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Simplifications of the Lorenz Attractor

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Abstract: *The Lorenz attractor was once thought to be the mathematically simplest autonomous dissipative chaotic flow, but it is now known that it is only one member of a very large family of such systems, many of which are even simpler. Even the system originally proposed by Lorenz is not in its simplest possible form. This paper will describe a number of simplifications that can be made to the Lorenz system that preserve its dynamics as well as a number of chaotic systems that are much simpler and hence can serve as alternate models of chaos.*

Key Words: Lorenz, chaos, attractor, Lyapunov exponent

INTRODUCTION

For many years, Ed Lorenz thought he had discovered the mathematically simplest system of ordinary differential equations capable of producing chaos. His equations (Lorenz, 1963) became a paradigm of chaos, and the accompanying strange attractor (Fig. 1), which serendipitously resembles the wings of a butterfly, became an emblem for early chaos researchers. Lorenz did not set out to discover chaos, but rather he was attempting to find a system of equations whose solutions were more complicated than periodic. When he found such a system, the sensitive dependence on initial conditions (the "butterfly effect") came as a surprise, but he quickly realized its significance.

The equations he derived after reducing an original 12-dimensional system for modeling atmospheric convection (Saltzman, 1962) to three dimensions are

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz \end{aligned} \tag{1}$$

where the overdot denotes a time derivative ($\dot{x} = dx/dt$, etc.). Three variables (x, y, and z) are needed because the Poincaré-Bendixson theorem (Hirsch, et al., 2004) states that the most complicated dynamic that can occur with only two variables is periodic. The fact that there are three parameters, which Lorenz took

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as $\sigma = 10$, $r = 28$, and $b = 8/3$, is no coincidence. One can linearly re-scale the variables x, y, z, and t so that four of the seven terms on the right-hand side of Eq. (1) have coefficients of 1.0. The choice of where to put the remaining coefficients is arbitrary, but Lorenz chose them to represent the Prandtl number (σ), the Rayleigh number (r), and the aspect ratio of the convection cylinders (b). For the values used by Lorenz, the Lyapunov exponents (Sprott, 2003) are $\lambda = (0.9056, 0, -14.5723)$. This system has been widely studied, and there is a whole book devoted to it (Sparrow, 1982), but it is not widely known that it can be simplified in several ways.

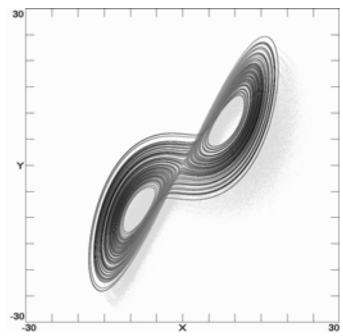

Fig. 1. The Lorenz attractor

A BETTER BUTTERFLY

A trivial simplification is to set one or more of the three parameters to 1.0. There are three ways to do this without destroying the chaos: $(r, \sigma, b) = (4, 16, 1)$, $(1, 16, 0.03)$, and $(1, -17, -1)$. The attractors for these three cases preserve the double-lobe structure of the Lorenz system as shown in Figs. 2 a-c, respectively. The Lyapunov exponents are given by $\lambda = (0.3359, 0, -6.3359)$, $(0.0531, 0, -2.0831)$, and $(0.0629, 0, -1.0629)$, respectively. Note that while all the figures in this paper are called “attractors” they are actually just a portion of a typical three-dimensional trajectory on the attractor projected onto a plane of two of the variables with the third variable displayed in shades of gray and with a subtle shadow to give an illusion of depth.

The Lorenz system is slightly inelegant in the sense that both the time scale and the attractor size depend on the parameters, each of which has dimensions of inverse time. Equation 1 can be put in dimensionless form by the linear transformation $x \to \sqrt{\sigma r}x$, $y \to \sqrt{\sigma r}y$, $z \to \sqrt{\sigma r}z + r$, and $t \to t/\sqrt{\sigma r}$, leading to the system

$$\begin{aligned}\dot{x} &= \alpha(y - x)\\ \dot{y} &= -xz - \gamma y\\ \dot{z} &= xy - \beta/\alpha - \beta z,\end{aligned} \tag{2}$$

where the new dimensionless parameters are given in terms of the old ones by $\alpha = \sqrt{\sigma / r}$, $\gamma = 1/\sqrt{\sigma r}$, and $\beta = b/\sqrt{\sigma r}$. While this system still has seven terms and two nonlinearities, the parameter space over which it is chaotic is now bounded, with its maximum Lyapunov exponent of λ = (0.0713, 0, −0.6493) at (α, γ, β) = (0.3, 0.028, 0.25) (Sprott, 2007). More significantly, Eq. 2 has chaotic solutions even for γ = 0 with values such as (α, γ, β) = (0.6, 0, 0.3) where the Lyapunov exponents are λ = (0.0645, 0, −0.9645) with an attractor as shown in Fig. 2d. Thus the Lorenz system can be reduced to one with only six terms and two parameters.

