

Spatiotemporal Chaos in Easter Island Ecology

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Abstract: *This paper demonstrates that a recently proposed spatiotemporal model for the ecology of Easter Island admits periodic and chaotic attractors, not previously reported. Such behavior may more realistically depict the population dynamics of general ecosystems and illustrates the power of simple models to produce the kind of complex behavior that is ubiquitous in such systems.*

Key Words: chaos, Easter Island, ecology, population dynamics, Turing instability

INTRODUCTION

A recent series of papers has modeled the population dynamics of Easter Island, in which the natives harvested most of the trees, leading to a rapid decline in the human population, perhaps exacerbated by a population of rats that ate the seeds from the trees (Basener, Brooks, Radin, & Wiandt, 2008a, 2008b, 2011; Basener & Ross, 2005; Brander & Taylor, 1998; Sprott, 2011). The most recent of these papers (Basener, Brooks, Radin, & Waindt, 2011) advances a three-species, continuous-time, spatiotemporal model and shows, counter-intuitively, that spatial diffusion can linearly destabilize the coexisting equilibrium through a Turing instability. Spatial effects were taken into account by assuming a one-dimensional ring of ten cells with each cell interacting only with its two nearest neighbors. Exhibited solutions were damped or growing oscillations or a growth followed by rapid extinction. The purpose of this paper is to extend that result to the continuum spatial limit and to show that periodic and chaotic solutions can also occur. It provides a new and potentially useful example of chaos in a simple system of partial differential equations with three spatial variables.

INVASIVE SPECIES MODEL

To include the effects of rats in the population dynamics, Basener et al. (2008b) proposed a three-component model with people (P), rats (R), and trees (T) given by

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$$\begin{aligned}\frac{dP}{dt} &= aP\left(1 - \frac{P}{T}\right) \\ \frac{dR}{dt} &= cR\left(1 - \frac{R}{T}\right) \\ \frac{dT}{dt} &= \frac{b}{1+fR}T\left(1 - \frac{T}{M}\right) - hP\end{aligned}$$

where T is in units of the number of trees required to support one human and R is in units of the number of rats that can be supported by one tree. The six parameters are typically taken as $a = 0.03$, $c = 10$, $b = 1$, $f = 0.001$, $M = 12,000$, and $h = 0.25$ with time in units of years. Note that the people and the rats do not interact directly but they compete for the trees, in the former case through harvesting ($-hP$) and in the latter case through a reduction in the growth rate of the forest (fR).

We can linearly rescale $P \rightarrow PbM/h$, $R \rightarrow MR$, $T \rightarrow MT$, and $t \rightarrow t/b$ to obtain an equivalent four-parameter model given by

$$\begin{aligned}\frac{dP}{dt} &= \gamma_P P\left(1 - \frac{P}{\eta_P T}\right) \\ \frac{dR}{dt} &= \gamma_R R\left(1 - \frac{R}{T}\right) \\ \frac{dT}{dt} &= \frac{T}{1 + \eta_R R}(1 - T) - P\end{aligned}$$

where $\gamma_P = a/b$ and $\gamma_R = c/b$ are the growth rates of the people and rats, respectively, in the presence of unlimited resources ($T \rightarrow \infty$), and $\eta_P = h/b$ and $\eta_R = fM$ are the rates at which people and rats consume the trees and their seeds, respectively. This system has four equilibria with only the one at $R = P/\eta_P = T = (1 - \eta_P)/(1 + \eta_P \eta_R)$ corresponding to coexistence of all three species, and it exists only if $\eta_P < 1$.

The parameters suggested by the authors give $(\gamma_P, \gamma_R, \eta_P, \eta_R) = (0.03, 10, 0.25, 12)$, but with considerable uncertainty, and they only show solutions that attract to one of the stable equilibria (either coexistence or extinction of one or more species). However, in a recent paper, Sprott (2011) showed that other choices of parameters such as $(\gamma_P, \gamma_R, \eta_P, \eta_R) = (0.1, 0.3, 0.47, 0.7)$ lead to chaotic solutions and strange attractors as well as stable limit cycles.

