Advances in Systems Science and Application(2012) Vol.12 No.4 44-48

A universal nonlinear control law for the synchronization of arbitrary 3-D continuous-time quadratic systems

Zeraoulia Elhadj[1] and J.C.Sprott[2]
[1]*Department of Mathematics, University of Tébessa, 12002, Algeria.*
[2]*Department of Physics, University of Wisconsin, Madison, WI 53706, USA*

Abstract
In this letter we present a universal nonlinear control law for the synchronization of arbitrary 3-D continuous-time quadratic systems. This control law does not require any type of conditions on the considered systems.
Keywords Synchronization, universal nonlinear control law, chaos. PACS numbers: 05.45.-a, 05.45.Gg

1 Introduction

Several methods have been successfully applied to chaos synchronization. For example, in [1] a method is introduced to synchronize two identical chaotic systems with different initial conditions. An adaptive control approach is presented in [2], a backstepping design was presented in [3], an active control method is presented in [4-6], and a nonlinear control scheme was given in [7-9]. Consequently, there are many applications of chaos synchronization in physical, chemical, and ecological systems, and in secure communications as shown in [1-2,10-13].

In this letter, we apply nonlinear control theory to synchronize two arbitrary 3-D continuous-time quadratic systems. The proposed control law does not need any conditions on the considered systems, and hence it is a universal synchronization approach for general 3-D continuous-time quadratic systems. In other words, the present letter is concerned with synchronization of nonlinear systems in the framework of nonlinear observers. The investigation is restricted to a pair of quadratic three dimensional systems, for which a control feedback can be chosen in such a way that global asymptotic stability of the error system can be established in the framework of classical Lyapunov theory. This restriction is justified by the importance of this type of systems in real applications [14] which is certainly a useful result.

2 Synchronization using a universal nonlinear control law

In this section, we consider two arbitrary 3-D continuous-time quadratic systems. The one with variables x_1, y_1, and z_1 will be controlled to be the new system given by

$$\begin{cases} x_1' = a_0 + a_1x_1 + a_2y_1 + a_3z_1 + f_1(x_1, y_1, z_1) \\ y_1' = b_0 + b_1x_1 + b_2y_1 + b_3z_1 + f_2(x_1, y_1, z_1) \\ z_1' = c_0 + c_1x_1 + c_2y_1 + c_3z_1 + f_3(x_1, y_1, z_1) \end{cases} \tag{1}$$

where

$$\begin{cases} f_1(x_1, y_1, z_1) = a_4x_1^2 + a_5y_1^2 + a_6z_1^2 + a_7x_1y_1 + a_8x_1z_1 + a_9y_1z_1 \\ f_2(x_1, y_1, z_1) = b_4x_1^2 + b_5y_1^2 + b_6z_1^2 + b_7x_1y_1 + b_8x_1z_1 + b_9y_1z_1 \\ f_3(x_1, y_1, z_1) = c_4x_1^2 + c_5y_1^2 + c_6z_1^2 + c_7x_1y_1 + c_8x_1z_1 + c_9y_1z_1 \end{cases} \tag{2}$$

and the one with variables x_2, y_2, and z_2 as the response system

$$\begin{cases} x_2' = d_0 + d_1x_2 + d_2y_2 + d_3z_2 + g_1(x_2, y_2, z_2) + u_1(t) \\ y_2' = r_0 + r_1x_2 + r_2y_2 + r_3z_2 + g_2(x_2, y_2, z_2) + u_2(t) \\ z_2' = s_0 + s_1x_2 + s_2y_2 + s_3z_2 + g_3(x_2, y_2, z_2) + u_3(t) \end{cases} \tag{3}$$

where

$$\begin{cases} g_1(x_2, y_2, z_2) = d_4x_2^2 + d_5y_2^2 + d_6z_2^2 + d_7x_2y_2 + d_8x_2z_2 + d_9y_2z_2 \\ g_2(x_2, y_2, z_2) = r_4x_2^2 + r_5y_2^2 + r_6z_2^2 + r_7x_2y_2 + r_8x_2z_2 + r_9y_2z_2 \\ g_3(x_2, y_2, z_2) = s_4x_2^2 + s_5y_2^2 + s_6z_2^2 + s_7x_2y_2 + s_8x_2z_2 + s_9y_2z_2 \end{cases} \tag{4}$$

