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A simple Jerk system with piecewise exponential nonlinearity

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Abstract

Third-order explicit autonomous differential equations in one scalar variable, sometimes called jerky dynamics, constitute an interesting subclass of dynamical systems that can exhibit chaotic behavior. In this paper, we investigated a simple jerk system with a piecewise exponential nonlinearity by numerical examination as well as dynamic simulation. Using the largest Lyapunov exponent as the signature of chaos, the region of parameter space exhibiting chaos is identified. The results show that this system has a period-doubling route to chaos and a narrow chaotic region in parameter space. The rescaled system is approximately described by a one-dimensional quadratic map. The parameters are fitted to a simple function to predict the values for which chaos occurs in the case of high nonlinearity where the region in parameter space that admits chaos is relatively small.

Keywords: Chaos; Jerk functions; Differential equations

1. Introduction

Since the Lorenz equations were discovered in 1963 [1], an immense effort has been devoted to identifying and understanding chaotic dynamics. However, there are still many open basic problems even for three-dimensional continuous-time dynamical systems. For example, during the last two decades, a number of workers [4-16] have attempted to find three-dimensional chaotic systems that are functionally as simple as possible. The minimal functional form of three-dimensional chaotic vector fields is not known and opens a largely unexplored field. In 1994, an investigation identifying minimal chaotic systems was reported by Sprott [2]. Using a numerical search, he found 19 distinct chaotic models (labeled A to S) that are algebraically simpler than the Lorenz system [1] and the Rössler system [3]. Subsequently, Hoover [4] pointed out that the only conservative system of those models was an already known special case of the Nose-Hoover thermostat dynamical system, which exhibits time-reversible Hamiltonian chaos. In 1996, Gottlieb [5] pointed out that this system can be recast into an explicit third-order form $\dddot{x} = J(x, \dot{x}, \ddot{x})$, which he called a 'jerk function', and asked a provocative question: 'What is the simplest jerk function that gives chaos?' In response, Linz [6] reported that the jerky dynamics for the Lorenz [1] and Rössler [3] model possess functionally complicated forms, and are not suitable candidates for the Gottlieb's simplest jerk function, and Sprott [7, 8] described a variety of simple jerk cases including two having a total of three terms with two quadratic nonlinearities and a particular case having three terms with a single quadratic nonlinearity, which are simpler than any previously known. Following Linz's study [6], Eichhorn *et al.* [9] found that fifteen of Sprott's chaotic flows can be recast into a jerk form, and classified these fifteen models, the Rössler toroidal model [10], and Sprott's minimal chaotic flow [8] into seven quadratic polynomial classes of jerky dynamics (labeled JD1 to JD7), and then examined the simple cases of JD1 and JD2 in detail and identified the regions of parameter space over which they exhibit chaos [11]. In addition to the work on jerk functions with polynomial nonlinearity, some authors [12-14] studied piecewise-linear jerk functions. For example, Linz and Sprott made a search for the algebraically simplest dissipative chaotic flow with a absolute-value nonlinearity [12]. Using electrical circuits, Sprott [13, 14] studied the general case $\dddot{x} = -A\ddot{x} - \dot{x} + G(x)$, where $G(x)$ is an elemental piecewise function. At the same time, Malasoma [15] investigated a dissipative chaotic jerk flow with a $x\dot{x}^2$ nonlinearity, which is parity invariant. Very recently, Patidar and Sud [16] studied the same family of jerk dynamical system, but where the nonlinearity is cubic, quartic, and quintic, and concluded that increasing the non-linearity does not increase the range of parameter space over which chaos occurs.

In this paper, we will focus on the case in which the nonlinearity is of the form $|\dot{x}|^b$, which represents a class of simple piecewise exponential nonlinear jerk systems. The plan of the paper is as follows: In Section 2, we present the simple piecewise exponential nonlinear jerk model and its basic properties. In Section 3, we describe the dynamics and bifurcations of the jerk system and present the dynamics in the parameter space after rescaling the system. Finally, we summarize the results and indicate future directions.

