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Non-existence of Shilnikov chaos in continuous-time systems*

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Abstract In this paper, a non-existence condition for homoclinic and heteroclinic orbits in n-dimensional, continuous-time, and smooth systems is obtained. Based on this result, and using an elementary example, it can be conjectured that there is a fourth kind of chaos in polynomial ODE systems characterized by the non-existence of homoclinic and heteroclinic orbits.

Key words homoclinic chaos, heteroclinic chaos, non-existence of Shilnikov chaos

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1 Introduction

Chaos means that a system will produce very different long-term behaviors when the initial conditions are perturbed only slightly. Chaos is used for some novel, time- or energy-critical interdisciplinary application. Examples include high-performance circuits and devices, liquid mixing, chemical reactions, biological systems, crisis management, secure information processing, and critical decision-making in politics, economics, as well as military applications, etc.

Homoclinic and heteroclinic orbits arise in the study of bifurcations and chaos[1], as well as in their applications to mechanics, biomathematics, and chemistry[2–3]. In some cases, it is necessary to determine the nature or type of chaotic behavior occurring in a dynamical system. One of the commonly used analytic criteria for proving chaos in autonomous systems is the work of Shilnikov[4–5] and its subsequent embellishments and slight extension[6–7]. The resulting chaos is called horseshoe type or Shilnikov chaos. By applying the undetermined coefficient method, homoclinic and heteroclinic orbits are found in some quadratic three dimensional autonomous systems[8–14]. These systems are said to have a Shilnikov type structure of chaos, and it is conjectured in [15] that the two Shilnikov theorems can be used to classify chaos in 3D polynomial ODE systems. For such systems, there only exist three kinds of chaos: homoclinic chaos, heteroclinic chaos, and a combination of homoclinic and heteroclinic chaos.

In this paper, we propose new conditions for obtaining the non-existence of homoclinic and heteroclinic orbits in an autonomous, n-dimensional, continuous-time, smooth system by means of a new criteria of the inequality type. Based on this result, we conjecture that there is a fourth kind of chaos in 3D polynomial ODE systems characterized by the non-existence of homoclinic and heteroclinic orbits.

Let us consider the nth-order autonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \tag{1}$$

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where the vector field $f = (f_1, f_2, \cdots, f_n)^{\mathrm{T}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ belongs to class $C^r (r \geqslant 1)$, and $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ is the state variable of the system, and $t \in \mathbb{R}$ is the time. Suppose that f has at least one equilibrium point P.

Definition 1 *The point $P = (p_1, p_2, \cdots, p_n)$ is called a hyperbolic saddle focus for system* (1) *if the eigenvalues of the Jacobian $A = Df(x)$, evaluated at P, are γ and $\alpha+i\beta$, where $\alpha\gamma < 0$ and $\beta \neq 0$.*

Definition 2 *A homoclinic orbit $\gamma(t)$ refers to a bounded trajectory of system* (1) *that is doubly asymptotic to an equilibrium point P of the system, i.e., $\lim_{t \longrightarrow -\infty} \gamma(t) = \lim_{t \longrightarrow +\infty} \gamma(t) = P$.*

The next definition requires the existence of at least two equilibrium points P_1 and P_2.

Definition 3 *A heteroclinic orbit $\delta(t)$ is similarly defined except that there are two distinct saddle foci P_1 and P_2 connected by the orbit, one corresponding to the forward asymptotic time, and the other to the reverse asymptotic time limit, i.e., $\lim_{t \longrightarrow +\infty} \delta(t) = P_1$ and $\lim_{t \longrightarrow -\infty} \gamma(t) = P_2$.*

The main motivation of the following theorem is to find sufficient conditions for the non-existence of homoclinic and heteroclinic orbits in system (1).