SPATIOTEMPORAL MODEL

In their recent paper, Basener et al. (2011) included spatial effects in the above model by assuming that the people or rats undergo spatial diffusion. They represented the diffusion by assuming each species resides in a ring of ten cells, individually governed by the above equations but with the people and rats

in each cell coupled to their two nearest neighbors by $D_P(P^{s-1} - 2P^s + P^{s+1})$ and $D_R(R^{s-1} - 2R^s + R^{s+1})$, respectively, where D_P and D_R are the corresponding (non-negative) constant diffusion rates and S is an index that denotes the number of the cell ($1 \leq S \leq 10$). The ring is closed by assuming that the cell with $S = 1$ is adjacent to the one with $S = 10$. One could imagine a rocky island where life exists only along the shore. Their goal was not so much to develop a realistic model of the Easter Island ecology, but to demonstrate the importance of spatial effects and to show that diffusion can destabilize the system, leading to the extinction of one or more species in contrast to intuition which says that diffusion should smooth out irregularities. This is an example of a Turing instability (Turing, 1952).

The spatially continuous description of such a system replaces the above three ordinary differential equations by three partial differential equations as follows:

$$\begin{aligned} \frac{\partial P}{\partial t} &= \gamma_P P \left(1 - \frac{P}{\eta_P T}\right) + D_P \frac{\partial^2 P}{\partial s^2} \\ \frac{\partial R}{\partial t} &= \gamma_R R \left(1 - \frac{R}{T}\right) + D_R \frac{\partial^2 R}{\partial s^2} \\ \frac{\partial T}{\partial t} &= \frac{T}{1 + \eta_R R} (1 - T) - P \end{aligned}$$

where P , R , and T are now densities of people, rats, and trees per unit length, respectively, and that depend on both space and time, and s is the spatial coordinate around the ring. Time is in units of $1/b$, which from earlier estimates (with $b = 1 \text{ year}^{-1}$) implies that it can be thought of as years. The two diffusion terms involving D_P and D_R tend to smooth out any spatial gradients in the population of people P and rats R .

This is an example of what in chemistry is called a reaction-diffusion system in which the spatially independent terms would represent the reactions of three chemicals, two of which are spatially diffusing. The most familiar such system is the Belousov-Zhabotinsky (BZ) reaction (Belousov, 1959; Zhabotinsky, 1964), which is known to have periodic and chaotic solutions as well as producing complex spatial patterns (Agladze, Krinsky, & Pertsov, 1984). In general, reaction-diffusion systems with three or more components have been relatively little studied and likely harbor phenomena yet undiscovered.

There are many methods for solving such a system, but we here employ the method of lines, which is identical to the method used by Basener et al. (2011), except that to more accurately approximate the continuum spatial limit, we use 100 cells and a five-point approximation of the second derivative given by

$$\partial^2 P / \partial s^2 \cong (-P^{s-2} + 16P^{s-1} - 30P^s + 16P^{s+1} - P^{s+2}) / 12,$$

and similarly for $\partial^2 R / \partial s^2$. To simplify the system and reduce the space of parameters, the two diffusion coefficients are taken equal: $D_P = D_R = D$. Similar results are obtained for either $D_P = 0$ or $D_R = 0$. Of course the trees could also diffuse but would probably do so at a much slower rate than the rats and people. The resulting system of 300 coupled ordinary differential equations is solved by a fourth-order Runge-Kutta method with an adaptive step size and an absolute allowed error of 10^{-12} per time step. Initial conditions are not critical, but are taken as $(P, R, T) = (0.01, 0.01, 1)$ for all cells but one for which $P = 0.02$ to perturb the spatial symmetry.

