International Journal of Bifurcation and Chaos, Vol. 23, No. 3 (2013) 1350046 (4 pages)

DOI: 10.1142/S0218127413500466

A RIGOROUS DETERMINATION OF THE OVERALL PERIOD IN THE STRUCTURE OF A CHAOTIC ATTRACTOR

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Received January 28, 2012; Revised July 12, 2012

There are many examples of nonconnected chaotic attractors consisting of several components. The determination of an overall period of such a system is typically done only by a numerical integration of the system. In this letter, we provide a rigorous proof that the exact value of the overall period of a particular 2-D chaotic attractor from an iterated map is two once the attractor has been partitioned and quantized into disconnected sets. As far as we know, there are no examples of a rigorous proof for such a property in the current literature.

Keywords: Rational chaotic map; nonconnectedness; overall period; rigorous proof.

1. Introduction

Firstly, we need the following definitions about *connected sets* and their *overall periods*:

Definition 1.1. A topological space X is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, X is said to be connected. A subset of a topological space is said to be connected if it is connected under its subspace topology.

Definition 1.2. The overall period of a set is equal to the number of connected components composing this set.

This definition of the overall period is a commonly used term by scientists and mathematicians. The origin of the concept is from population biology, where the corresponding strange attractors are composed of several pieces. The disconnected picture describes the reality observed in nature in the distributions, dynamics, and the locations of multiple animals.

Secondly, following this observation, we note that the pictures of many chaotic attractors appear to be a single set. Many others seem to be composed of several distinct pieces. The only confirmation of their geometry is numerical simulation. There are many examples of nonconnected chaotic attractors consisting of several components [Cazelles & Ferriere, 1992; Barahona & Poon, 1996]. An example of a connected chaotic set can be found in [Peitgen *et al.*, 1982]. Some examples of nonconnected attractors from [Zeraoulia & Sprott, 2011] are shown in Fig. 1. As far as we know, there is no mathematical proof of such a property for any examples of such systems.

