On a scattering dissipative map

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Abstract

Scattering systems are important in physical problems.We perform an anatomy of the phase space structure to identify the rich dynamical structure produced by the interaction of Hamiltonian and dissipative dynamics.

1 Introduction

There exists a variety of physically interesting situations described by smooth maps. Dissipative systems received only limited attention, in part because it was observed in these systems that orbits eventually tended toward stable fixed points or periodic cycles. Interesting behavior appeared in the study of van der Pol's equation, which describes an oscillator with nonlinear damping. Cartwright and Littlewood found that for certain parameter values, this equation had periodic orbits of different periods and exhibited a rich array of dynamical behaviors. Their results showed the existence of an attracting set more complicated than a fixed point or an invariant curve. Levinson performed detailed analysis for a simplified model. His work inspired Smale, who introduced the general idea of a horseshoe, which Levi used later to explain the observed phenomena.

We investigate basins of attraction in a scattering map, a planar quadratic map which exhibits a panoply of interesting phenomena as dissipation is added. A great effort has been devoted to the characterization of this kind of chaos. Many authors have insisted on the presence of unstable periodic trajectories and homoclinic orbits as a consequence of chaotic behavior in dissipative and Hamiltonian systems. From a mathematical point of view, a small part of the phase space corresponds to a stable regular domain, outside of which there is an infinity of unstable orbits that form the backbone of a nonattracting chaotic set. The appearance of a multiplicity of periodic orbits is the hallmark of the existence of a chaotic set.

We study a planar quadratic scattering map [6],

$$T_1 : \begin{cases} x_{n+1} = a[x_n - \frac{1}{4}(x_n + y_n)^2 - b(x_n + y_n)] \\ y_{n+1} = \frac{1}{a}[x_n + \frac{1}{4}(x_n + y_n)^2] \end{cases} \tag{1}$$

where x_n, y_n are real variables and a, b are real positive parameters. T_1 has a constant Jacobian determinant $J = 1 - b$. Thus for $b = 0$, the map is conservative and globally area-preserving with a weak dissipative effect for small positive values of b. The map (1) has only two fixed points, $(\frac{4(-a^2+a-ab)(ab+1-a)}{(1-a^2)(1+a)}, \frac{4(ab+1-a)^2}{(1-a^2)(1+a)})$ and $(0, 0)$. Both of these fixed points undergo bifurcations in parameter space for positive and negative values of a, but we will consider the case constrained by the parameter $a > 1$. It is easy to verify that T_1 has a unique inverse expressed as follows:

$$T^{-1} : \begin{cases} x_{n+1} = \frac{1}{a}[x_n + \frac{1}{4a(1-b)^2}(x_n + a^2 y_n)^2 - \frac{b}{1-b}(x_n + a^2 y_n)] \\ y_{n+1} = -\frac{1}{a}[x_n + \frac{1}{4a(1-b)^2}(x_n + a^2 y_n)^2 - \frac{1+b}{1-b}(x_n + a^2 y_n)] \end{cases} \tag{2}$$

Seoane *et al.* have determined the fractal dimension of this dynamical system, revealed a *crossover phenomenon* (see Seoane *et al.* for more detail in [6-7]), and insisted on the influence and the importance of unstable periodic orbits in understanding the scattering process. They pointed out the fractal structure by using a mathematical technique previously used in studying chaotic scattering via the construction of a Cantor set. The physical interest of these results, according to the authors, can be potentially useful for particle advection in open chaotic domains. Chaos has physical significance. The authors explained that escaping particles in such systems can be seen, for instance, as inertial particles in fluids with open flows where dissipation plays the role of the mass of the inertial particles, because it is well known that the advective dynamics of idealized particles in two-dimensional and incompressible flows can be described as Hamiltonian.

The dynamics of Eq.1 are sensitive to dissipative perturbations from the precise Hamiltonian structure. We investigate the type of bifurcations that will occur in such a map with a principal attractor that dominates the attraction basin. On the one hand, we have the dissipation which can lead to several coexisting attractors for some values of the parameter a. These periodic attractors are generated through saddle-node (fold) bifurcations as a is varied. On the other hand, we have to deal with homoclinic bifurcations. The dynamical system is smooth and described by a Hamiltonian map which contains Kol'mogorov-Arnol'd-Moser (KAM) islands. This paper is about the routes to chaotic scattering, and it explains the phenomenon of attractor generation.