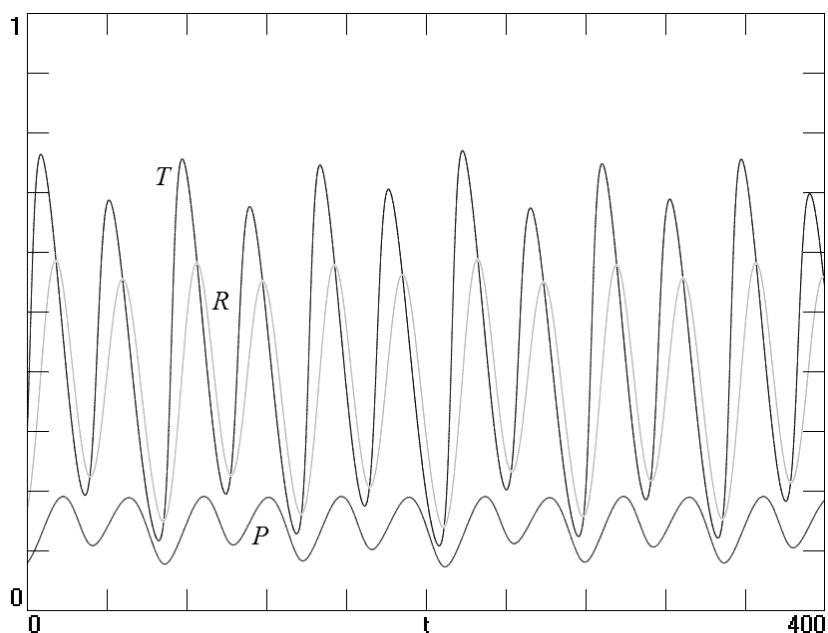


Fig. 1. Population density of people (P), rats (R), and trees (T) as a function of time at a fixed position around the ring. Time is in units of years. Parameters are $(\gamma_P, \gamma_R, \eta_P, \eta_R, D) = (0.1, 0.21, 0.42, 1.11, 0.01)$ with initial conditions $(P, R, T) = (0.01, 0.01, 1)$ for all cells but one for which $P = 0.02$.

SPATIOTEMPORAL CHAOS

A computer search through thousands of combinations of the five parameters seeking solutions with a positive Lyapunov exponent (Sprott, 2003) revealed a number of chaotic solutions, a typical example of which has $(\gamma_P, \gamma_R, \eta_P, \eta_R, D) = (0.1, 0.21, 0.42, 1.11, 0.01)$. The spatiotemporal chaos can be illustrated in a number of ways. Figure 1 shows the population densities in a representative cell as a function of time. In this and the subsequent figures, the initial transient, which depends on the initial conditions, has been allowed to

decay so that the figures show only the dynamics on the attractor. There is a dominant period of about $T = 35.5$ years, which is perhaps a bit short to be biologically realistic for Easter Island, but with an apparently chaotic modulation. Calculations show that the chaotic behavior persists for at least a million years. Figure 2 shows the population densities at a representative time as a function of distance around the ring indicating the spatial asymmetry. This is an example of symmetry breaking since a completely symmetric model has a spatially asymmetric solution.

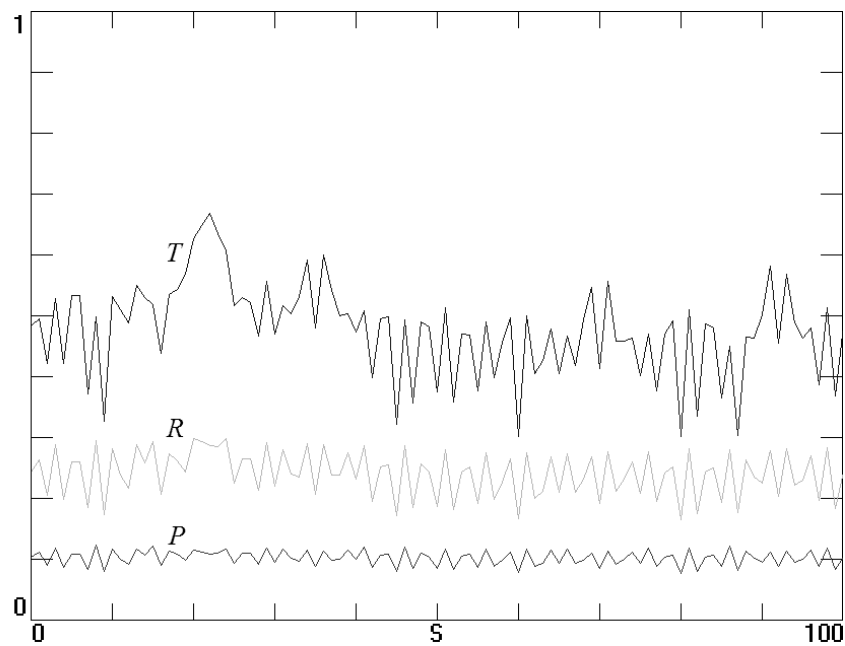


Fig. 2. Population density of people (P), rats (R), and trees (T) as a function of position around the ring at a fixed time. Parameters are $(\gamma_P, \gamma_R, \eta_P, \eta_R, D) = (0.1, 0.21, 0.42, 1.11, 0.01)$ with initial conditions $(P, R, T) = (0.01, 0.01, 1)$ for all cells but one for which $P = 0.02$.

Whereas the two previous figures show the temporal and spatial variation of the populations at a fixed position and time, respectively, Fig. 3 is a gray scale plot of the population density of people as a function of both space and time in which the lighter areas represent the higher population densities. Plots of rats and trees are not shown but are qualitatively similar. The vertical stripes indicate the tendency of the regions of the island to synchronize, and this tendency becomes progressively less evident as the number of cells is increased (not shown), corresponding to a larger island, with everything else held constant.

With a sufficiently large island, the spatial asymmetry grows to the point where the system becomes unstable, causing the extinction of one or more of the species.