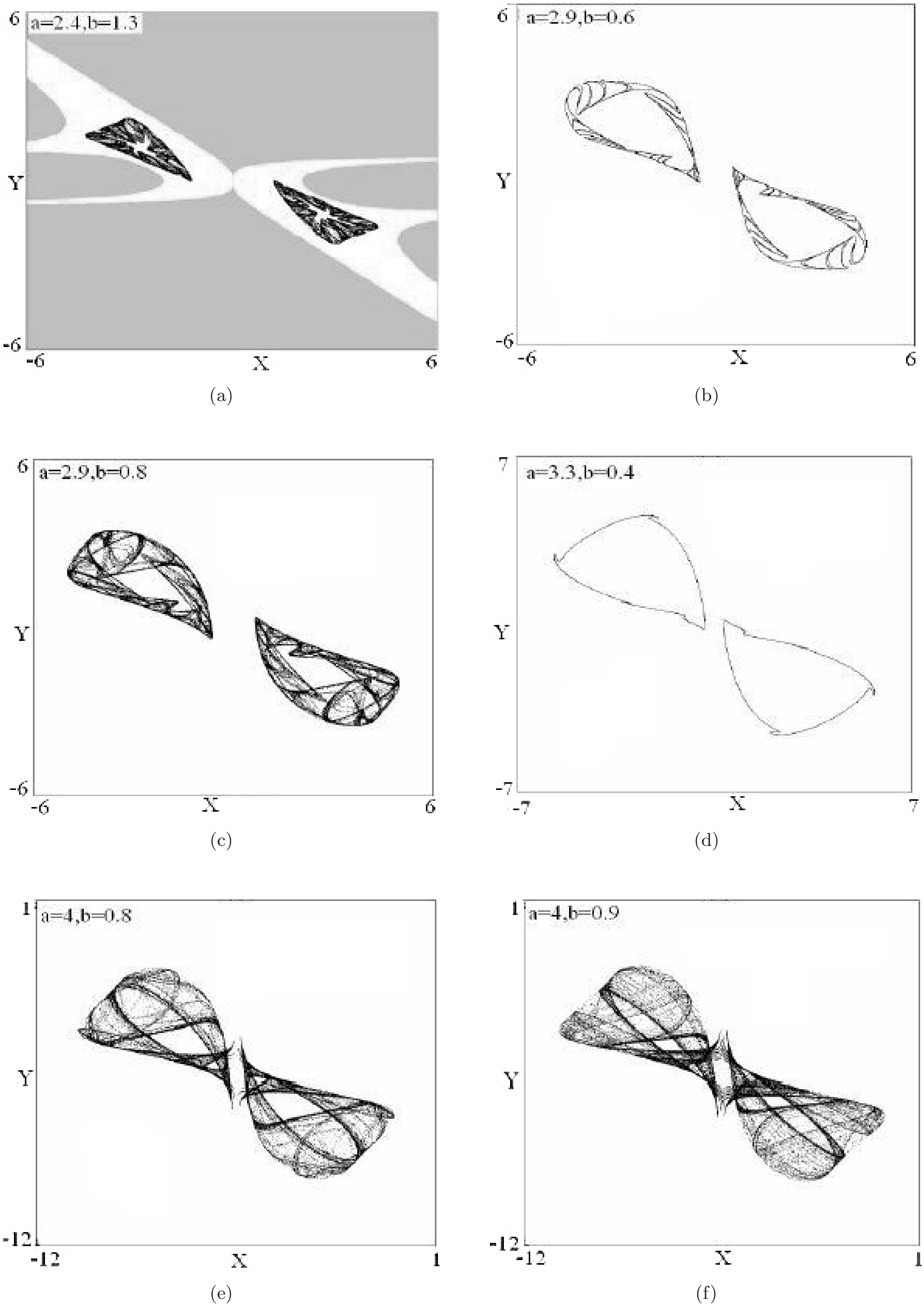

Fig. 1. Attractors of the map (1) with (a) $a = 2.4$, $b = 1.3$, (b) $a = 2.9$, $b = 0.6$, (c) $a = 2.9$, $b = 0.8$, (d) $a = 3.3$, $b = 0.4$, (e) $a = 4$, $b = 0.8$, (f) $a = 4$, $b = 0.9$. From [Zeraoulia & Sprott, 2011].

Population dynamics in ecological systems have several real-world applications. For example, one problem is concerned with whether fishermen can get a fairly uniform catch throughout the year. Typically, these applications include population bursts and population collapse. On the other hand, in ecological models whose chaotic component is coupled with strong periodicity [Cazelles & Ferriere, 1992; Barahona & Poon, 1996], the corresponding strange attractors are split into several disconnected pieces. The iteration and therefore the location of any point follows a precise order before falling back into the initial piece. This scenario creates an overall period in the structure of these attractors, which is equal to the number of pieces. Generally, this pattern occurs in some special discrete dynamical systems [Curry & Yorke, 1978]. The determination of an overall period of a system is done only by a numerical integration of such a system. For example, the system in [Cazelles & Ferriere, 1992; Barahona & Poon, 1996] has a nonconnected chaotic attractor consisting of seven disconnected components. Hence the overall period for this case is seven. In many other cases, the overall periodicity is not evident.

In this letter, we provide a rigorous proof that the exact value of the overall period of a chaotic attractor produced by a particular 2-D discrete iterative system with a rational fraction is two. As far as we know, there is no rigorous proof for such a property in the current literature.

2. A Rigorous Determination of the Overall Period

Rational chaotic systems are relatively rare in theory and practice [Zeraoulia & Sprott, 2011; Lu *et al.*, 2004; Chang *et al.*, 2005]. In [Lu *et al.*, 2004] a 1-D discrete iterative system with a rational fraction of the form $g(x) = \frac{1}{0.1+x^2} - ax$ was discovered in a study of evolutionary algorithms.The dynamics of this map is much more complicated than the dynamics of the logistic system. In [Chang *et al.*, 2005] an extended version of the 1-D map given in [Lu *et al.*, 2004] to 2-D was given as $h(x, y) = (\frac{1}{0.1+x^2} - ay, \frac{1}{0.1+y^2} + bx)$. This map has more complicated dynamical behavior than the previous 1-D map. A relatively new and very simple 2-D map, characterized by the existence of only one rational fraction with no vanishing denominator is constructed in [Zeraoulia & Sprott, 2011] and given by

$$\begin{cases} x_{n+1} = \dfrac{-a}{1+y_n^2} x_n \\ y_{n+1} = x_n + by_n, \end{cases} \tag{1}$$

where $a \geq 0$ and $0 < b < 1$ are bifurcation parameters. Note also that some nonconnected chaotic systems are relevant for the modeling of special ecological systems, whereas map (1) typically has no direct application to particular physical systems, but it serves to exemplify these kinds of dynamical behaviors. Thus an analytical and numerical study is warranted. Map (1) produces several chaotic attractors obtained by the quasi-periodic route to chaos [Zeraoulia & Sprott, 2011]. Some of them along with quasi-periodic orbits are shown in Fig. 1. It is remarkable that all these phase portraits are composed of two separated components. In this case, it seems that the overall period of system (1) is two. This result is based on the pictures shown in Fig. 1. In this case, we will show rigorously that the overall period of the chaotic attractors produced by map (1) is two.