2 Main results

Theorem 1 *Suppose that there exists at least one integer $j \in \{1, 2, \cdots, n\}$ such that the component $f_j(x)$ satisfies $\exists \alpha < 0 : f_j(x) \geqslant \alpha, \forall x \in \mathbb{R}^n$. Then system* (1) *cannot have homoclinic and heteroclinic orbits.*

Proof First, we note that the main assumption of Theorem 1 do not contradict to the assumption that the system (1) has an equilibrium point. Hence, let P be an equilibrium point of system (1), then if there exists a homoclinic orbit $\gamma(t) = (\gamma_1(t), \gamma_2(t), \cdots, \gamma_n(t))^{\mathrm{T}}$, then $\lim_{t \longrightarrow +\infty} \gamma(t) = \lim_{t \longrightarrow -\infty} \gamma(t) = P = (p_1, p_2, \cdots, p_n)$. We have $x_j'(t) = f_j(x) \geqslant \alpha, \forall x \in \mathbb{R}^n$, where $\alpha < 0$. This implies by a simple integration from t_0 to t that

$$x_j(t) \geqslant \alpha(t - t_0) + x_j(t_0), \tag{2}$$

where t_0 is the initial time such that $t \geqslant t_0$, thus, using (2) one has $\gamma_j(t) \geqslant \alpha(t - t_0) + \gamma_j(t_0)$, and $\lim_{t \longrightarrow -\infty} \gamma_j(t) = +\infty \neq p_j$, that is, at least one component of $\gamma(t)$ is not bounded. Now, let P_1, P_2 be saddle foci equilibrium points of system (1), then from inequality (2) one has $\lim_{t \longrightarrow -\infty} \delta_j(t) = +\infty \neq p_j$, thus, at least one component of $\delta(t)$ is not bounded. Therefore, the system (1) has no nomoclinic and heteroclinic orbits.

From Theorem 1, it is important to remark that if system (1) is chaotic, then its chaos is not of the horseshoe type.

3 Example

In this section, we give an elementary example of a 3D polynomial ODE system characterized by the existence of a chaotic attractor without homoclinic and heteroclinic orbits. Indeed, consider the following system:

$$\begin{cases} x' = a(y - x), \\ y' = -ax - byz, \\ z' = -c + y^2, \end{cases} \tag{3}$$

where a, b, and c are positive bifurcation parameters.

The system (3) is chosen as an example satisfying the conditions of Theorem 1, and it is unknown if this system is equivalent to one of the 3D quadratic continuous-time systems with two equilibria, such as the Rössler system. The system (3) does not have a homoclinic orbit on the attractor according to Theorem 1. This result is confirmed numerically by an independent variation of a, b, and c (one at a time). The calculations were performed using a fourth-order Runge-Kutta algorithm with adaptive step size. Then, to determine the long-time behavior, we numerically computed the largest Lyapunov exponent as the usual test for chaos. For $a = 40$, $b = 33$, and $c = 10$, the system (3) has the chaotic attractor shown in Fig. 1, and the Lyapunov exponents for these values are (2.6721, 0, −15.7588). This shows the chaoticity of the attractor. From this elementary example, we conjecture that there is a fourth kind of chaos in 3D polynomial ODE systems characterized by the non-existence of homoclinic and heteroclinic orbits. Additionally, we note that there are values of the parameters where the system (3) exhibits multistability, for example $(a, b, c) = (40, 25, 10)$.

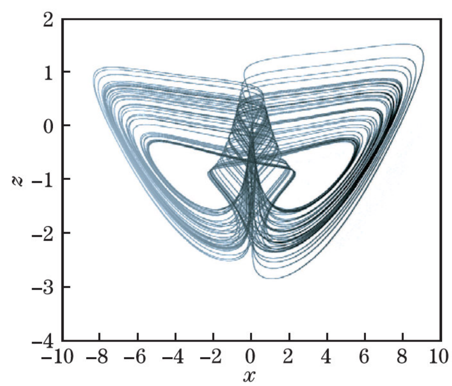

Fig. 1 Projection onto xz-plane of chaotic attractor obtained from system (3) for $a = 40$, $b = 33$, and $c = 10$

4 Conclusions

This work presents a simple criterion for the non-existence of homoclinic and heteroclinic orbits in continuous-time dynamical systems in any dimension. It is conjectured that there is a fourth kind of chaos in 3D polynomial ODE systems characterized by the non-existence of homoclinic and heteroclinic orbits as shown by the elementary example in this paper.

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