Boundedness of the Lorenz–Stenflo system

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April 19, 2010

Abstract

In this letter, we find upper and lower bounds for the Lorenz–Stenflo system. In particular, we find large regions in the bifurcation parameter space where this system is bounded.

Keywords: Lorenz–Stenflo system, lower and upper bounds
PACS numbers: 11.10.St, 05.45.-a, 05.45.Gg.

1 Introduction

Bounded chaotic attractors and the estimate of their bounds is important in chaos control, chaos synchronization, and their applications [Chen, 1999]. Such an estimation is quite difficult to achieve technically, however. Several works on this topic were realized for some 3-D quadratic continuous-time systems [Leonov *et al.*, 1987; Pogromsky *et al.*, 2003; Li *et al.*, 2005; Zeraoulia & Sprott, 2010; and references therein]. In this letter, we find upper and lower bounds for the Lorenz–Stenflo system [Stenflo, 1996] given by

$$\left\{\begin{array}{c} x' = -\sigma\left(x-y\right)+sw \\ y' = -xz+rx-y \\ z' = xy-bz \\ w' = -x-\sigma w \end{array}\right. \tag{1}$$

These bounds are obtained based on multivariable function analysis concerned with locating maxima and minima. In particular, we find large regions in the bifurcation parameter space $(\sigma, r, b, s) \in \mathbb{R}^4$ where system (1) is bounded. The Lorenz–Stenflo system (1) describes finite-amplitude, low-frequency, short-wavelength, acoustic gravity waves in a rotational system [Stenflo, 1996]. Several results about the dynamics of system (1) have been reported in [Yu & Yang, 1996; Yu *et al.*, 1996; Zhou *et al.*, 1997; Yu, 1999; Banerjee *et al.*, 2001]. In a recent paper [Xavier & Rech, 2010], the precise locations for pitchfork and Hopf bifurcations of fixed points were determined along with a numerical characterization of periodic and chaotic attractors.