2. The simple piecewise exponential nonlinear jerk model

Consider the so-called simplest dissipative chaotic flow studied by Sprott [7] for $b = 2$ in the generalized form

$$\dddot{x} + a\ddot{x} - |\dot{x}|^b + x = 0. \quad (1)$$

Here, a and b are bifurcation parameters. Equation (1) can be equivalently written as three first-order ordinary differential equations

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = -az + |y|^b - x. \quad (2)$$

This three-dimensional dynamical system has only five terms and a single nonlinearity. It is more general than the simplest dissipative chaotic flow which involves a quadratic nonlinearity. The attractor of system (1) is shown in Fig. 1 for $a = 2$ and b=1.5. For these parameters, the Lyapunov exponents are (0.0256, 0, −2.0256), and the Kaplan-Yorke dimension is D_{KY} = 2.0126.

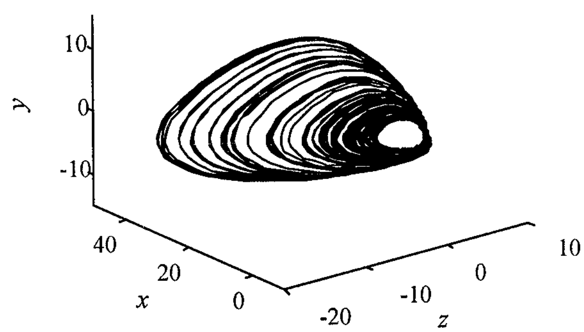


Fig. 1: Chaotic attractor for system (1) with a = 2 and b=1.5.

The system (1) has the following features:

(i) It is the simplest form of a jerk model with the fewest terms and a single nonlinearity.
(ii) For b = 2, it is the algebraically simplest dissipative chaotic flow studied in Ref. [7].
(iii) The chaotic range is increasingly narrower as b increases.
(iv) The attractor size increases exponentially when b approaches 1, and it is necessary to linearly rescale the variable x.
(v) It exhibits a period-doubling route to chaos and is approximately described by a one-dimensional quadratic map rescaling.
(vi) According to the topological definition by Vaněček and Čelikovský [17], the linearization of system (1) about the origin produces a 3×3 constant matrix of partial derivatives, $A = [a_{ij}]_{3\times 3}$, in which the sign of $a_{12}a_{21}$ distinguishes nonequivalent topologies. According to this criterion, $a_{12}a_{21} = 0$.

1) Dissipation and the existence of an attractor

The rate of volume contraction is given by the Lie derivative

$$\frac{1}{V}\frac{dV}{dt}=\sum_i \frac{\partial \dot{\phi}_i}{\partial \phi_i},\ i=1,2,3,$$
$$\phi_1 = x\ ,\ \phi_2 = y\ ,\ \phi_3 = z\ . \quad (3)$$

For dynamical system (2), we obtain

$$\frac{1}{V}\frac{dV}{dt}=\frac{\partial \dot{x}}{\partial x}+\frac{\partial \dot{y}}{\partial y}+\frac{\partial \dot{z}}{\partial z}=-a\ , \quad (4)$$

which can be solved to yield

$$V(t)=V(0)e^{-at}\ . \quad (5)$$

For $a>0$, the dynamical system (2) is dissipative with solutions for $t\to\infty$ that contract at an exponential rate a onto an attractor of zero volume that may be an equilibrium point, a limit cycle, or a strange attractor.

2) Equilibria and stability

The equilibria of Eq. (2) can be found by solving the three equations $\dot{x}=\dot{y}=\dot{z}=0$, which lead to $y=0$, $z=0$, and $-az+|y|^b-x=0$. Therefore, the only equilibrium $S_0(0,0,0)$ is at the origin.

