On the existence of non-horseshoe-type chaos in 3-D quadratic continuous-time systems

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Abstract

In this paper, we obtain non-existence conditions for horseshoe-type chaos in 3-D quadratic continuous-time systems. This kind of chaos in polynomial ODE systems is characterized by the non-existence of homoclinic and heteroclinic orbits.

Keywords: homoclinic chaos, heteroclinic chaos, non-existence of Shilnikov chaos, 3-D quadratic autonomous systems

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1 Introduction

It is well known that the famous analytic method for proving chaos in autonomous systems employs the two Shilnikov theorems [1-2], and their subsequent embellishments and slight extensions given in [3-4]. The resulting chaos in this case is called *horseshoe-type* or *Shilnikov chaos.* Several works have proved the existence of chaos in some 3-D quadratic autonomous systems [5-6-7-8-9-10-11]. The analysis is based on the method of undetermined coefficients to find homoclinic and heteroclinic orbits. It was conjectured in

[11] that the two Shilnikov theorems can be used to classify chaos in 3-D polynomial ODE systems. The conjecture claimed that for such systems, there exist four kinds of chaos: homoclinic chaos, heteroclinic chaos, a combination of homoclinic and heteroclinic chaos and chaos of other types. Examples includes the attractor described in [13]. This example is a 3D quadratic flow which simply does not have any fixed point (thus, cannot have any homoclinic and heteroclinic trajectories).

In this paper, we propose non-existence conditions for horseshoe-type chaos in 3-D quadratic continuous-time systems. The obtained conditions are four simple inequalities. Based on this result, we give some examples of chaotic attractors in polynomial ODE systems characterized by the existence of equilibrium points and by the non-existence of homoclinic and heteroclinic orbits.

As a motivation for this result, let us consider the n^{th}-order autonomous system: $x' = f(x)$, where the vector field $f = (f_1, f_2, ..., f_n)^T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ belongs to class $C^r(r \geq 1)$, $x = (x_1, x_2, ..., x_n)^T$ is the state variable of the system, and $t \in \mathbb{R}$ is the time. Suppose that f has at least one equilibrium point P. Then we have the following definitions: (a) The point $P = (p_1, p_2, ..., p_n)$ is called a hyperbolic saddle focus for system (1) if the eigenvalues of the Jacobian $A = Df(x)$ evaluated at P are $\gamma, \alpha + i\beta$, where $\alpha\gamma < 0$ and $\beta \neq 0$. (b) Consider the n^{th}-order autonomous system $x' = f(x)$, where the vector field $f = (f_1, f_2, ..., f_n)^T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ belongs to class $C^r(r \geq 1)$, $x = (x_1, x_2, ..., x_n)^T$ is the state variable of the system, and $t \in \mathbb{R}$ is the time. (c) Consider the n^{th}-order autonomous system $x' = f(x)$, where the vector field $f = (f_1, f_2, ..., f_n)^T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ belongs to class $C^r(r \geq 1)$, $x = (x_1, x_2, ..., x_n)^T$ is the state variable of the system, and $t \in \mathbb{R}$ is the time. (d) A homoclinic orbit $\gamma(t)$ refers to a bounded trajectory of system (1) that is doubly asymptotic to an equilibrium point P of the system, i.e., $\lim_{t \longrightarrow -\infty} \gamma(t) = \lim_{t \longrightarrow +\infty} \gamma(t) = P$. The next definition requires the existence of at least two equilibrium points P_1 and P_2. (e) A heteroclinic orbit $\delta(t)$ is similarly defined except that there are two distinct saddle foci, P_1 and P_2, being connected by the orbit, one corresponding to the forward asymptotic time, and the other to the reverse asymptotic time limit, i.e., $\lim_{t \longrightarrow +\infty} \delta(t) = P_1$ and $\lim_{t \longrightarrow -\infty} \gamma(t) = P_2$.