2 Estimating the bounds for the Lorenz–Stenflo system

To estimate the bound for the Lorenz–Stenflo system (1), we consider the Lyapunov function $V(x,y,z,w)$ defined by $V(x,y,z,w) = \frac{\frac{1}{s}x^2+y^2+\left(z-\left(r+\frac{\sigma}{s}\right)\right)^2+w^2}{2}$. The derivative of V along the solutions of (1) is given by $\frac{dV}{dt} = -\frac{\sigma}{s}x^2 - y^2 - b\left(z-\frac{\sigma+rs}{2s}\right)^2 - \sigma w^2 + \frac{b\left(r+\frac{\sigma}{s}\right)^2}{4}$. Let $H(x,y,z,w) = \frac{x^2}{\frac{b(\sigma+rs)^2}{4\sigma s}} + \frac{y^2}{\frac{b\left(r+\frac{\sigma}{s}\right)^2}{4}} + \frac{\left(z-\frac{\sigma+rs}{2s}\right)^2}{\frac{(\sigma+rs)^2}{4s^2}} + \frac{w^2}{\frac{b(\sigma+rs)^2}{4\sigma s^2}} - 1$. Thus to prove the boundedness of system (1), we assume that it is bounded, and then we find its bound, i.e., assume that σ, s, and b are strictly positive and $r \geq 0$. Then if system (1) is bounded, the function $\frac{dV}{dt}(x,y,z,w)$ has a maximum value, and the maximum point (x_0,y_0,z_0,w_0) satisfies $H(x_0,y_0,z_0,w_0) = 0$. Now consider the 4-D ellipsoid defined by $\Gamma = \{(x,y,z,w) \in \mathbb{R}^4 : H(x,y,z,w) = 0, \sigma > 0, s > 0, b > 0, r \geq 0\}$, and define the function $F(x,y,z,w) = G(x,y,z,w) + \lambda H(x,y,z,w)$, where $G(x,y,z,w) = x^2+y^2+z^2+w^2$ and $\lambda \in \mathbb{R}$ is a finite parameter. We have $\max_{(x,y,z,w)\in\Gamma} G = \max_{(x,y,z,w)\in\Gamma} V$ and $\frac{\partial F}{\partial x} = \frac{2\left(br^2s^2+2brs\sigma+4\lambda s\sigma+b\sigma^2\right)}{b(\sigma+rs)^2}x$, $\frac{\partial F}{\partial y} = \frac{2\left(br^2s^2+2brs\sigma+4\lambda s^2+b\sigma^2\right)}{b(\sigma+rs)^2}y$, $\frac{\partial F}{\partial z} = \frac{2\left(r^2s^2+2rs\sigma+4\lambda s^2+\sigma^2\right)}{(\sigma+rs)^2}z - \frac{2\left(2r\lambda s^2+2\sigma\lambda s\right)}{(\sigma+rs)^2}$, and $\frac{\partial F}{\partial w} = \frac{2\left(br^2s^2+2brs\sigma+4\lambda s^2\sigma+b\sigma^2\right)}{b(\sigma+rs)^2}w$. In this case, the Hessian matrix of the function F is diagonal with the elements (eigenvalues) $\frac{2\left(br^2s^2+2brs\sigma+4\lambda s\sigma+b\sigma^2\right)}{b(\sigma+rs)^2}$, $\frac{2\left(br^2s^2+2brs\sigma+4\lambda s^2+b\sigma^2\right)}{b(\sigma+rs)^2}$, $\frac{2\left(r^2s^2+2rs\sigma+4\lambda s^2+\sigma^2\right)}{(\sigma+rs)^2}$, and $\frac{2\left(br^2s^2+2brs\sigma+4\lambda s^2\sigma+b\sigma^2\right)}{b(\sigma+rs)^2}$. Thus the scalar function F has a maximum point if all eigenvalues of the corresponding Hessian matrix are strictly negative, that is, $\lambda < \min\left(\frac{-b(\sigma+rs)^2}{4s\sigma}, \frac{-b(\sigma+rs)^2}{4s^2}, \frac{-(\sigma+rs)^2}{4s^2}, \frac{-b(\sigma+rs)^2}{4s^2\sigma}\right)$. If $s \geq 1$, $0 < \sigma \leq s$, and $0 < \sigma < bs$, we have $\frac{-b(\sigma+rs)^2}{4s\sigma} - \left(\frac{-b(\sigma+rs)^2}{4s^2}\right) = \frac{-b(s-\sigma)(\sigma+rs)^2}{4s^2\sigma} \leq 0$, $\frac{-b(\sigma+rs)^2}{4s\sigma} - \left(\frac{-(\sigma+rs)^2}{4s^2}\right) = \frac{-(\sigma+rs)^2(-\sigma+bs)}{4s^2\sigma} \leq 0$, and $\frac{-b(\sigma+rs)^2}{4s\sigma} - \left(\frac{-b(\sigma+rs)^2}{4s^2\sigma}\right) = \frac{-b(s-1)(\sigma+rs)^2}{4s^2\sigma} \leq 0$. Thus $\lambda < \frac{-b(\sigma+rs)^2}{4s\sigma}$. Then the only critical point of F is $x_0 = 0, y_0 = 0, z_0 = \frac{2s(\sigma+rs)\lambda}{\sigma^2+4s^2\lambda+r^2s^2+2rs\sigma}$, and $w_0 = 0$, and hence $\max_{(x,y,z,w)\in\Gamma} G = \left(\frac{2s\sigma\lambda+2rs^2\lambda}{\sigma^2+4s^2\lambda+r^2s^2+2rs\sigma}\right)^2 = f(\lambda)$. In this case, there exists a parameterized family (in λ) of bounds of system (1). We remark that for different values of λ, one can get different estimates for system (1). Some calculations lead to $f'(\lambda) = \frac{(\sigma+rs)^4 8s^2\lambda}{(r^2s^2+2rs\sigma+4\lambda s^2+\sigma^2)^3}$. We have $r^2s^2+2rs\sigma+4\lambda s^2+\sigma^2 < 0$ for all $-\infty < \lambda < \frac{-b(\sigma+rs)^2}{4s\sigma}$, and hence $f'(\lambda) > 0$, which means that $f(\lambda)$ is an increasing function, that is, $\lim_{\lambda\to-\infty}\left(\frac{2s\sigma\lambda+2rs^2\lambda}{\sigma^2+4s^2\lambda+r^2s^2+2rs\sigma}\right)^2 = Q^2 = \frac{\left(2rs^2+2\sigma s\right)^2}{16s^4} < f(\lambda) < \frac{1}{4}b^2\frac{(\sigma+rs)^2}{(-\sigma+bs)^2} = R^2$. Finally, we have $\max_{(x,y,z,w)\in\Gamma}(x^2+y^2+z^2+w^2) < \frac{b^2(\sigma+rs)^2}{4(-\sigma+bs)^2} = R^2$, which is the upper bound for the Lorenz–Stenflo system (1). For the other values of $(\sigma, r, b, s) \in \mathbb{R}^4$, the same logic applies.

Finally, we have proved the following result:

Theorem 1 *The Lorenz–Stenflo system (1) is contained in part of the 4-D ellipsoid defined by* $\Omega = \left\{(x,y,z,w)\in\mathbb{R}^4 : Q^2 < \frac{\frac{1}{s}x^2+y^2+\left(z-\left(r+\frac{\sigma}{s}\right)\right)^2+w^2}{2} \leq R^2\right\}$ *for all* $r \geq 0, b > 0$, $s \geq 1, 0 < \sigma \leq s, \sigma < bs$, *and all initial conditions, where* $Q^2 = \frac{\left(2rs^2+2\sigma s\right)^2}{16s^4}$ *and* $R^2 = \frac{b^2(\sigma+rs)^2}{4(bs-\sigma)^2}$.

We remark that if $\sigma \to bs$, then the upper bound converges to infinity. The volume of the resulting set in $\mathbb{R}^4$ is $\frac{1}{4}b^2\frac{(\sigma+rs)^2}{(-\sigma+bs)^2} - \left(\frac{\left(2rs^2+2\sigma s\right)^2}{16s^4}\right) = \frac{\sigma(\sigma+rs)^2(2bs-\sigma)}{4s^2(bs-\sigma)^2} > 0$ since $bs > \sigma$.

3 Conclusion

Using multivariable function analysis, we find upper and lower bounds for the Lorenz–Stenflo system. In particular, we find large regions in the bifurcation parameter space where this system is bounded.

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