We begin by considering a two-dimensional map that contains the key features of chaotic scattering (see Yalcinkaya and Lai [8]). The map is given by:

$$T_2 : \begin{cases} x_{n+1} = a[x_n - \frac{1}{4}(x_n + y_n)^2] \\ y_{n+1} = \frac{1}{a}[x_n + \frac{1}{4}(x_n + y_n)^2] \end{cases} \tag{3}$$

The target region is centered at the origin. The parameter a is taken greater than 1. This map captures most of the important features of conservative chaotic scattering experiments. The authors provide the essential ingredient necessary for understanding this conservative Hamiltonian system.

This quadratic map has one fixed point at the origin $(0,0)$ (unstable) and another at $x = ay$, $y = 4(a-1)/(a+1)^2$ (stable for $1 < a < 3 + \sqrt{8}$ and unstable for $a > 3 + \sqrt{8}$). The map (3) is discussed in [4] by Lau *et al.* who proved that the dynamics are associated with the phenomenon known under chaotic scattering. It is singular and nonhyperbolic due to the presence of KAM tori. For $a = 4$, they observed the disappearance of the period-4 island that surrounds the central region of the stable fixed point, and similarly at $a = 5.1$, the central island is destroyed by the period-3 bifurcation.

We emphasize that our aim is to study the map (1), and we focus on the effects of perturbations to its Hamiltonian structure. This map can be considered as a model giving rise to an interesting set of bifurcations with a fractal structure, and with a constant birth of island chains of periods varying between 6 to 19, forming an attracting set of saddles and foci.

Our numerical evidence includes the following: First we establish the hierarchy of fixed points and k-order periodic points. Then we describe the parameter plane in which these points undergo a saddle-node (fold) bifurcation (instead of a saddle-center bifurcation) or period doubling (flip) bifurcation (see Fig. 1). Finally, we consider negative values for a and study the properties of the map including KAM tori and period doublings.

The two fixed points do not undergo identical sets of bifurcations in the parameter plane. We first choose the parameters so that the origin (0,0) is a saddle, while the second equilibrium can experience bifurcations, and we do this by having $a > 1, b > 0$. We also choose negative values of a, which is instructive, with the occurrence of a period-doubling bifurcation so that the origin remains a saddle while the other point bifurcates.

From the points of view of [6], a chaotic scattering set may be envisioned as the intersection of a stable manifold and an unstable manifold where the stable or the unstable one consists of an uncountably many fractal set that provides chaos and is responsible for this phenomenon. The two manifolds oscillate more and more wildly in a homoclinic tangle (also known as a stochastic layer, see Fig. 2) causing the sensitive dependence on initial conditions in this layer. There is much interest in describing the motion in the layer and in estimating its width as a function of the perturbation.