Here $(a_i, b_i, c_i)_{0\le i\le 9} \subset \mathbb{R}^{30}$ and $(d_i, r_i, s_i)_{0\le i\le 9} \subset \mathbb{R}^{30}$ are bifurcation parameters, and $u_1(t), u_2(t), u_3(t)$ are the unknown (to be determined) nonlinear controller such that two systems (1) and (3) can be synchronized.

First, let us define the following quantities depending on the above two systems in which we can proceed with our proposed method:

$$\begin{cases} \xi_1 = a_1 + d_1 + a_4(x_1 + x_2) + d_4(x_1 + x_2) + a_7y_1 + a_8z_1 + d_7y_2 + d_8z_2 \\ \xi_2 = a_2 + d_2 + a_5(y_1 + y_2) + d_5(y_1 + y_2) + a_9z_1 + d_9z_2 \\ \xi_3 = a_3 + d_3 + a_6(z_1 + z_2) + d_6(z_1 + z_2) \\ \xi_4 = \eta_1 + \eta_2 + \eta_3 \\ \xi_5 = b_1 + r_1 + b_4(x_1 + x_2) + r_4(x_1 + x_2) + b_7y_1 + b_8z_1 + r_7y_2 + r_8z_2 \end{cases} \tag{5}$$

and

$$\begin{cases} \xi_6 = b_2 + r_2 + b_5(y_1 + y_2) + r_5(y_1 + y_2) + b_9z_1 + r_9z_2 \\ \xi_7 = b_3 + r_3 + b_6(z_1 + z_2) \\ \xi_8 = \eta_4 + \eta_5 + \eta_6 \\ \xi_9 = c_1 + s_1 + c_4(x_1 + x_2) + s_4(x_1 + x_2) + c_7y_1 + c_8z_1 + s_7y_2 + s_8z_2 \\ \xi_{10} = c_2 + s_2 + c_5(y_1 + y_2) + s_5(y_1 + y_2) + c_9z_1 + s_9z_2 \\ \xi_{11} = c_3 + s_3 + c_6(z_1 + z_2) + s_6(z_1 + z_2) \\ \xi_{12} = \eta_7 + \eta_8 + \eta_9 \end{cases} \tag{6}$$

where

$$\begin{cases} \eta_1 = d_4x_1^2 + d_7x_1y_2 + d_8x_1z_2 + d_1x_1 - a_4x_2 - a_7x_2y_1 - a_8x_2z_1 \\ \eta_2 = -a_1x_2 + d_5y_1^2 + d_9y_1z_2 + d_2y_1 - a_5y_2^2 - a_9y_2z_1 - a_2y_2 \\ \eta_3 = d_6z_1^2 + d_3z_1 - a_6z_2^2 - a_3z_2 - a_0 + d_0 \\ \eta_4 = r_4x_1^2 + r_7x_1y_2 + r_8x_1z_2 + r_1x_1 - b_4x_2^2 - b_7x_2y_1 - b_8x_2z_1 \\ \eta_5 = -b_1x_2 + r_5y_2^2 + r_9y_1z_2 + r_2y_1 - b_5y_2^2 - b_9y_2z_1 - b_2y_2 \\ \eta_6 = r_6z_1^2 + r_3z_1 - b_6z_2^2 - b_3z_2 - b_0 + r_0 \\ \eta_7 = s_4x_1^2 + s_7x_1y_2 + s_8x_1z_2 + s_1x_1 - c_4x_2^2 - c_7x_2y_1 - c_8x_2z_1 \\ \eta_8 = -c_1x_2 + s_5y_1^2 + s_9y_1z_2 + s_2y_1 - c_5y_2^2 - c_9y_2z_1 - c_2y_2 \\ \eta_9 = s_6z_1^2 + s_3z_1 - c_6z_2^2 - c_3z_2 - c_0 + s_0 \end{cases} \tag{7}$$

