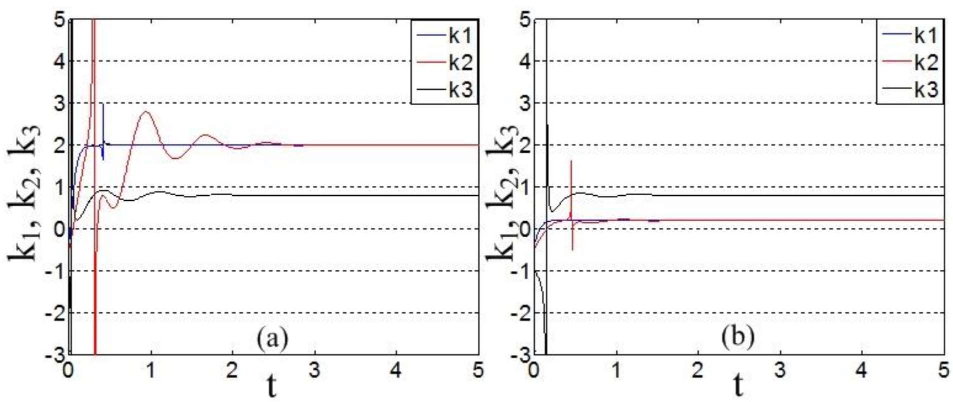

Fig.5. Behavior of the drive and response systems for same-phase projective synchronization (a) $k_1 = k_2 = 2$, $k_3 = 4/5$ (b) $k_1 = k_2 = 1/5$, $k_3 = 4/5$.

For other cases, we get the same results. For cases, $m < 0$, $m + p < 0$, $n > 0$, $n + q > 0$ and $m > 0$, $m + p > 0$, $n < 0$, $n + q < 0$, the drive system is modified as

$$\begin{cases} \dot{x}_1 = -ax_1 - \dfrac{a\sqrt{n(n+q)}}{\sqrt{m(m+p)}} y_2, \\ \dot{x}_2 = -mx_1x_3 + rx_1 - x_2, \\ \dot{x}_3 = (m+p)x_1x_2 - bx_3, \end{cases} \tag{13}$$

Correspondingly, the error equation is

$$\begin{cases} e_1 = \sqrt{m(m+p)}x_1 + \sqrt{n(n+q)}y_1, \\ e_2 = \sqrt{m(m+p)}x_2 + \sqrt{n(n+q)}y_2, \\ e_3 = mx_3 - ny_3, \end{cases} \tag{14}$$

Then we get the same evolution for the difference as in the former cases. Under these conditions, the amplitude parameters *m*, *n*, *p*, and *q* give the scale factors $k_{1,2} = x_1 / y_1 = x_2 / y_2 = \dfrac{-\sqrt{n(n+q)}}{\sqrt{m(m+p)}}, k_3 = x_3 / y_3 = n / m$, which shows that the synchronization is anti-phase.

4. Discussion and conclusions

We point out that the scale factors in chaotic systems for projective synchronization overlap the amplitude parameters. Based the fact that the amplitude of chaotic systems can be controlled partially or totally, we discussed several different regimes for projective synchronization. Chaotic systems can achieve projective synchronization of same-phase or anti-phase signals using the amplitude parameters. The inherent amplitude parameters provide a flexible means for projective synchronization.

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