Linearizing Eq. (2) about the equilibrium S_0 provides three eigenvalues, which are solutions of the characteristic equation

$$f(\lambda)=\lambda^3+a\lambda^2+1=0\ . \quad (6)$$

An elementary study proves that this polynomial has only one real root $\lambda_r<-2a/3$, which is therefore negative for $a > 0$. Since the characteristic equation is cubic with real coefficients, without loss of generality, we have $f(\lambda)=(\lambda-\lambda_r)(\lambda-\lambda_c)(\lambda-\bar{\lambda}_c)$ where λ_c is a complex number. After expanding the above equation and equating the coefficients with those of the original characteristic equation, we conclude that $\mathrm{Re}(\lambda_c)=2^{-1}\lambda_r^{-2}$ is positive, and thus the equilibrium is unstable. The negative real eigenvalue is associated with a one-dimensional stable manifold, whereas the complex conjugate eigenvalues, with positive real parts, are associated with a two-dimensional unstable manifold in which trajectories are spiraling outward. Thus the origin of the phase space is a saddle-focus with an instability index of 2.

3. Dynamical behaviors

3.1 Bifurcations and route to chaos

To study the dynamics of system (1), two cases were considered as follows:

1) Fix b = 1.58, and vary a. The system is calculated numerically with $a\in[2.05, 2.4]$ for an increment of Δa = 0.001. The bifurcation diagram is shown in Fig. 2(a). With an increase in the value of the parameter a, a reverse period-doubling is observed. The largest Lyapunov exponent of the system (1) was computed numerically for the same conditions and is shown in Fig.2(b). It is evident that chaos exists for $2.07\le a\le 2.14$ when $b=1.58$.

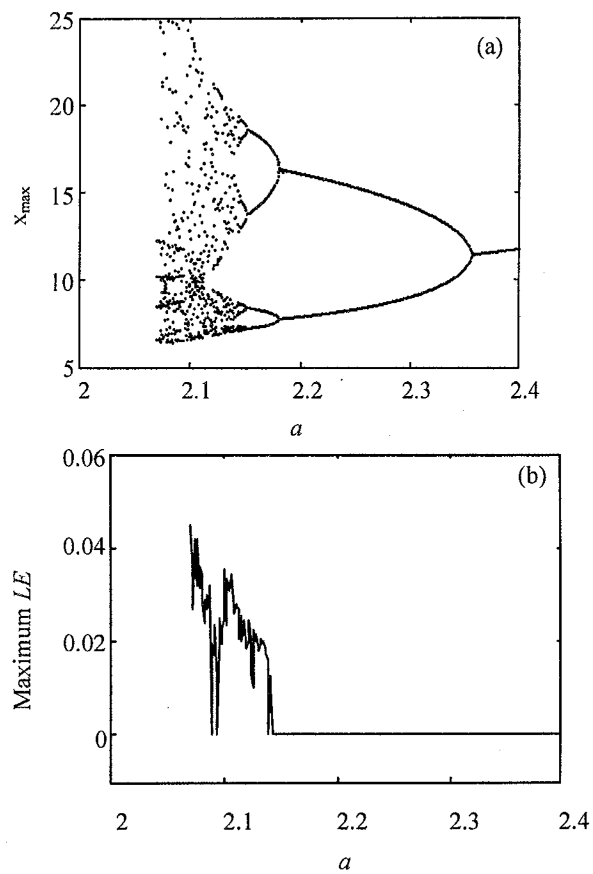


Fig. 2: Bifurcation and the largest Lyapunov exponent of system (1) with parameter a. (step size: 0.001, b = 1.58, initial conditions: (10.8058, -2.8207, -4.291)).

2) Fix a = 2.15, and vary b. The system is calculated numerically with $b \in [1.50, 1.75]$ for an increment of Δb = 0.001. The bifurcation diagram is shown in Fig. 3(a) and (b). With an increase in b, a period-doubling is observed. The largest Lyapunov exponent of the system (1) was computed numerically for the same range and step size with b and the result is shown in Fig.3(c). It is evident that chaos exists for $1.60 \le b \le 1.63$ with a periodic-6 window at $b \in (1.635, 1.645)$. This window is expanded with steps of $\Delta b = 0.00005$ in Fig. 3(d), showing a period-doubling route to chaos.