The above quantities comes from the formulation of the problem as the system in (8) below. Now let the error states be $e_1 = x_2 - x_1, e_2 = y_2 - y_1$, and $e_3 = z_2 - z_1$. Then the error system is given by

$$\begin{cases} e_1' = \xi_1e_1 + \xi_2e_2 + \xi_3e_3 + \xi_4 + u_1(t) \\ e_2' = \xi_5e_1 + \xi_6e_2 + \xi_7e_3 + \xi_8 + u_2(t) \\ e_3' = \xi_9e_1 + \xi_{10}e_2 + \xi_{11}e_3 + \xi_{12} + u_3(t) \end{cases} \tag{8}$$

We propose the following universal control law for the system (3):

$$\begin{cases} u_1 = -(\xi_1 + 1)e_1 - (\xi_2 + \xi_5)e_2 - \xi_4 \\ u_2 = -(\xi_6 + 1)e_2 - (\xi_7 + \xi_{10})e_3 - \xi_8 \\ u_3 = -(\xi_3 + \xi_9)e_1 - (\xi_{11} + 1)e_3 - \xi_{12} \end{cases} \tag{9}$$

Then the two 3-D continuous-time quadratic systems (1) and (3) approach synchronization for any initial condition. Indeed, the error system (8) becomes

$$\begin{cases} e_1' = -e_1 - \xi_5e_2 + \xi_3e_3 \\ e_2' = \xi_5e_1 - e_2 - \xi_{10}e_3 \\ e_3' = -\xi_3e_1 + \xi_{10}e_2 - e_3 \end{cases} \tag{10}$$

and if we consider the Lyapunov function $V = \frac{e_1^2+e_2^2+e_3^2}{2}$, then it is easy to verify the asymptotic stability of the error system (10) by Lyapunov stability theory since we have $\frac{dV}{dt} = -e_1^2 - e_2^2 - e_3^2 < 0$ for all $(a_i, b_i, c_i)_{0\leq i\leq 9} \subset R^{30}$, $(d_i, r_i, s_i)_{0\leq i\leq 9} \subset R^{30}$ and for all initial conditions. In particular, if the two systems (1) and (3) are chaotic, then the control law (9) guaranties also their synchronization for any initial condition. A practical example of this situation can be found in [8]. On the other hand, any 3-D continuoustime quadratic chaotic system can be stabilized (controlled) to a stable 3-D continuous-time quadratic system that converges to an equilibrium point (to a 3-D continuous-time quadratic system that converges to a periodic solution). Furthermore, any 3-D continuous-time quadratic system can be chaotified to a chaotic 3-D continuous-time quadratic system.

3 Example

The most known example of 3-D quadratic systems, is the original Lorenz system given by:

$$\left\{\begin{array}{c} x_1' = a_1x_1 - a_1y_1 \\ y_1' = b_1x_1 - y_1 - x_1z_1 \\ z_1' = -c_3z_1 + x_1y_1 \end{array}\right. \tag{11}$$

To apply the above method, we consider the one with variables x2, y2, and z2 as the response system

$$\left\{\begin{array}{c} x_2' = d_1x_2 - d_1y_2 \\ y_2' = r_1x_1 - y_2 - x_2z_2 + u_2(t) \\ z_2' = -s_3z_2 + x_1y_2 + u_3(t) \end{array}\right. \tag{12}$$

