A Search for the Simplest Chaotic Partial Differential Equation

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Abstract

A search for chaos in partial differential equations concludes that the Kuramoto-Sivashinsky equation is likely the simplest one that permits chaos. All of the possible equations with one quadratic nonlinearity and no explicit time dependence that are "simpler" than the Kuramoto-Sivashinsky equation are tested, but none show signs of chaos. As the simplest chaotic partial differential equation, the Kuramoto-Sivashinsky equation bears insight into what essential elements are needed for chaos.

I. Introduction

What causes chaos in partial differential equations? One way to answer this question is to determine the minimum requirements for chaos to occur. By eliminating superfluous elements while still retaining the chaotic behavior, the elements that remain will be the ones essential for chaos. Thus more than mere aesthetics motivate the search for the simplest chaotic equations, for these uniquely simple equations can reveal insight into the causes of chaos.

While a search for the simplest chaotic ordinary differential equation has previously been done (6), such an endeavor has not been done for partial differential equations (PDE's), an area not well studied in general. Before embarking on this project, one relatively simple partial differential equation was known to be chaotic[1]:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - \frac{1}{R}\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4}, \tag{1.1}$$

where $u(x,t)$ is a scalar function and R is a real number. This PDE, known as the Kuramoto-Sivashinsky equation (1, 4, 5), was known to be chaotic for $R = 2$ (see Fig. 1).

[1] Saying an equation is chaotic is shorthand for saying there exist coefficients such that the solution to the equation is chaotic.

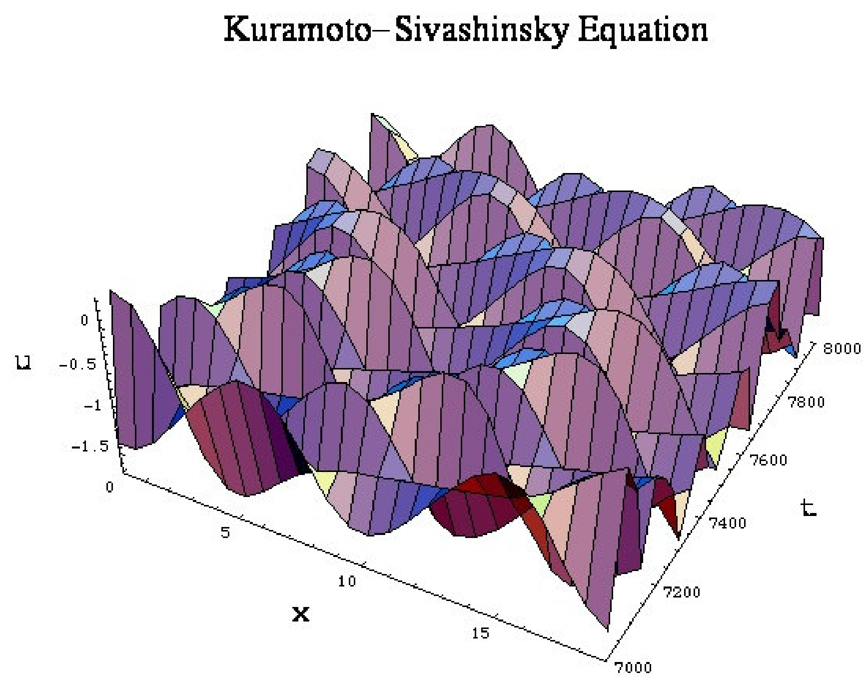

Figure 1. Chaotic behavior is apparent in this three-dimensional plot of the Kuramoto-Sivashinsky equation, solved here with R = 2, periodic boundary conditions, and one complete sine wave for the initial condition (i.e., $u(x,0) = \sin(\frac{2\pi}{19}x)$, and the size of the system is 19). Space is displayed horizontally, time into the page ($7000 \le t \le 8000$), and the state values vertically.

We decided to search in the space of equations that have the same form as the Kuramoto-Sivashinsky equation. Specifically, we considered partial differential equations with periodic boundary conditions that have the form

$$\frac{\partial u(x,t)}{\partial t} = F(u(x,t)), \tag{1.2}$$

where $F(u(x,t))$ can consist of derivatives in space but not in time, can contain a constant term, and must contain exactly one quadratic nonlinearity (e.g., u^2 or $u \cdot \partial^3 u / \partial x^3$ etc.). (In the future this search could be extended to other nonlinearities such as cubic ones, but for the sake of simplicity and time this paper considers only quadratic nonlinearities.)

The goal was to find equations that are somehow "simpler" than the Kuramoto-Sivashinsky equation yet still chaotic. But first, how does one define the "simplicity" of an equation? Since there is no universal, accepted definition, we created our own, which works as follows:

- Arrange the $\partial u / \partial t$ term on the left-hand side of the equation and all the other terms on the right-hand side;
- Sum the number of terms, the values of the powers (only those ≥ 2), and degrees of the derivatives (only those ≥ 1) on the right-hand side;
- The sum of those three quantities is the "complexity" of the equation.

The term $u \cdot \partial^3 u / \partial x^3$, for example, would add 4 to the complexity (1 for being a term, 3 for the third derivative), while $u^2 \cdot \partial^3 u / \partial x^3$ (which happens to be a cubic nonlinearity) would add 6 to the complexity (1 for being a term, 3 for the third derivative, 2 for the power of 2 on u). The Kuramoto-Sivashinsky equation has three terms, one first derivative, one second derivative, and one fourth derivative, so it has a complexity of $3+1+2+4=10$.