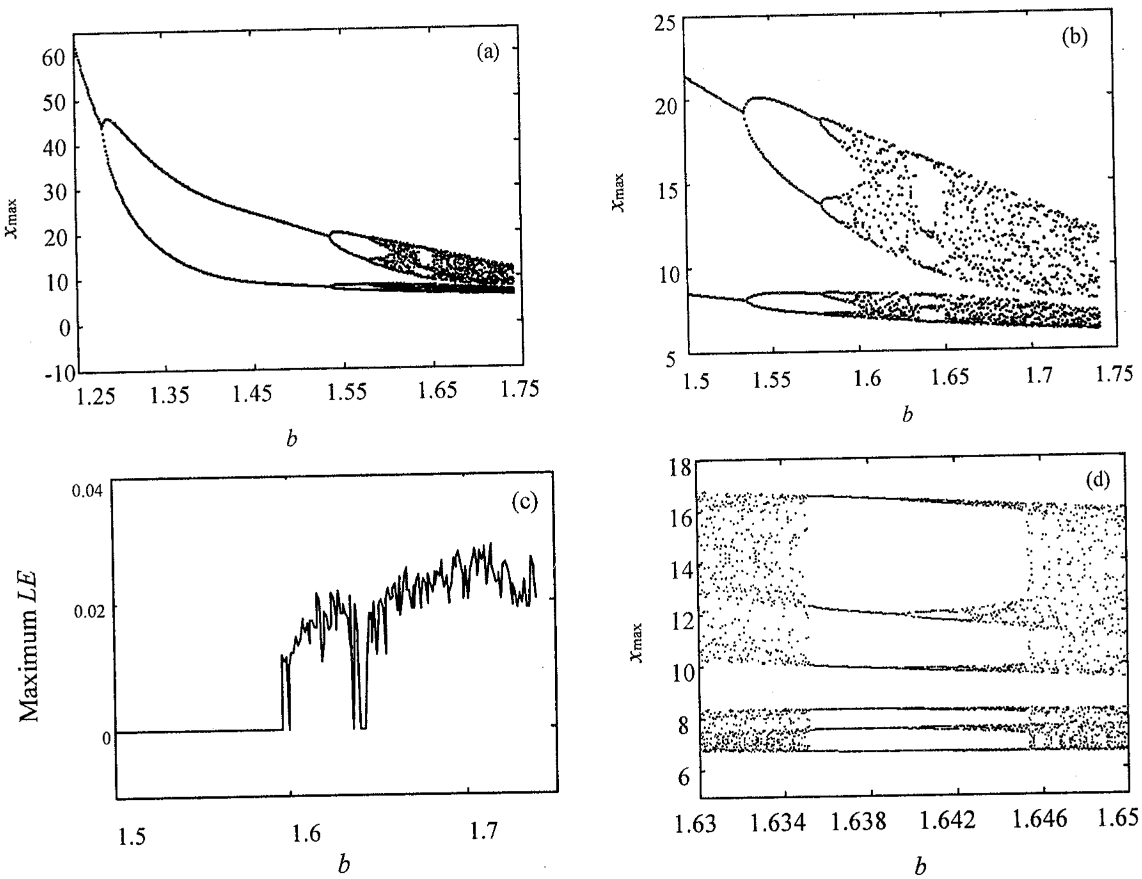


Fig. 3: Bifurcation diagrams of system (1) as a function of b. (step size: 0.001, a = 2.15, initial conditions: (-1.1706, 0.2984, 0.6057)) (a) $b \in (1.25, 1.75)$ (b) $b \in (1.5, 1.75)$ (c) the largest Lyapunov exponent of system (1) for $b \in (1.5, 1.75)$ (d) details of the periodic window at $b \in (1.63, 1.65)$.

From the two cases above, one concludes that selection of appropriate values for the system parameters can be used to suppress or generate chaos, and this system and most of the other cases found by Sprott [2, 7] share a common route to chaos, i.e. period-doubling bifurcations.