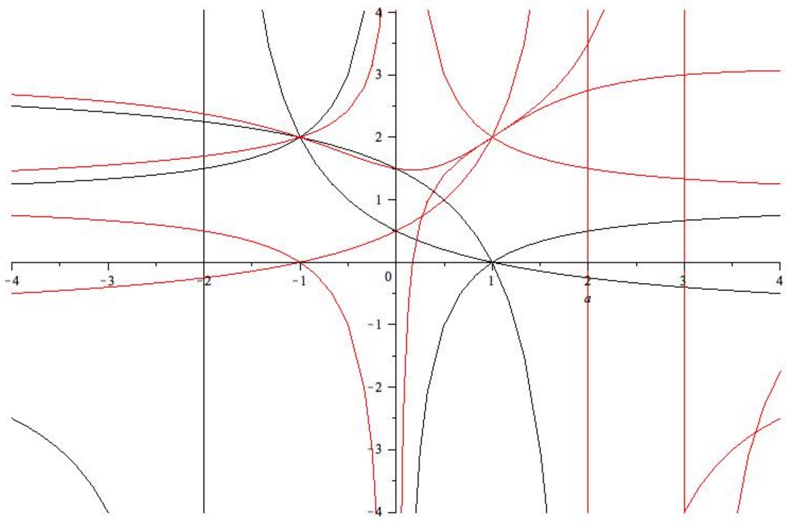


Fig. 1: Saddle-node Bifurcation curves in black and flip in red

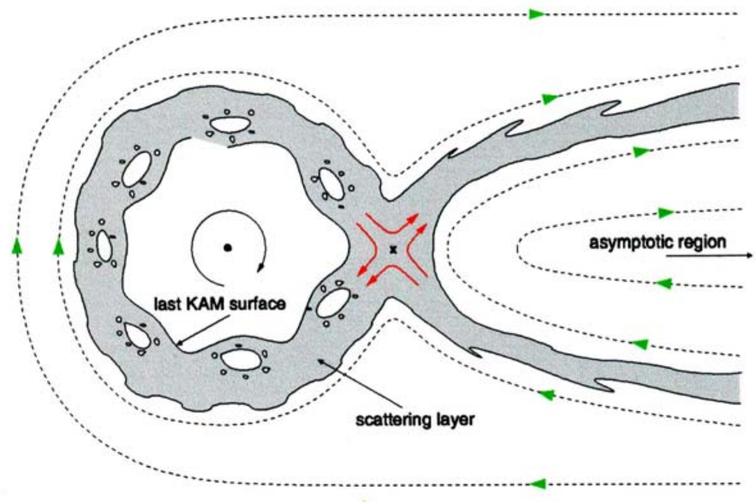


Fig. 2: A scattering layer, KAM surfaces with a binary horseshoe (see [3])

2 Some properties of invariant curves

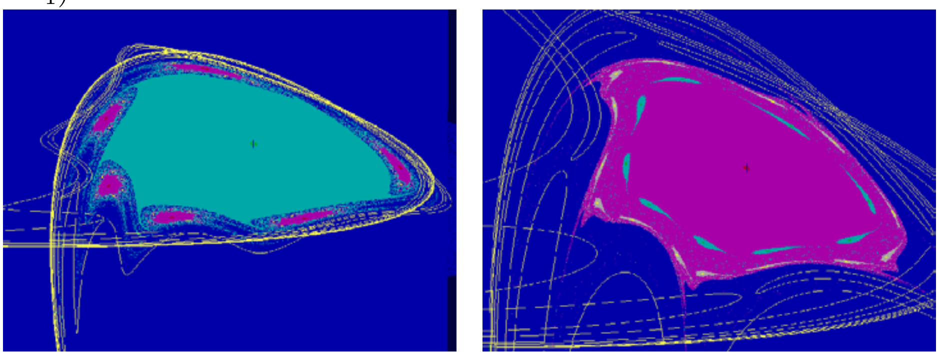
Hénon in [2] showed that any second-degree area-preserving planar map

$$\begin{cases} x_{n+1} = ax_n + by_n + cx_n^2 + dx_ny_n + ey_n^2 \\ y_{n+1} = ex_n + fy_n + gx_n^2 + hx_ny_n + iy_n^2 \end{cases} \tag{4}$$

with a center at the origin can be reduced to this map by a linear change of coordinates:

$$
\begin{cases} u_{n+1} = u_n \cos\alpha - (v_n - u_n^2)\sin\alpha \\ v_{n+1} = u_n \sin\alpha - (v_n - u_n^2)\cos\alpha \end{cases} \tag{5}
$$

Hamiltonian system orbits are regular. They are smooth curves in the phase plane referred to as KAM circles or KAM tori. KAM theory tells us that KAM tori with irrational rotation numbers persist in the perturbed system but are gradually destroyed as the perturbation is increased. Tori with rational numbers disintegrate immediately. The Poincaré–Birkhoff theorem explains that only an even number of periodic points remain, forming Birkhoff periodic orbits. Fixed points and periodic points are either centers or saddles, and they alternate on the chains of periodic points in Birkhoff periodic trajectories, forming structures known as island chains. Around each centre are KAM tori interspersed with more island chains. The stable and unstable manifolds of a saddle become tangled, generating homoclinic points from their intersections (see Fig. 3). Under the effect of the perturbation, KAM curves become scarce, and island chains predominate. Hénon showed that the island chains visible in the phase portraits depend on the choice of the parameter α (corresponding to the choice of the two parameters a and b in T_1).



