On some universal dynamics of a 2-D Hénon-like mapping with an unknown bounded function

Zeraoulia Elhadj[1], J. C. Sprott[2]
[1]Department of Mathematics, University of Tébessa, (12002), Algeria.
E-mail: zeraoulia@mail.univ-tebessa.dz and zelhadj12@yahoo.fr.
[2] Department of Physics, University of Wisconsin, Madison, WI 53706, USA.
E-mail: sprott@physics.wisc.edu.

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Abstract

This paper investigates the dynamics of a 2-D Hénon-like mapping with an unknown bounded function. The values of parameters and the range of initial conditions for which the dynamics of this equation is bounded or unbounded are rigorously derived. The results given here are universal and do not depend on the expression of the nonlinearity in the considered map.

Key words: 2-D Hénon-like mapping with unknown bounded function, bounded orbits, unbounded orbits.

1 Introduction

One of the best known 2-D discrete models is the Hénon map [1] given by

$$H(x,y) = \begin{pmatrix} 1 - ax^2 + by \\ x \end{pmatrix}. \tag{1}$$

There are many works that model the original Hénon map, for example [6-8]. Moreover, it is possible to change the form of the Hénon map to obtain other new chaotic attractors with interesting properties [2-3-4-5-7-9]. Applications include secure communications using the notions of chaos [11-12].

A 2-D Hénon-like mapping with an unknown bounded function has the following form:

$$g(x,y) = \begin{pmatrix} 1 - af(x) + by \\ x \end{pmatrix}, \tag{2}$$

where a and b are the control parameters and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an unknown nonlinear bounded function, not necessarily continuous. The map (2) can be rewritten as a second order nonlinear difference equation given by

$$x_{n+1} = 1 - af(x_n) + bx_{n-1}, \tag{3}$$

in which, together with some specified values of initial conditions, defines a sequence $(x_n)_n$. Difference equations have a variety of applications as in computer science and approximations in numerical analysis [13-16-17]. The asymptotic behavior of solutions of a difference equation (3) generally depends on both the parameter values and the initial conditions. We are particularly interested in the asymptotic behavior of solutions, that is $n \longrightarrow +\infty$. Analyzing equation (3) can be quite difficult. Since f is a nonlinear function, it may be impossible to solve equation (3) in any simple closed form. This means that either one is limited to analyzing it using numerical simulations and generalizing from the limited number of cases that can be done [13-14-17], or else using the standard analysis methods for dynamical systems to produce a general picture of what happens in the system.

2 Asymptotic behaviors

The essential motivation of this work is to derive rigorously universal regions for the control parameters a and b and the initial conditions x_0 and x_1 (*i.e.*, y_0 for the map (2)) for which the dynamics of the new map (2) is bounded or unbounded using its equivalent transformation to the difference equation (3). Note that the results obtained in this paper speak only to the existence of bounded and unbounded orbits. To obtain information about the occurrence of other behaviors such as periodic orbits and chaos, one has to consider the map (2) with a specific function, which cannot be obtained in the general form of (2) or (3).

In all proofs given here, we use the following standard results:

Theorem 1 *Let* $(x_n)_n$ *and* $(z_n)_n$ *be two real sequences. If* $|x_n| \leq |z_n|$ *and* $\lim_{n \longrightarrow +\infty} |z_n| = A < +\infty$*, then* $\lim_{n \longrightarrow +\infty} |x_n| \leq A$*, or if* $|z_n| \leq |x_n|$ *and* $\lim_{n \longrightarrow +\infty} |z_n| = +\infty$*, then* $\lim_{n \longrightarrow +\infty} |x_n| = +\infty$*.*

Proof. The proof is available in the standard mathematics books and will not be given here. ■

We use this result to construct a sequence $(z_n)_n$ that satisfies the above conditions for determining whether the difference equation (3) has bounded or unbounded orbits.