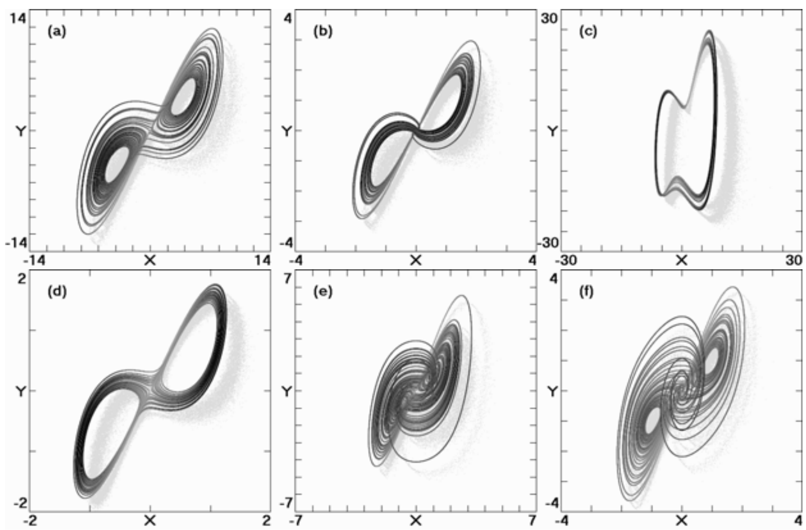

Fig. 2. Simplified variants of the Lorenz attractor

It turns out that one can do even better by transforming Eq. 1 as follows: $x \to \sigma x$, $y \to \sigma y$, $z \to \sigma z + r$, and $t \to t/\sigma$. Then take $r, \sigma \to \infty$ but in such a way that $R = br/\sigma^2$ remains finite, leading to the diffusionless Lorenz system (van der Schrier and Maas, 2000)

$$\begin{aligned} \dot{x} &= y - x \\ \dot{y} &= -xz \\ \dot{z} &= xy - R \end{aligned} \qquad (3)$$,

which is chaotic for a wide range of the single parameter R including R = 1 as shown in Fig. 2e with Lyapunov exponents λ = (0.2101, 0, −1.2101). Its attractor has the familiar two-lobe structure of the Lorenz system, but with a higher Kaplan-Yorke dimension (Kaplan & Yorke, 1979) of D_{KY} = 2.1736 in contrast to the case in Fig. 1 whose dimension is D_{KY} = 2.062. In fact, the system in Eq. 3 has its maximum dimension of D_{KY} = 2.2354 at R = 3.4693 (Sprott, 2007), which is significantly higher than any other variant of the Lorenz system. A similar two-lobe attractor as shown in Fig. 2f is obtained if the *xy* term in Eq. 3 is replaced by y^2. Both of these forms were first discovered in a systematic search for chaotic systems with only five terms and two quadratic nonlinearities (Sprott, 1994). They represent a significant simplification of the original Lorenz system since they have two fewer terms and a single parameter (R) with chaos for $R = 1$.

FOLDED BANDS

It is not clear how aware Lorenz was of these simplifications to his system, but he knew that chaotic systems existed with seven terms and a single quadratic nonlinearity (Lorenz, 1993, p. 148):

> One other study left me with mixed feelings. Otto Rössler of the University of Tübingen had formulated a system of three differential equations as a model of a chemical reaction. By this time, a number of systems of differential equations with chaotic solutions had been discovered, but I felt I still had the distinction of having found the simplest. Rössler changed things by coming along with an even simpler one. His record still stands.

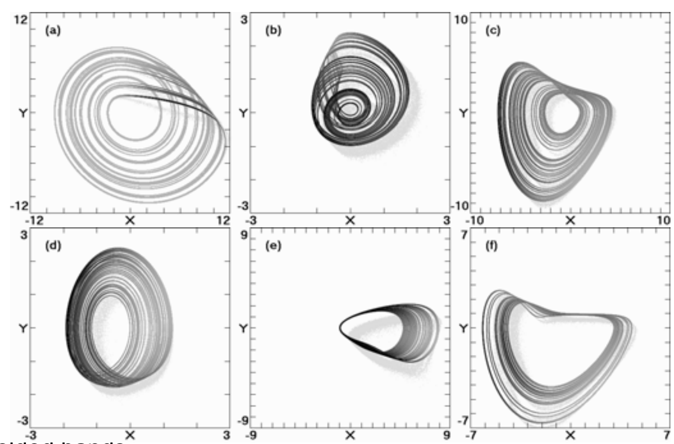

Fig. 3. Folded bands

The system to which Lorenz referred is (Rössler, 1976)

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned} \tag{4}$$

which is chaotic for $a = b = 0.2$ and $c = 5.7$ with an attractor as shown in Fig. 3a and with Lyapunov exponents $\lambda = (0.0714, 0, -5.3943)$.