$a = 3;\ b = 0.001$ $\qquad$ $a = 3.5,\ b = 0.001$

Fig. 3: Two phase portraits of the map 1. Island chains prominent are of periods 6, 9, and 14

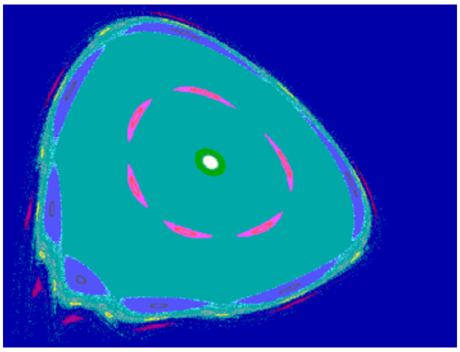
A numerical plot of the stable and unstable manifolds of the saddle point at $(0, 0)$ shows that the orbit sweeps around the stable fixed point $(\frac{4(-a^2+a-ab)(ab+1-a)}{(1-a^2)(1+a)}, \frac{4(ab+1-a)^2}{(1-a^2)(1+a)})$. These manifolds of the period-1 saddle in T_1 are shown intersecting in a homoclinic tangle. The horseshoe is clearly shown in the neighborhood of the saddle point. The map T_1 exhibits the complexity that was glimpsed by Hénon of Hamiltonian dynamical structure in the Hénon area-preserving map (5). This complexity is explained in the Poincaré–Birkhoff theorem and KAM theorem [1, 5].

One has two fixed points and periodic points, the first is a stable node whose basin of attraction has for its external boundary the stable invariant manifold of the saddle point $S\ (0,0)$. Island chains of period-8 (represented in red small pink domains in Fig. 3) and period-9 (represented by the gray domains) are most prominent, with the stable and unstable invariant manifolds in yellow color.

2.1 Complexity and Basins

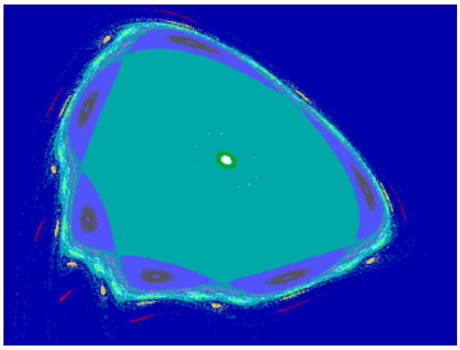
A sequence of phase portraits of the map (1) is considered, varying the two parameters a and b. In Fig. 4, complicated structures in dissipative and conservative cases are observed for very small values of b.

$a = 2.7, b = 0.00008$



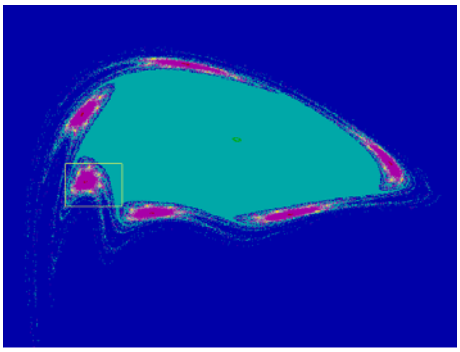
(a)

$a = 2.9, b = 0.0001$



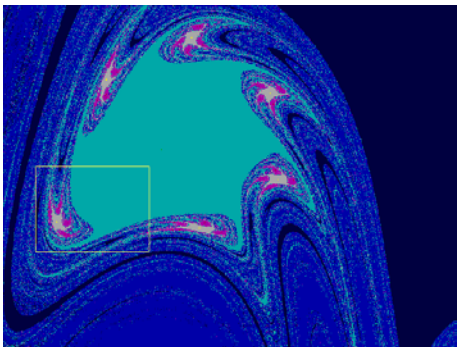
(b)