Theorem 2 *Suppose that* f *is a bounded function over its definition set such that* $\sup_x |f(x)| = \delta$*. Then for every* $n > 1$ *and all values of* a *and* b*, the sequence* $(x_n)_n$ *given in (3) satisfies the following inequality:*

$$|1 - x_n + b x_{n-2}| \leq |a| \delta. \tag{4}$$

Proof. We have for every $n > 1$ that $x_n = 1 - a f(x_{n-1}) + b x_{n-2}$. Then one has that

$$|-x_n + 1 + b x_{n-2}| = |a f(x_{n-1})| \leq |a| \delta \tag{5}$$

since $\sup_x |f(x)| = \delta$. ■

Theorem 3 *For every* $n > 1$*, and all values of* a *and* b*, and for all values of the initial conditions* $(x_0,\ x_1) \in \mathbb{R}^2$*, the sequence* $(x_n)_n$ *satisfies the following equalities:*

First, if $b \neq 1$*, then*

$$x_n = \begin{cases} \frac{b^{\frac{n-1}{2}} - 1}{b-1} + b^{\frac{n-1}{2}} x_1 - a \sum\limits_{p=1}^{p=\frac{n-1}{2}} b^{p-1} f(x_{n-(2p-1)}), \text{ if } n \text{ is odd} \\ \frac{b^{\frac{n}{2}} - 1}{b-1} + b^{\frac{n}{2}} x_0 - a \sum\limits_{p=1}^{p=\frac{n}{2}} b^{p-1} f(x_{n-(2p-1)}), \text{ if } n \text{ is even.} \end{cases} \tag{6}$$

Second, if $b = 1$*, then*

$$x_n = \begin{cases} \frac{n-1}{2} + x_1 - a \sum\limits_{p=1}^{p=\frac{n-1}{2}} f(x_{n-(2p-1)}), \text{ if } n \text{ is odd} \\ \frac{n}{2} + x_0 - a \sum\limits_{p=1}^{p=\frac{n}{2}} f(x_{n-(2p-1)}), \text{ if } n \text{ is even.} \end{cases} \tag{7}$$

Proof. We have for every $n > 1$, the following equalities:

$$x_n = 1 - af(x_{n-1}) + bx_{n-2}, \tag{8}$$

$$x_{n-2} = 1 - af(x_{n-3}) + bx_{n-4}, \tag{9}$$

$$x_{n-4} = 1 - af(x_{n-5}) + bx_{n-6}... \tag{10}$$

Then the results in (6) and (7) are obtained by successive substitutions of, for example, (9) and (10)... into (8)..., for all $k = n-2, n-4, ..., 2$. ■

Theorem 4 *The fixed points* (l, l) *of the map (2) exist if one of the following conditions holds:*

(i) If $a \neq 0$ *and* $b \neq 1$*, then* l *satisfyies the following conditions:*

$$\begin{cases} 1 - af(l) + (b-1)\, l = 0 \text{ and } l \leq \frac{1+|a|\delta}{1-b}, \text{ if } b > 1 \\ \frac{1+|a|\delta}{1-b} \leq l, \text{ if } b < 1. \end{cases} \tag{11}$$

(ii) If $b = 1$*, and* $a \neq 0$*, then* l *is given by* $f(l) = \frac{1}{a}$.

(iii) If $b \neq 1$ *and* $a = 0$*, then* l *is given by* $l = \frac{1}{1-b}$.

(iv) If $a = 0$ *and* $b = 1$*, then there are no fixed points for the map (2).*

Proof. The proof is direct except for the case (i) where we use Theorem 2, and therefore, one concludes that all fixed points of the map (2) are confined to the interval $\left]-\infty, \frac{1+|a|\delta}{1-b}\right]$ if $b > 1$, and to $\left[\frac{1+|a|\delta}{1-b}, +\infty\right[$ if $b < 1$. On the other hand, case (iii) gives a simple linear second-order difference equation $x_n = 1 + bx_{n-2}$. This situation is very standard. ■

Next, we state the main results of the paper as follows:

2.1 Existence of bounded orbits

In this subsection, we determine sufficient conditions for which the map (2) has bounded solutions. This case is very interesting since almost periodic, quasi-periodic, and chaotic orbits are bounded. Hence we prove the following theorem:

Theorem 5 *Consider the map (2), and assume that* f *is a bounded function. Then for* $|b| < 1$ *and all* $a \in \mathbb{R}$ *and all initial conditions* $(x_0, x_1) \in \mathbb{R}^2$*, the orbits of the map (2) are bounded.*

Proof. From equation (2) and the fact that f is a bounded function, one has the followings inequalities for all $n > 1$:

$$|x_n| \leq 1 + |a|\,\delta + |bx_{n-2}|\,, \tag{12}$$

$$|x_{n-2}| \leq 1 + |a|\,\delta + |bx_{n-4}|\,, \tag{13}$$

$$...\,|x_2| \leq 1 + |a|\,\delta + |bx_0|\,. \tag{14}$$

This implies from (12), (13), (14), ... that

$$|x_n| \leq 1 + |a|\,\delta + |bx_{n-2}|\,, \tag{15}$$

$$|x_n| \leq (1 + |a|)\,\delta + |b|\,(1 + |a|\,\delta + |bx_{n-4}|)\,, \tag{16}$$

$$|x_n| \leq (1 + |a|\,\delta) + (1 + |a|\,\delta)\,|b| + |b|^2\,|x_{n-4}|\,, ... \tag{17}$$

Hence from (13) and (17) one has

$$|x_n| \leq (1 + |a|\,\delta) + (1 + |a|\,\delta)\,|b| + |b|^2\,(1 + |a|\,\delta) + |b|^3\,|x_{n-6}|\,, ... \tag{18}$$

Since $|b| < 1$, then the use of (18) and the induction about some integer k and the use of the sum of a geometric growth formula leads to the following inequality for every $n > 1$:

$$|x_n| \leq (1 + |a|\,\delta)\left(\frac{1 - |b|^k}{1 - |b|}\right) + |b|^k\,|x_{n-2k}|\,, \tag{19}$$

where k is the largest integer j such that $j \leq \frac{n}{2}$. Thus one has the following two cases:

(1) If n is odd, *i.e.*, $\exists m \in \mathbb{N}$ such that $n = 2m+1$, then the largest integer $k \leq \frac{n}{2}$ is $k = \frac{n-1}{2}$, for which $(x_n)_n$ satisfies the following inequality:

$$|x_{2m+1}| \leq (1 + |a|\,\delta)\left(\frac{1 - |b|^m}{1 - |b|}\right) + |b|^m\,|x_1| = z_m. \tag{20}$$

(2) If n is even, *i.e.*, $\exists m \in \mathbb{N}$ such that $n = 2m$, then the largest integer $k \leq \frac{n}{2}$ is $k = \frac{n}{2}$, for which x_n satisfies the following inequality:

$$|x_{2m}| \leq (1 + |a|\,\delta)\left(\frac{1 - |b|^m}{1 - |b|}\right) + |b|^m\,|x_0| = u_m. \tag{21}$$

Thus, since $|b| < 1$, then the sequences $(z_m)_m$ and $(u_m)_m$ are bounded, *i.e.*,

$$\begin{cases} z_m \leq \frac{(1+|a|\delta)}{1-|b|} + \left| |x_1| - \frac{(1+|a|\delta)}{1-|b|} \right|, \text{for all } m \in \mathbb{N} \\ u_m \leq \frac{(1+|a|\delta)}{1-|b|} + \left| |x_0| - \frac{(1+|a|\delta)}{1-|b|} \right|, \text{for all } m \in \mathbb{N}. \end{cases} \tag{22}$$

Thus equations (20) and (21) and inequalities (22) give the following bounds for the sequence $(x_n)_n$:

$$|x_n| \leq \max\left(\frac{(1+|a|\,\delta)}{1-|b|} + \left| |x_0| - \frac{(1+|a|\,\delta)}{1-|b|} \right|, \frac{(1+|a|\,\delta)}{1-|b|} + \left| |x_1| - \frac{(1+|a|\,\delta)}{1-|b|} \right| \right). \tag{23}$$