Thus, the universal control law for the system (12) is given by:

$$\left\{\begin{array}{c} u_1(t) = -(\xi_1 + 1)e_1 - (\xi_2 + \xi_5)e_2 - \xi_4 \\ u_2(t) = e_2 - \xi_8 \\ u_3(t) = -\xi_9e_1 - (\xi_{11} + 1)e_3 - \xi_{12} \end{array}\right. \tag{13}$$

where

$$\left\{\begin{array}{c} \xi_1 = a_1 + d_1, \xi_2 = -a_1 - d_1, \xi_4 = d_1x_1 - a_1x_2 - d_1y_1 + a_1y_2 \\ \xi_5 = -z_1 - z_2, \xi_6 = -2, \xi_8 = -x_1z_2 + z_1x_2 - y_1 + y_2 \\ \xi_9 = y_1 + y_2, \xi_{11} = -c_3 - s_3, \xi_{12} = x_1y_2 - x_2y_1 - s_3z_1 + c_3z_2 \end{array}\right. \tag{14}$$

In particular, if the two systems (11) and (12) are chaotic, then the control law (13) guaranties their synchronization for any initial condition.

4 Conclusion

We have presented a universal nonlinear control law (without any conditions) for the synchronization of arbitrary 3-D continuous-time quadratic systems. This universal law (9) can be considered either as a stabilization, or a control, or as a chaotification approach for the system under consideration.

References

[1] L.M. Pecora and T.L. Carroll.(1990), *Synchronization in chaotic systems,* Phys. Rev. Lett. 64,pp.821-824.

[2] L. Kocarev and U. Parlitz.(1995), *General approach for chaotic synchronization with application to communication,* Phys. Rev. Lett. 74,pp.5028-5031.

[3] X. Tan, J. Zhang and Y. Yang.(2003), *Synchronizing chaotic systems using backstepping design,* Chaos, Solitons & Fractals 16,pp.37-45.

[4] H. K. Chen, T. N. Lin and J. H. Chen.(2003), *The stability of chaos synchronization of the Japanese attractors and its application,* Jpn. J. Appl. Phys. 42, pp.7603-7610.

[5] M. C. Ho and Y. C. Hung.(2002), *Synchronization two different systems by using generalized active control,* Phys. Lett. A 301,pp.424-428.

[6] M. T.Yassen.(2005), *Chaos synchronization between two different chaotic systems using active control,* Chaos, Solitons & Fractals 23,pp.131-140.

[7] H. K. Chen.(2005), *Global chaos synchronization of new chaotic systems via nonlinear control,* Chaos, Solitons & Fractals, 23, pp.1245-1251.

[8] J. H. Park.(2005), *Chaos synchronization of a chaotic system via nonlinear control,* Chaos, Solitons & Fractals 25,pp.579-584.

[9] L. Huang, R. Feng and M. Wang.(2004), *Synchronization of chaotic systems via nonlinear control,* Phys. Lett. A 320,pp.271-275.

[10] A. Pikovsky, M. Rosenblum and J. Kurths.(2001), *Synchronization, a universal concept in nonlinear science,* Cambridge University Press, Cambridge.

[11] E. Mosekilde, Y. Mastrenko and D. Postnov.(2002), *Chaotic synchronization. Applications for living systems,* World Scienctific, Singapore.

[12] H. K. Chen.(2005), *Synchronization of two different chaotic systems: A new system and each of the dynamical systems Lorenz,* Chen and Lu. Chaos, Solitons & Fractals 25,pp.1049-1056.

[13] J. Lu, X. Wu and J. L.(2002), *Synchronization of a unified chaotic system and the application in secure communication,* Phys. Lett. A 305, pp.365-370.

[14] G. Chen.(1999), *Controlling Chaos and Bifurcations in Engineering Systems,* CRC Press, Boca Raton, FL.

Corresponding author

Zeraoulia Elhadj can be contacted at:zeraoulia@mail.univ-tebessa.dz, and zelhadj12@yahoo.fr.