3.2 Rescaling the system

Since the attractor size increases exponentially, and the solution of Eq. (1) is periodic as b approaches 1 (see Fig. 3), we can rescale the system by assuming $x = A(1+\sin \omega t)$. Then $\dot{x} = A\omega \cos \omega t$, $\ddot{x} = -A\omega^2 \sin \omega t$, and $\dddot{x} = -A\omega^3 \cos \omega t$. Substituting these into Eq. (1) gives

$$-A\omega^3 \cos \omega t - aA\omega^2 \sin \omega t - |A\omega \cos \omega t|^b + A(1+\sin \omega t) = 0 \quad (7)$$

Equating $\sin \omega t$ terms above gives

$$\omega^2 = 1/a. \quad (8)$$

Averaging Eq. (7) over one cycle, we have

$$\langle |A\omega\cos\omega t|^b\rangle = \langle A(1+\sin\omega t)\rangle. \quad (9)$$

Then

$$(A\omega)^b \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos^b(\theta)d\theta = A, \quad (10)$$

$$\because \quad \lim_{b\to 1}\int_{-\pi/2}^{\pi/2}\cos^b(\theta)d\theta \approx 2. \quad (11)$$

Thus we have

$$A^b\omega^b \approx 0.5\pi A, \quad (12)$$

$$\omega \approx [0.5\pi A^{1-b}]^{\frac{1}{b}}. \quad (13)$$

From Eq. (8) and Eq. (13), we have

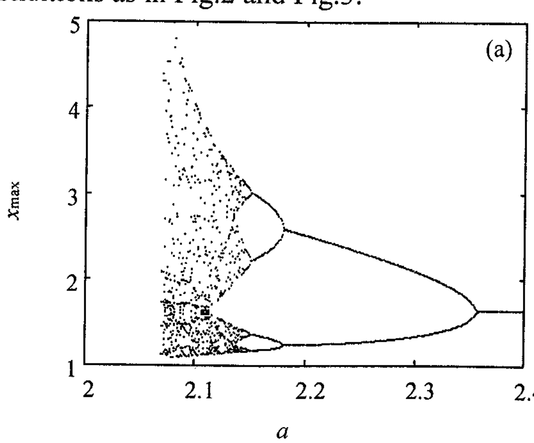
$$A = (0.5\pi a^{\frac{b}{2}})^{\frac{1}{b-1}}. \quad (14)$$

For example, if $a = 2.15$ and $b = 1.25$, then $A = 41.2644$ and $x_{max} = 82.5288$. This value agrees roughly with the result shown in Fig.3(a).

Let $b = 1+\varepsilon$ and $y = xe^{\frac{1}{1-b}} = xe^{-1/\varepsilon}$, then $x = ye^{1/\varepsilon}$. Substituting into Eq. (1) gives

$$\dddot{y} + a\ddot{y} - c|\dot{y}|^b + y = 0, \quad (15)$$

where $c = 0.5\pi a^{b/2}$. The bifurcation diagrams of the rescaled system are presented in Fig.4 with the same parameter values and initial conditions as in Fig.2 and Fig.3.



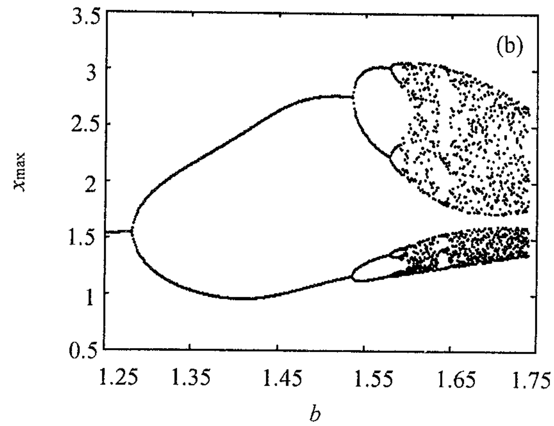


Fig. 4: Bifurcation diagrams of system (15) with different parameters. (a) $a \in (2.07, 2.4)$, $b = 1.58$, step size: 0.001, and IC: (10.8058, -2.8207, -4.291) (b) $b \in (1.25, 1.75)$, $a = 2.15$, step size: 0.001, IC: (-1.1706, 0.2984, 0.6057).