Finally, for all values of a and for values of b satisfying $|b| < 1$ and for all values of the initial conditions $(x_0, x_1) \in \mathbb{R}^2$, one has that all orbits of the map (2) are bounded, *i.e.*, in the subregion of $\mathbb{R}^4$:

$$\Omega_1 = \left\{ (a, b, x_0, x_1) \in \mathbb{R}^4 / \ |b| < 1 \right\}. \tag{24}$$

Hence the proof is completed. ■

2.2 Existence of unbounded orbits

In this subsection, we determine sufficient conditions for which the orbits of the map (2) are unbounded. Hence we prove the following theorem:

Theorem 6 *Consider the map (2), and assume that f is a bounded function. Then the map (2) possesses unbounded orbits in the following subregions of $\mathbb{R}^4$:*

$$\Omega_2 = \left\{ (a, b, x_0, x_1) \in \mathbb{R}^4 / \ |b| > 1, \ \text{and both } |x_0|, |x_1| > \frac{|a|\,\delta + 1}{|b| - 1} \right\} \tag{25}$$

and

$$\Omega_3 = \left\{ (a, b, x_0, x_1) \in \mathbb{R}^4 / \ |b| = 1, \ \text{and } |a| < \frac{1}{\delta} \right\}. \tag{26}$$

Proof. (a) For every $n > 1$, we have $x_n = 1 - af(x_{n-1}) + bx_{n-2}$. Then, $|bx_{n-2} - af(x_{n-1})| = |x_n - 1|$ and $||bx_{n-2}| - |af(x_{n-1})|| \leq |x_n - 1|$. (We use the inequality $|x| - |y| \leq ||x| - |y|| \leq |x - y|$.) This implies that

$$|bx_{n-2}| - |af(x_{n-1})| \leq |x_n| + 1. \tag{27}$$

Since $|f(x_{n-1})| \leq |\delta|$, this implies that $-|af(x_{n-1})| \geq -|a|\,\delta$ and $|bx_{n-2}| - |af(x_{n-1})| \geq |bx_{n-2}| - |a|\,\delta$. Finally, one has from (27) that

$$|bx_{n-2}| - (|a|\,\delta + 1) \leq |x_n|\,. \tag{28}$$

Then by induction as in the previous section, one has

$$|x_n| \geq \begin{cases} \left(\frac{-(|a|\delta+1)}{|b|-1} + |x_1|\right)|b|^{\frac{n-1}{2}} + \frac{|a|\delta+1}{|b|-1}, \text{ if } n \text{ is odd,} \\ \\ \left(\frac{-(|a|\delta+1)}{|b|-1} + |x_0|\right)|b|^{\frac{n}{2}} + \frac{|a|\delta+1}{|b|-1}, \text{ if } n \text{ is even.} \end{cases} \tag{29}$$

Thus, if $|b| > 1$ and both $|x_0|$, $|x_1| > \frac{|a|\delta+1}{|b|-1}$), then one has $\lim_{n\longrightarrow+\infty} |x_n| = +\infty$.

(b) For $b = 1$, one has

$$|x_n| \geq \begin{cases} (1 - |a|\,\delta)\left(\frac{n-1}{2}\right) + x_1, \text{ if } n \text{ is odd,} \\ \\ (1 - |a|\,\delta)\left(\frac{n}{2}\right) + x_0, \text{ if } n \text{ is even.} \end{cases} \tag{30}$$

Hence, if $|a| < \frac{1}{\delta}$, then one has $\lim_{n\longrightarrow+\infty} x_n = +\infty$.

For $b = -1$, one has from Theorem 3 the following inequalities:

$$x_n \leq \begin{cases} -\left(\frac{n-1}{2}\right) + x_1 + \left|\sum_{p=1}^{p=\frac{n-1}{2}} a\ \ (-1)^{p-1} f(x_{n-(2p-1)})\right|, \text{ if } n \text{ is odd,} \\ \\ -\left(\frac{n}{2}\right) + x_0 + \left|\sum_{p=1}^{p=\frac{n}{2}} a\ \ (-1)^{p-1} f(x_{n-(2p-1)})\right|, \text{ if } n \text{ is even.} \end{cases} \tag{31}$$