What Lorenz did not realize was that Rössler himself had proposed a simpler system with only six terms, two parameters, and a single quadratic nonlinearity (Rössler, 1979):

$$\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x\\ \dot{z} &= a(y - y^2) - bx\end{aligned} \tag{5}$$

(called the prototype-4 system), which is chaotic for $a = b = 0.5$ with an attractor as shown in Fig. 3b and with Lyapunov exponents $\lambda = (0.0938, 0, -0.5938)$.

Systems such as Eq. 5 are not rare, and Sprott (1994) reported 14 additional examples that were not previously known, which prompted Gottlieb (1997) to ask "What is the simplest jerk function that gives chaos?" by which he meant a system of the form $\dddot{x} = J(x, \dot{x}, \ddot{x})$. The term "jerk" comes from the fact that in a mechanical system in which x is the displacement, successive time derivatives of x are velocity, acceleration, and jerk (Schot, 1978). The third derivative is the minimum necessary for chaos in an autonomous flow since it is always possible to write such a system in terms of three variables as

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= z\\ \dot{z} &= J(x, y, z)\end{aligned} \tag{6}$$

Linz (1997) showed that both the Lorenz system and the Rössler system could be written in jerk form, but the equations are complicated and inelegant. Eichhorn, Linz, and Hänggi, P. (1998) showed that all 14 of the systems discovered by Sprott (1994) as well as the Rössler prototype-4 system could be reduced to a hierarchy of seven quadratic jerk equations of increasing complexity, the simplest two of which are

$$\dddot{x} = -1.8\ddot{x} - 2x - x\dot{x} - 1 \tag{8}$$

and

$$\dddot{x} = -0.4\ddot{x} - 2.1\dot{x} + x^2 - 1 \tag{9}$$

where parameters have been chosen to produce chaos as shown by the attractors in Figs. 3c and d with Lyapunov exponents of $\lambda = (0.0647, 0, -1.8647)$ and $(0.0856, 0, -0.4856)$, respectively. These two systems have been studied by Eichhorn, Linz, and Hänggi, P. (2002).

Meanwhile Sprott (1997) showed that there is an even simpler jerk system of the form:

$$\dddot{x} = -a\ddot{x} + \dot{x}^2 - x, \tag{10}$$

which is chaotic for a small range of the single parameter a around 2.02 with an attractor as shown in Fig. 3e where the Lyapunov exponents are $\lambda = (0.0486, 0,$

−2.0686). Equation 10 can be cast into an equivalent form by differentiating each term with respect to time and making the substitution $\dot{x} \to x/2$ to get

$$\dddot{x} = -a\ddot{x} + x\dot{x} - x \tag{11}$$

whose attractor is shown in Fig. 3f and whose Lyapunov exponents are the same as for Eq. 10. Zhang and Heidel (1997) have rigorously proved that there can be no simpler chaotic flow.

All of these systems with a single quadratic nonlinearity have basically the same features – a single dominant frequency and a simple topology, usually called a “folded band,” but that might better be called a “twisted band.” In this sense, they do not resemble the double-lobe Lorenz attractor with its relatively broadband power spectrum. The reason is that the Lorenz system has three equilibrium points, whereas systems with a single quadratic nonlinearity can have only two. Thus they are not so much simplifications of the Lorenz attractor, but rather they are different systems worthy of study in their own right.

JERKY LORENZ-LIKE SYSTEMS

It is reasonable to ask whether there are simple jerk systems with different nonlinearities whose attractors more closely resemble the Lorenz attractor. One example with a cubic nonlinearity was discovered long ago by Moore and Spiegel (1966) as a model for the irregular variable luminosity of stars and is given in slightly simplified form by

$$\dddot{x} = -\ddot{x} + 9\dot{x} - x^2\dot{x} - 5x. \tag{12}$$

Its attractor is shown in Fig. 4a with Lyapunov exponents of λ = (0.0652, 0, −1.0652). It has only a single equilibrium point at the origin. Moore and Spiegel noted the aperiodic behavior of their solutions and the sensitive dependence on initial conditions but were apparently unaware of Lorenz’s similar work. Had they published a few years earlier and commented on the implications of their results to predictability, they might have become as famous as Lorenz.