The main motivation of the theorem proved in [12] is the search of sufficient conditions for the non-existence of homoclinic and heteroclinic orbits in a system of the form $x' = f(x)$ as follows: Suppose that there exists at least one integer $j \in \{1, 2, ..., n\}$ such that the component $f_j(x)$ satisfies

$\exists\alpha < 0 : f_j(x) \geq \alpha, \forall x \in \mathbb{R}^n$. Then system (1) cannot have homoclinic and heteroclinic orbits. From this result it is important to remark that if system (1) is chaotic, then its chaos is not of horseshoe-type. We note that the assumptions of the theorem proved in [12] do not contradict the assumption that the system $x' = f(x)$ has an equilibrium point.

2 On the non-existence of horseshoe-type chaos in 3-D quadratic continuous-time systems

The most general 3-D quadratic continuous-time system is given by

$$\left\{\begin{array}{l} x' = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz \\ y' = b_0 + b_1x + b_2y + b_3z + b_4x^2 + b_5y^2 + b_6z^2 + b_7xy + b_8xz + b_9yz \\ z' = c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5y^2 + c_6z^2 + c_7xy + c_8xz + c_9yz \end{array}\right. \tag{1}$$

where $(a_i, b_i, c_i)_{0\leq i\leq 9} \in \mathbb{R}^{30}$ are the bifurcation parameters. Note that in this paper we use the following simple result available in most textbooks on linear algebra: The polynomial $Ax^2 + Bx + C$ has no real zeros if and only if $A > 0$ and $B^2 - 4AC < 0$, or $A < 0$ and $B^2 - 4AC < 0$, and thus this polynomial is strictly positive or strictly negative.

By using the theorem proved in [12], it is evident that any system of the form (1) with the components $z' = c_0 + c_4x^2, z' = c_0 + c_5y^2$, $z' = c_0 + c_6z^2$ with $c_0 < 0$ cannot have homoclinic and heteroclinic orbits. The same result holds true for the cases: $x' = a_0 + a_4x^2, x' = a_0 + a_5y^2$, $x' = a_0 + a_6z^2$ and $y' = b_0 + b_4x^2, y' = b_0 + b_5y^2$, $y' = b_0 + b_6z^2$. Also, the same result is true for the total and partial combinations of these cases.

For the general case (1) and by using the above criterion proved in [12] for $j = 3$, we have

$$c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5y^2 + c_6z^2 + c_7xy + c_8xz + c_9yz \geq \alpha \tag{2}$$

with $\alpha < 0$ for all $(x, y, z) \in \mathbb{R}^2$, that is

$$c_4x^2 + (c_1 + yc_7 + zc_8)\, x + \left(c_5y^2 + c_9yz + c_2y + c_6z^2 + c_3z - \alpha + c_0\right) \geq 0 \tag{3}$$

Inequality (3) is possible for all $(x, y, z) \in \mathbb{R}^3$ if and only if the coefficient c_4 of the term x^2 is positive and the discriminant Δ_1 of the null equation corresponding to (3) is strictly negative, i.e.,

$$\left\{\begin{array}{c} c_4 > 0 \\ \Delta_1 = \xi_1 y^2 + \xi_2 y + \xi_3 < 0 \end{array}\right. \tag{4}$$

The discriminant Δ_1 is strictly negative if $\xi_1 < 0$ and $\Delta_2 = \xi_2^2 - 4\xi_1\xi_3 < 0$, that is

$$\left\{\begin{array}{c} \xi_1 < 0 \\ \Delta_2 = \xi_4 z^2 + \xi_5 z + \xi_6 < 0 \end{array}\right. \tag{5}$$