Since $\left|a\ \ (-1)^{p-1} f(x_{n-(2p-1)})\right| \leq |a|\,\delta$, then one has the following:

$$x_n \leq \begin{cases} (|a|\,\delta - 1)\left(\frac{n-1}{2}\right) + x_1, \text{ if } n \text{ is odd,} \\ \\ (|a|\,\delta - 1)\left(\frac{n}{2}\right) + x_0, \text{ if } n \text{ is even.} \end{cases} \tag{32}$$

Thus, if $|a| < \frac{1}{\delta}$, then one has $\lim_{n \longrightarrow +\infty} x_n = -\infty$.

Note that there is no similar proof for the following subregions of $\mathbb{R}^4$:

$$\Omega_4 = \left\{ (a, b, x_0, x_1) \in \mathbb{R}^4 \ / \ |b| > 1, \text{and both } |x_0|, |x_1| \leq \frac{|a| \delta + 1}{|b| - 1} \right\}, \quad (33)$$

$$\Omega_5 = \left\{ (a, b, x_0, x_1) \in \mathbb{R}^4 \ / \ |b| = 1, \text{and } |a| \geq \frac{1}{\delta} \right\}. \quad (34)$$

Hence the proof is completed. ■

It can be seen from the above results that a Hénon-like map of the form (2) with any bounded function may exhibit with respect to the parameter b the following dynamics:

(i) If $|b| < 1$, then the map (2) is bounded (see Theorem 5).

(ii) If $|b| \geq 1$, then the map (2) is unbounded (see Theorem 6).

3 Some examples

3.1 $f(x) = \sin(x)$

In this subsection, we give an elementary example of the above situation, where we choose the function $f(x) = \sin(x)$. Hence we show numerically that the map (2) with a sine function is capable of generating *multi-fold* strange attractors as shown in Fig. 2 [2] obtained by a period-doubling route to chaos as shown in Fig. 1. In this case, we have $\delta = 1$, and if we set $a = 4$ and $|b| < 1$, then one can see that the orbits of the map (2) with a sine function are all bounded as shown in Fig. 1, *i.e.*, there is a bounded stable fixed point, as well as periodic and chaotic orbits. New in this example is that we obtain *multi-fold* strange attractors by a C^{∞}-modification of the Hénon map as shown in [18]. This effect in some ways simplifies the study of such maps and avoids some problems related to the lack of continuity in the derivative of the map as in [2]. As a suggested application, the picture of a chaotic model is structured around a constituted *skeleton* of unstable periodic orbits that are together dense in the attractor, as well as the passing in-transit orbits between them, which form the chaotic attractor. The ideas for using chaos in the security of multi-user communications [11] are often based on the control and use of the unstable periodic orbits, the main idea being that they act as the skeleton of the chaotic attractor and provide a reservoir of secure communication channels. In this way, a number of users

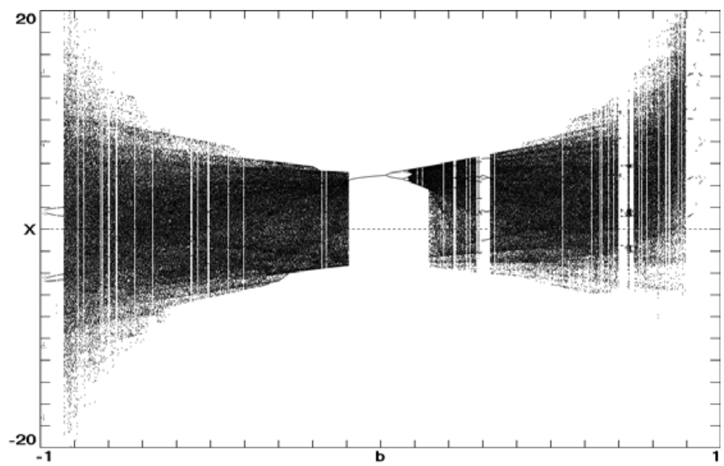


Figure 1: Bifurcation diagram of the map (2) with a sine function obtained for $-1 < b < 1$ and $a = 4$.