Other similar chaotic systems but with three equilibrium points along the x-axis are given by

$$\dddot{x} = -\ddot{x} - 4\dot{x} - x^3 + 5x \tag{13}$$

$$\dddot{x} = -\ddot{x} - 5\dot{x} - x^5 + 4x \tag{14}$$

$$\dddot{x} = -\ddot{x} - 6\dot{x} - 2\tan x + 5x \tag{15}$$

$$\dddot{x} = -\ddot{x} - 2\dot{x} - \sinh x + 3x \tag{16}$$

$$\dddot{x} = -\ddot{x} - \dot{x} + 3\sin x - x \tag{17}$$

$$\dddot{x} = -\ddot{x} - \dot{x} + 6\arctan x - 2x \tag{18}$$

$$\dddot{x} = -\ddot{x} - \dot{x} + 7\tanh x - 2x \tag{19}$$

and

$$\dddot{x} = -\ddot{x} - \dot{x} + \operatorname{sgn} x - 2x \tag{20}$$

with attractors as shown in Figs. 4b-i and Lyapunov exponents of λ = (0.1722, 0, −1.1722), (0.2218, 0, −1.2218), (0.1925, 0, −1.1925), (0.1234, 0, −1.1234), (0.1083, 0, −1.1083), (0.0512, 0, −1.0512), (0.1138, 0, −1.1138), and (0.1766, 0, −1.1766), respectively. Note that the piecewise linear system in Eq. 20 is especially suited for electronic circuit implementation since the signum function can be simply implemented with a saturating operational amplifier (Sprott, 2000). What these systems have in common is that the last two terms sum to zero at two non-zero values of x, one positive and one negative, in addition to the zero at x = 0. The two non-zero equilibrium points are spiral saddles, just as with the Lorenz system. Hence they qualify as simplifications of the Lorenz attractor and provide potentially useful models of chaos in much the same way as the Lorenz system has in the past.

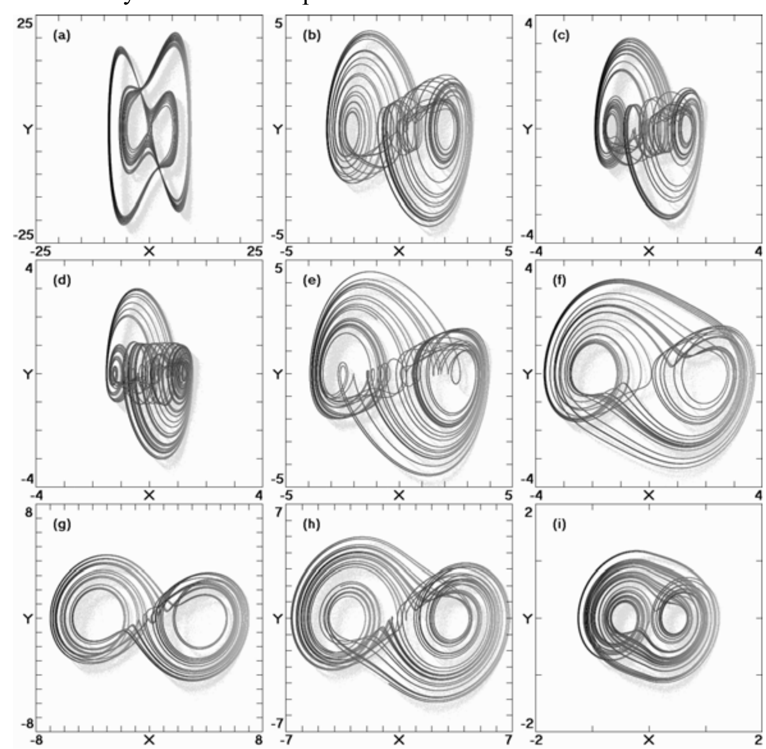

Fig. 4. Jerky Lorenz-like systems

SUMMARY AND CONCLUSIONS

While the Lorenz system is historically important and useful as a paradigm of chaos in continuous-time dissipative systems, it is by no means the simplest such example. Its double-lobe structure and chaotic behavior can be replicated in a number of other elegant systems containing fewer terms, fewer parameters, and a single nonlinearity. Lorenz would certainly have applauded these extensions of his seminal work, but he rightly deserves the credit for opening the door to such explorations and heralding the implications of chaos to the predictability of dynamical systems.

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