The discriminant Δ_2 is strictly negative if

$$\left\{\begin{array}{c} \xi_4 < 0 \\ \Delta_3 = \xi_5^2 - 4\xi_4\xi_6 < 0 \end{array}\right. \tag{6}$$

where

$$\left\{\begin{array}{c} \xi_1 = c_7^2 - 4c_4c_5 \\ \xi_2 = 2c_7\left(c_1 + zc_8\right) - 4c_4\left(c_2 + zc_9\right) \\ \xi_3 = \left(c_1 + zc_8\right)^2 - 4c_4\left(c_6z^2 + c_3z - \alpha + c_0\right) \\ \xi_4 = \left(4c_4c_9 - 2c_7c_8\right)^2 - 4\xi_1\left(c_8^2 - 4c_4c_6\right) \\ \xi_5 = -2\left(2c_1c_7 - 4c_2c_4\right)\left(4c_4c_9 - 2c_7c_8\right) - 4\left(2c_1c_8 - 4c_3c_4\right)\xi_1 \\ \xi_6 = -16c_4\xi_1\alpha + \xi_7 \\ \xi_7 = \left(2c_1c_7 - 4c_2c_4\right)^2 - 4\left(c_1^2 - 4c_0c_4\right)\xi_1 \end{array}\right. \tag{7}$$

Thus the conditions for (3) are

$$\left\{\begin{array}{c} c_4 > 0 \\ \xi_1 < 0 \\ \xi_4 < 0 \\ \Delta_3 = \xi_5^2 - 4\xi_4\xi_6 < 0 \end{array}\right. \tag{8}$$

We have $\Delta_3 = 64\xi_1\xi_4c_4\alpha + \left(\xi_5^2 - 4\xi_4\xi_7\right) < 0$ if $\alpha < \frac{-\left(\xi_5^2 - 4\xi_4\xi_7\right)}{64\xi_1\xi_4c_4}$. In terms of the bifurcation parameters $(c_i)_{0\leq i\leq 9} \in \mathbb{R}^{10}$, the inequalities of (8) are equivalent to

$$\left\{\begin{array}{c} c_4 > 0 \\ c_5 > \frac{c_7^2}{4c_4} \\ c_6 > \frac{-\left(\left(4c_4c_9 - 2c_7c_8\right)^2 - 4\xi_1c_8^2\right)}{16\xi_1c_4} \\ \alpha < \frac{-\left(\xi_5^2 - 4\xi_4\xi_7\right)}{64\xi_1\xi_4c_4} \end{array}\right. \tag{9}$$

since $-\left(\xi_5^2 - 4\xi_4\xi_7\right) > 0, c_4\xi_1 < 0$ and $64\xi_1\xi_4 c_4 > 0$. If conditions (9) hold, then system (1) cannot have homoclinic and heteroclinic orbits and therefore cannot have horseshoe-type chaos if it is chaotic. We note that the above result still holds true if we replace the conditions for $(c_i)_{0\leq i\leq 9} \in \mathbb{R}^{10}$ in (9) by the conditions for $(a_i)_{0\leq i\leq 9} \in \mathbb{R}^{10}$ or $(b_i)_{0\leq i\leq 9} \in \mathbb{R}^{10}$. Also, the above analysis is true if the above conditions are commenced from the inequalities $c_5 > 0$ or $c_6 > 0$ just like the case for c_4. An elementary example that satisfies the above conditions (9) is the one studied in [12] and given by

$$\begin{cases} x' = a(y-x) \\ y' = -ax - byz \\ z' = -c + y^2 \end{cases} \tag{10}$$

For system (10) we have $a_1 = -a, a_2 = a, a_0 = a_i = 0, i = 3, ..., 9, b_0 = b_i = 0,$ $i = 2, .., 8, b_1 = -a, b_9 = -b,$ $c_0 = \alpha = -c, c_5 = 1, c_i = 0, i = 1, ..., 9$ with $i \neq 5$. For $a = 40, b = 33$, and $c = 10$, the system (10) has a chaotic attractor, and the Lyapunov exponents for these values are $\{2.6721, 0, -15.7588\}$.

3 Conclusion

In this paper, we obtain non-existence conditions for horseshoe-type chaos in 3-D quadratic continuous-time systems. Proving rigorously such a kind of chaos is a very interesting challenge for future investigation since the two Shilnikov theorems cannot apply for those systems in the absence of homoclinic and heteroclinic orbits.

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