$a = 3, b = 0.001$



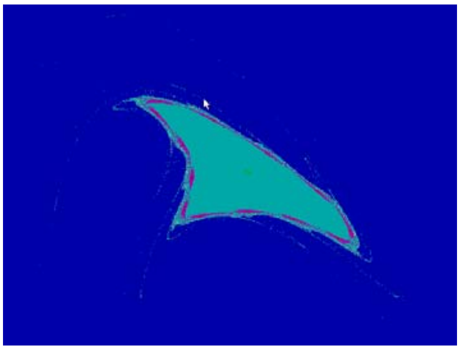
(c)

$a = 3, b = 0.001$



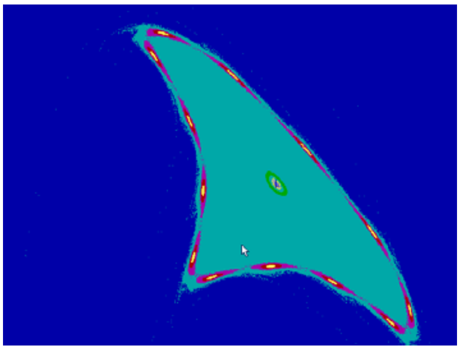
(d)

$a = 4.52, b = 0.001$



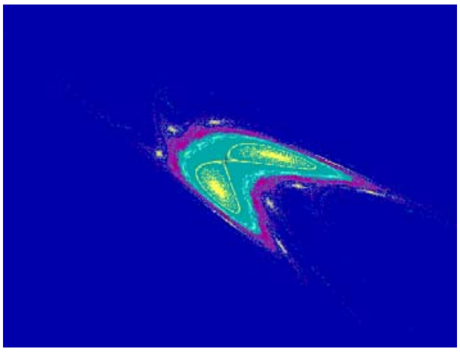
(e)

$a = 4.52, b = 0.0001$



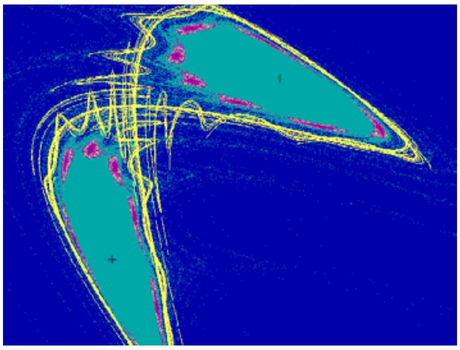
(f)

$a := -1.3; b = 0.001$



(g)

$a := -.7; b = 0.001$

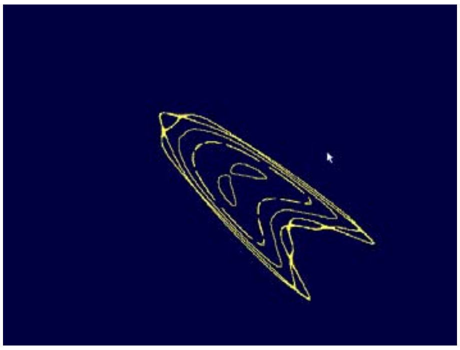


(h)

$a := -.67; b = 0.0001$

(i)

$a := -.67; b = 0$



(j)

Fig. 4: The sequence shows phase portraits. (d) is a zoom of a part of (c). (j) invariant circles formed and island chains.

The evolution of the phase portraits is given directly in the figures for the parameters (a, b). We can see the period-doubling bifurcation which is evident for negative values of a. For $a < 0$, the behavior of the map changes, and the stable fixed point bifurcates into a period-2 orbit coexisting with the principal saddle at the origin, and in (j) the saddle connection is clearly visible with a double binary horseshoe, which depends sensitively on the map and the parameter a.

3 Conclusion

In this paper, specific bifurcations arising in Hamiltonian maps have been studied. These bifurcations concern the evolution of nonattracting chaotic sets, and one sees the creation and annihilation of Birkhoff periodic orbits at saddle-node bifurcations, with their own basins. In such maps, chaotic sets can occur at bifurcations, giving rise to intersections of invariant stable and unstable manifolds of the principal saddle. The map's dynamics are extremely rich with its structure influenced by Hamiltonian and dissipative characters and the interaction of their effects and dominated by the chaotic scattering. Important changes can occur with small changes of parameter values.

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