Theorem 1. *The overall period of the chaotic attractors produced by map (1) is two for all* $a > 0$.

Proof. For the case with $a = 0$, we observe that for any starting value $x_0 \in \mathbb{R}$, we have $x_n = 0$ for any integer $n > 0$. Thus, the attractor is limited to the y-axis with a fast (exponential) convergence towards the fixed point $(0, 0)$. Thus the resulting set (fixed point $(0, 0)$) is connected, but the corresponding periodicity in this particular case is one. Hence, we must assume that $a > 0$. In the following proof we make use of inequality constraints to determine the trajectory of iterations of the map among the four quadrants of the x_n–y_n plane. Indeed, from map (1), it is easy to show that $\text{sgn}(x_{n+1}) = -\text{sgn}(x_n)$ since $\frac{-a}{1+y_n^2} < 0$ for all $a > 0$ and $\text{sgn}(y_{n+1} - by_n) = \text{sgn}(x_n)$. Here $\text{sgn}(\cdot)$ is the standard signum function that gives ± 1 according to the sign of its argument. If $x_n \geq 0$, then $y_{n+1} \geq by_n$, and if $x_n \leq 0$, then $y_{n+1} \leq by_n$. Thus, if $x_n \geq 0$ and $y_n \geq 0$, then $y_{n+1} \geq 0$, and if $x_n \leq 0$ and $y_n \leq 0$, then $y_{n+1} \leq 0$. Also, $\text{sgn}(by_{n+1} - y_{n+2}) = \text{sgn}(x_n)$. If $x_n \geq 0$, then $y_{n+2} \leq by_{n+1}$, and if $x_n \leq 0$, then $y_{n+2} \geq by_{n+1}$. Thus, if $x_n \geq 0$ and $y_{n+1} \leq 0$, then $y_{n+2} \leq 0$, and if $x_n \leq 0$ and $y_{n+1} \geq 0$, then $y_{n+2} \geq 0$.

These inequalities determine the positions of the points $(x_n, y_n), (x_{n+1}, y_{n+1})$ and (x_{n+2}, y_{n+2}), and we remark that the values of $(y_n)_n$ are distributed according to the signs of $(x_n)_n$. The next points (x_{n+k}, y_{n+k}) and (x_{n+k+1}, y_{n+k+1}) do not have the same location since $\text{sgn}(y_{n+1+k} - by_{n+k}) = \text{sgn}(x_n)$ if k is even and $\text{sgn}(y_{n+k+2} - by_{n+k+1}) = -\text{sgn}(x_n)$ if k is odd. This means that (x_{n+1}, y_{n+1}) and (x_{n+2}, y_{n+2}) cannot be on the same side of the plane $x_n - y_n$ for all $n \in \mathbb{N}$, and the case where $x_{n+1} = x_n$ implies that $x_n = 0$ for all $a > 0$. This means that any values (x_n, y_n) and (x_m, y_m) in the graph of a chaotic attractor of map (1) are equal if and only if $x_n = x_m = 0$ and $y_n = y_m = 0$. Thus the only common point between the two different pieces is the fixed point $P = (0, 0)$. Thus we have proved that the distribution of all the points (x_n, y_n) in the graph of a chaotic attractor of map (1) are located on two different curves with no intersection except at the fixed point $P = (0, 0)$. Thus any chaotic attractor of map (1) must be composed of two different connected components (curves here), and hence it is not connected. Finally, the overall period of map (1) is exactly two by the above rigorous proof. ■

3. Conclusion

In this paper, we provide a rigorous proof that the exact value of the overall period of a particular chaotic attractor produced by a 2-D discrete iterative system with a rational fraction is two. As far as we know, there is no rigorous proof of such a property in the current literature. This result opens some new directions in studies of the geometry of chaotic attractors such as possible laws for the distribution of points in space occupied by such an attractor.

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