3.3 Range of chaos and the case of large nonlinearity

From the calculated results of the Lyapunov exponent spectrum for different values of the parameters a and b, we have characterized the regions of parameter space where chaos exists. The calculations were performed using a fourth-order Runge-Kutta integrator with a variable step size following the algorithm of Wolf *et al* [18]. For all the calculations, the parameters are slowly changed without altering the initial conditions. The results of the Lyapunov calculations for the region $1 \le a \le 3$ and $1 \le b \le 3$ with a resolution of 800×800 are shown in Fig.5. The meaning of different shades of colors are as follows: white shows unbounded solutions, i.e. non-existence of any stable attractor, black and grey represent the existence of chaotic strange attractors and limit cycles, respectively. The small horizontal black strip near $b = 1.9588$ is an island with bounded dynamics (limit cycles and strange attractors) located within the unbounded region. The bifurcation in this island is shown in Fig.6. A coexisting limit cycle and a chaotic attractor are also observed in this region.

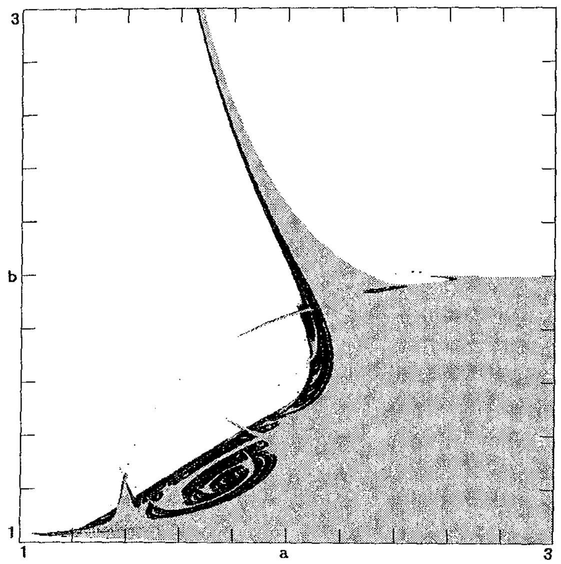


Fig. 5: Regions of parameter space with different dynamics (white = unbounded solutions, gray = limit cycles, black = chaos) for the simple jerk dynamical system (15).

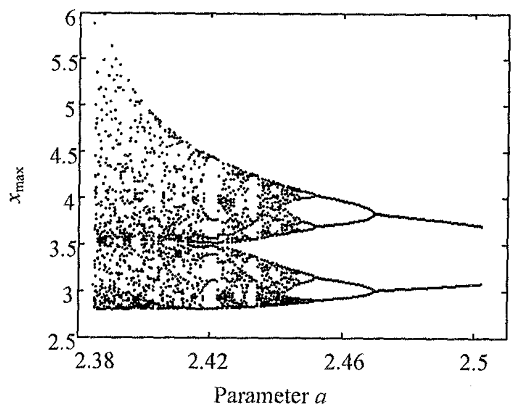


Fig. 6: Bifurcation diagram of the island at b = 1.9588, IC: (0.3687, 0.4164, 0.1267).

To understand further the dynamical behavior of system (15), we investigated the bifurcations at b =3 shown in Fig. 7(a) by plotting the return map for successive maxima of x as shown in Fig. 7(b). The similarity of Fig. 7(a) and Fig. 4 to a Feigenbaum diagram suggests that the sequence of local maxima should approximately follow a quadratic map. At low resolution, the return map appears to be one-dimensional and strongly resembles a parabola. The curve is inverted with respect to the usual logistic map, and a least squares fit to a quadratic polynomial gives

$$x_{n+1} = 14.3684x_n^2 - 49.8555x_n + 44.8524\,. \quad (16)$$

Using the linear transformation $x_n = -0.272y_n + 1.8709$, Eq. (16) can be transformed into the logistic map $y_{n+1} = \mu y_n(1-y_n)$ with $\mu = 3.9082$, which is well in the chaotic region and slightly below the crisis at $\mu = 4$. Thus it is hardly surprising that the bifurcations and route to chaos resemble those found in the logistic map.