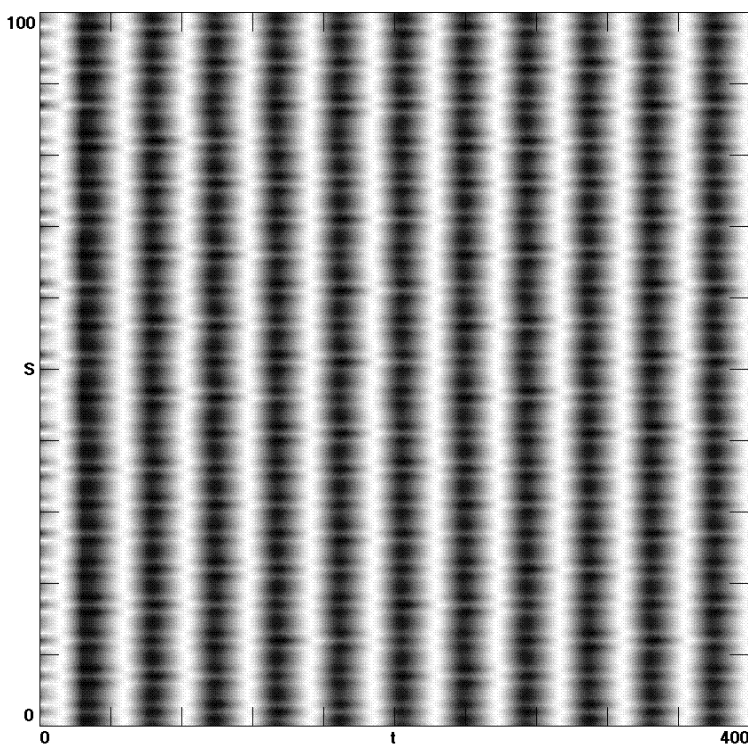


Fig. 3. Population density of people as a function of position around the ring (s) and time (t). Time is in units of years, and the lighter regions correspond to greater population. Parameters are $(\gamma_P, \gamma_R, \eta_P, \eta_R, D) = (0.1, 0.21, 0.42, 1.11, 0.01)$ with initial conditions $(P, R, T) = (0.01, 0.01, 1)$ for all cells but one for which $P = 0.02$.

Yet another way to exhibit the chaos is to plot a portion of the trajectory that lies on a strange attractor as shown in Fig. 4. This case shows the trajectory of the population density of rats as a function of the population density of people at a typical position around the ring, with the population of trees indicated by a shadow that gives the impression of a third dimension. This plot is a projection of an object in a 300-dimensional space onto a space of three dimensions, but it suggests that the attractor dimension is small, perhaps even less than three. This presumption is supported by the relatively small largest Lyapunov exponent of 0.001 year^{-1} and the large sum of the 300 Lyapunov exponents of -23.45 year^{-1} . Thus the system is strongly dissipative and only weakly chaotic with information lost on a time scale of about 1000 years.

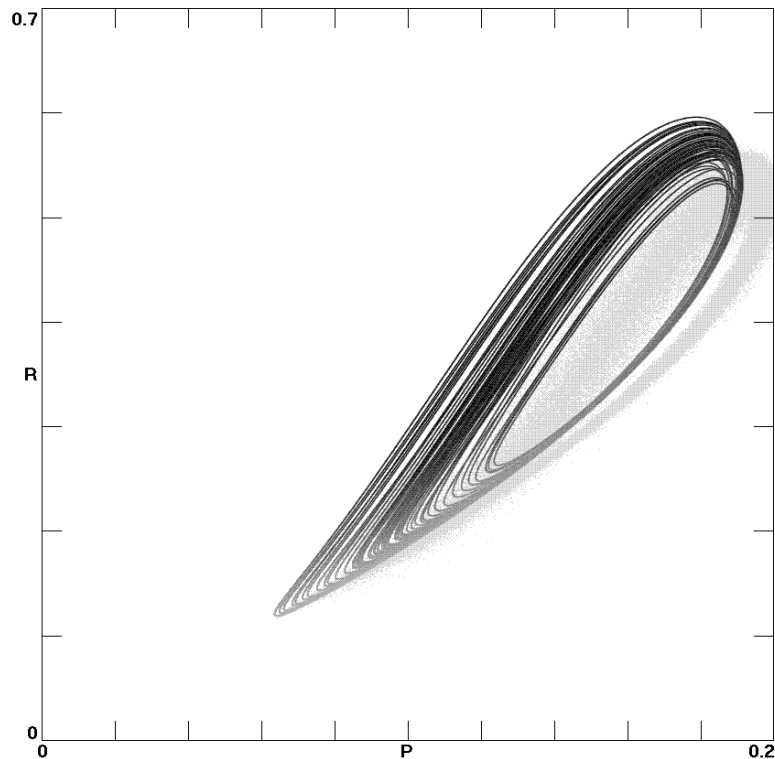


Fig. 4. A projection of the strange attractor at a typical place around the ring, showing the population density of rats (R) versus the population density of people (P) with the population density of trees indicated by a shadow.