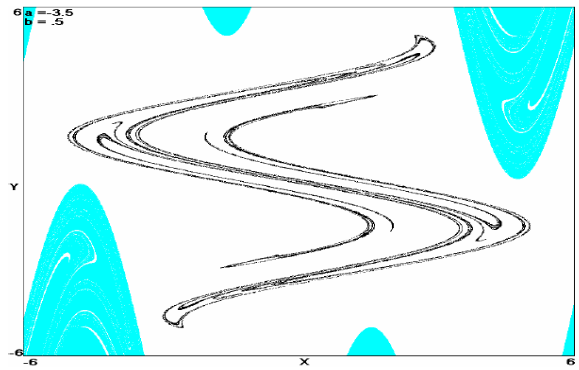
can each be provided a clean code in the same channel. Therefore, the interest in *multi-fold* attractors resides in the possibility that they permit one to generate shorter orbits and thus a faster transmission of the messages, as well as better security in the communications. Furthermore, the same phenomena are observed for the function $f(x) = \cos(x)$.

3.2 $f(x) = \mathrm{sgn}(x)$

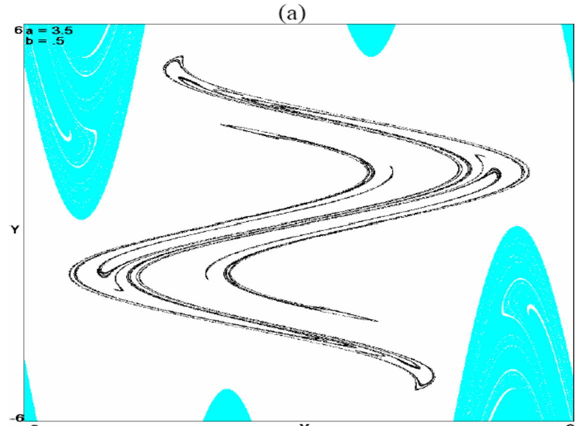
In this subsection, we choose the function $f(x) = \mathrm{sgn}(x)$, where sgn(.) is the standard signum function that gives ± 1 depending on the sign of its argument. Hence we show numerically that the map (2) with the signum function converges to a period-2 orbit for all $|b| < 1$ and $a = 4$ as shown in Fig. 3.

4 Conclusion

We have reported some universal results relevant to the dynamics of a 2-D Hénon-like mapping with an unknown bounded function. Sufficient conditions for which this map is bounded or unbounded are rigorously derived. Elementary examples are also given and discussed.



(a)



(b)

Figure 2: Different multi-fold chaotic attractors (with their bassins of attraction in white) obtained from map (2) with a sine function, observed for the initial condition $(x_0, y_0) = (0.01, 0.01)$ and (a) $a = -3.5$ $b = 0.5$. (b) $a = 3.5, b = 0.5$.

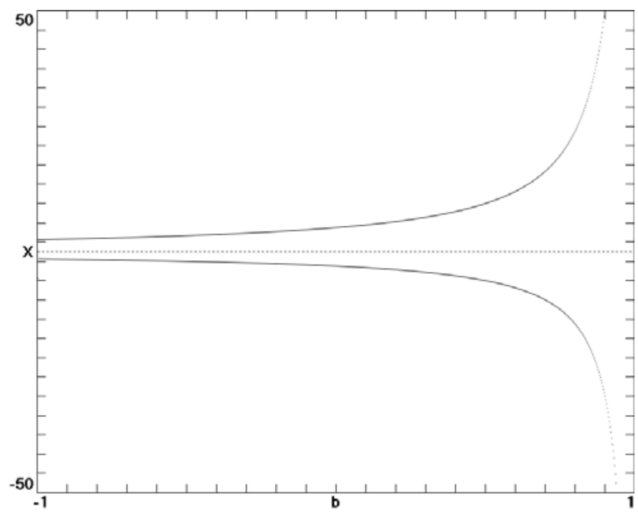


Figure 3: Bifurcation diagram of the map (2) with a signum function obtained for $-1 < b < 1$ and $a = 4$.

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