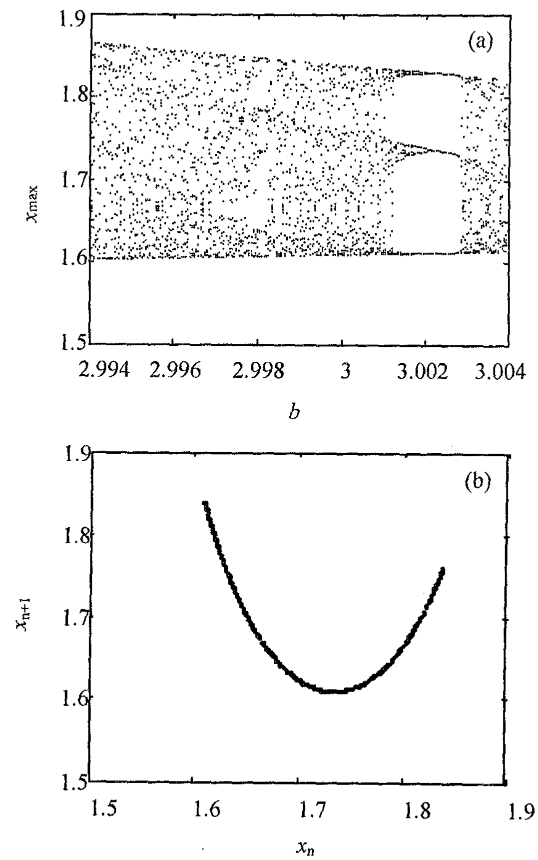


Fig. 7: (a) Bifurcations in the region of b = 2.994~3.004, a = 1.655, IC: (-0.02, -0.05, -0.06), step size: 0.0005. (b) Return map with each value of $x_{\max}$ plotted versus the previous value of $x_{\max}$ for a = 1.655, b = 3, IC: (-0.02, -0.05, -0.06).

As shown in Fig. 5, the range over which chaos exists becomes smaller as b increases. To predict the chaos in the limit of infinite b, we determined the minimum value of a for which chaos occurs at various values of b and initial conditions for each case as listed in Table 1. An exponential fit as shown in Fig.8 has the form

$$a = 0.0656e^{-0.0127b} + 0.8212. \quad (17)$$

Thus we conclude that in the limit of infinite b, where the nonlinearity approaches a square well (and the Lyapunov exponent is hard to calculate), chaos is localized to the vicinity of $a = 0.8212$. We also studied the case of b less than 1, and found it is unbounded for $0 < b < 1$.

Table 1. Parameters value for chaos at large b.

b	$a_{\min}$	LE	IC: (x_0, y_0, z_0)
110	0.837	0.0891	(0.4108, -0.5423, 0.4301)
120	0.835	0.0870	(0.8424, 0.5337, -0.3298)
130	0.834	0.0736	(-0.343, -0.2836, 0.3322)
140	0.832	0.0711	(-0.3505, -0.2717, 0.3338)
150	0.831	0.0810	(0.6479, 0.7369, -0.0789)
160	0.830	0.0782	(-0.3374, 0.2991, 0.4374)
170	0.829	0.0744	(-0.2385, 0.3998, 0.3941)
180	0.828	0.0879	(1.3462, 0.0056, -0.9828)
190	0.827	0.0859	(0.9898, -0.7871, -1.1484)
200	0.826	0.0939	(0.8688, 0.5438, -0.3436)