ROUTE TO CHAOS

It was previously shown (Sprott, 2011) that in the absence of diffusion, the route to chaos as the harvesting rate of trees η_p increases is by way of a Hopf bifurcation followed by period-doubling of the resulting limit cycle followed by a growth in size of the chaotic attractor until stability is lost through a global bifurcation. A similar scenario occurs for the spatiotemporal model with D as the bifurcation parameter as shown in Fig. 5. With $D = 0$, the system has stable periodic oscillations for this choice of the other parameters as evidenced by a largest Lyapunov exponent of zero to a precision of $1 \times 10^{-5} \text{ year}^{-1}$. As D increases, the limit cycle period-doubles, eventually leading to chaos. Even higher values of D lead to instability and extinction. Because of the complexity of the system, it is difficult to quantify exactly the bifurcation points, but the chaotic region is relatively narrow and occurs just prior to extinction as was also

the case for the spatially independent model. In the chaotic region, the largest Lyapunov exponent increases with increasing D up to the point where the population crashes, which occurs somewhere between $D = 0.010$ and $D = 0.011$.

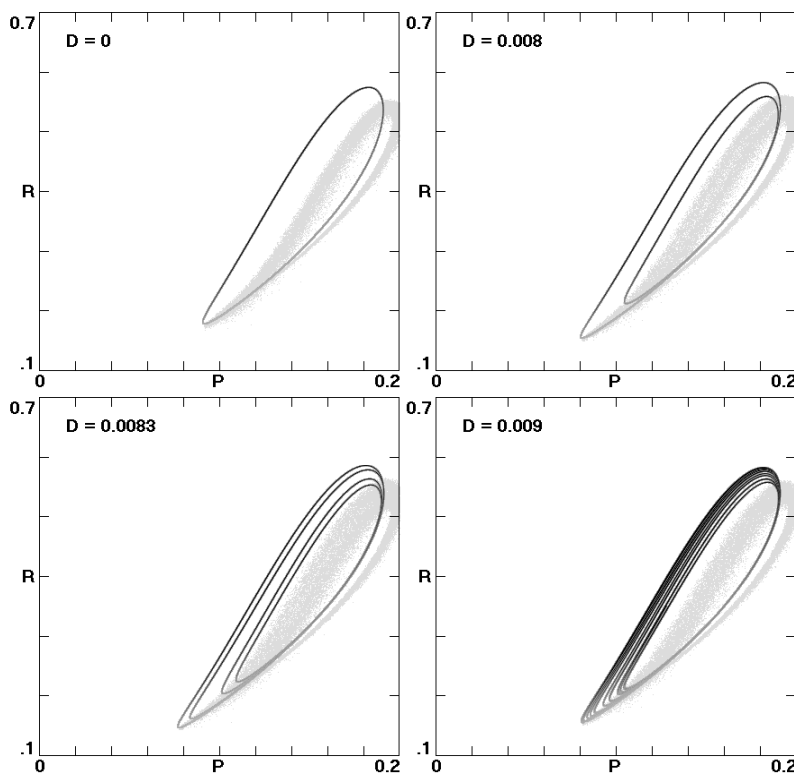


Fig. 5. Attractors showing the progression from a simple limit cycle at $D = 0$ to a period-2 limit cycle at $D = 0.008$ to a period-4 limit cycle at $D = 0.0083$ to chaos at $D = 0.009$.

CONCLUSION

It has been shown here that a simple three-component spatiotemporal model of Easter Island ecology admits periodic and chaotic attractors with complex spatial structure, not previously reported. These solutions exist over a relatively narrow range of parameters, although there may be mechanisms not included in the model such as human learning that would keep the system in a state of weak chaos (sometimes called “the edge of chaos”). These results are possibly relevant to what actually happened on Easter Island, but more importantly, they show a counter-intuitive example of a situation in which

diffusion causes chaos and instability, and they provide a new three-species reaction-diffusion model that is perhaps of more general interest. An obvious next step is to extend the calculation to two spatial dimensions, but such a study would be very computationally intensive and might not shed any additional light on what happened on Easter Island.

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