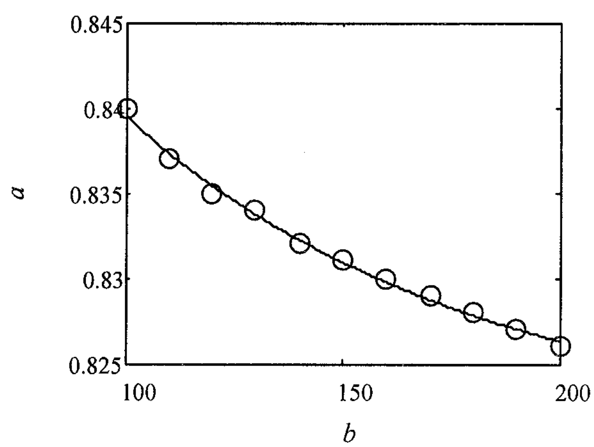


Fig. 8: Parameter values at which chaos occurs at large b, along with the exponential fit in Eq. (17).

4. Conclusions

In this paper, we have investigated the dynamics of a class of simple piecewise exponential jerk systems, which can exhibit chaotic behavior at some values of the system parameters. The region of chaos in the parameter space is recognized by using analytical methods, bifurcation diagrams, and extensive Lyapunov exponent calculations. Several results were obtained as follows: (1) The region of chaos in parameter space is narrow. (2) The size of the strange attractor expands exponentially as b approaches 1. (3) The system parameters can be used to suppress or generate chaos. (4) The route to chaos of the system is by period-doubling bifurcations, and it has the same behavior as in the logistic equation. (5) Increasing the nonlinearity does not generally lead to more chaos, and the chaotic region can be predicted by a fitting function. There are additional interesting features of this system in terms of control, synchronization, circuit implementation, and its application to secure communications that deserve further study.

Acknowledgments

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References

[1] Lorenz, E. N. Deterministic nonperiodic flow, J. Atmos. Sci. 20 (1963)130~141

[2] Sprott, J. C. Some simple chaotic flows, Phys. Rev. E 50(2)(1994)R647~R650

[3] Rössler, O. E. An equation for continuous chaos, Phys. Lett. A 57(1976)397~398

[4] Hoover, W. G. Remark on 'some simple chaotic flows', Phys. Rev. E 51(1)(1995)759~760

[5] Gottlieb, H. P. W. What is the simplest jerk function that gives chaos? Am. J. Phys. 64(1996)525

[6] Linz, S. J. Nonlinear dynamical models and jerky motion, Am. J. Phys. 65(1997) 523~526

[7] Sprott, J. C. Simplest dissipative chaotic flow, Phys. Lett. A 228(1997)271~274

[8] Sprott, J. C. Some simple chaotic jerk functions, Am, J. Phys. 65 (1997)537~543

[9] Eichhorn, R. Linz, S. J. and Hänggi, P. Transformations of nonlinear dynamical systems to jerky motion and its application to minimal chaotic flow, Phys. Rev. E 58 (1998)7151~7164

[10] Rössler, O. E. Continuous chaos—four prototype equations, Ann. (N.Y.) Acad. Sci. 316 (1979)376~392

[11] Eichhorn, R. Linz, S. J. and Hänggi, P. Simple polynormial classes of chaotic jerky dynamics, Chaos, Solitons & Fractals 13(2002)1~15

[12] Linz, S. J. and Sprott, J. C. Elementary chaotic flow, Phys. Lett. A 259 (1999)240~245

[13] Sprott, J. C. Simplest chaotic systems and circuits, Am. J. Phys. 68 (2000)758~763

[14] Sprott, J. C. A new class of chaotic circuit, Phys. Lett. A 266(2000)19~23

[15] Malasoma, J. M. What is the simplest dissipative chaotic jerk equation which is parity invariant? Phys. Lett. A 264 (2000)383~389

[16] Patidar, V. and Sud, K. K. Bifurcation and chaos in simple jerk dynamical systems, Pramana – Journal of physics, 64(1) (2005)75~93

[17] Vaněček, A. & Čelikovský, S. Control Systems: From Linear Analysis to Synthesis of Chaos (Prentice-Hall, London). (1996)

[18] Wolf, A. Swift, J. Swinney, H. and Vastano, J. Determining Lyapunov exponents from a time series, Physica D 16(1985)285~317