Enumerating all of the equations of the form in Equation (1.2) that have a complexity strictly less than 10 (the complexity of the Kuramoto-Sivashinsky equation) yields a total of 210 equations. After deciding to consider only linearly dissipative equations, or equations that have a friction-like element that slowly removes energy from the system (a term such as $-u$, $+u_{xx}$, $-u_{xxxx}$, $+u_{xxxxxx}$, etc.),[2] this number is reduced to 195 equations.

II. The Search Method

The standard test for chaos is the largest Lyapunov exponent (3), which measures the average exponential rate at which nearby initial conditions spread apart. If this Lyapunov exponent is positive, then small perturbations grow exponentially, predictability is lost, and the system is chaotic. By calculating the largest Lyapunov exponent of the Kuramoto-Sivashinsky equation for many values of R, we concluded that the PDE is most chaoic for R near 1.2 (see Fig. 2) The procedure used to numerically calculate the largest Lyapunov exponent was similar to the one outlined in (7, p. 116-117) with a perturbation of $\Delta \mathrm{R} = 10^{-6}$.

However, since this procedure was written for finite systems, it had to be modified slightly for partial differential equations, which are infinite systems. To calculate the Lyapunov exponent, one repeatedly perturbs the system and computes the difference between the perturbed and unperturbed trajectories. These operations are straightforward for finite systems such as maps, flows and ordinary differential equations because the state variables live in $\mathbf{R}^{\mathbf{k}}$. The states of these partial differential equations, however, are one-dimensional functions $u(x,t)$, with t fixed and x varying from 0 to the size of the system. Since we know how to calculate the Lyapunov exponent when the state variable is a number or vector, we convert these state functions $u(x,t)$ into vectors by collecting the values at each whole number position in space, and then we proceed to use the finite procedure for calculating the Lyapunov exponent.

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[2] Here we use the shorthand $u_{xx} = \partial^2 u / \partial x^2$, etc.

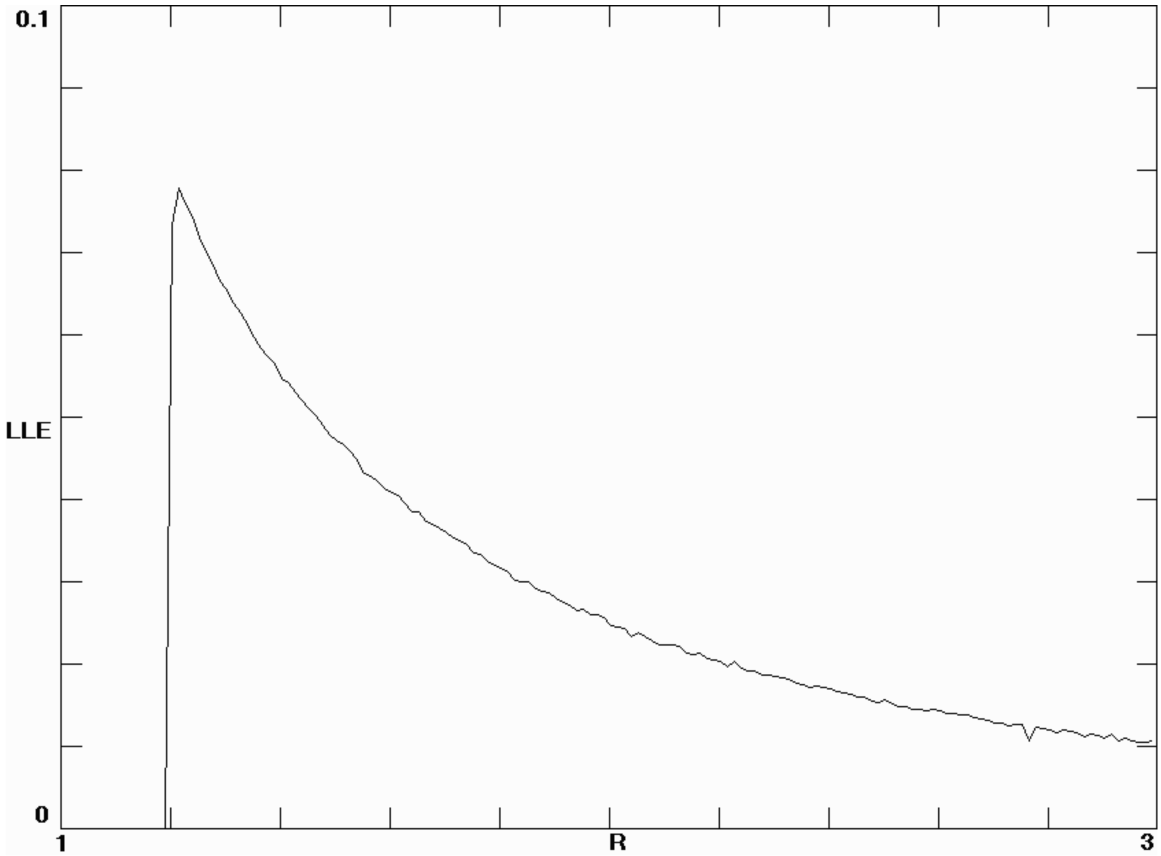

Figure 2. Plot of the largest Lyapunov exponent of the Kuramoto-Sivashinsky equation versus the parameter R. (Positive LLE indicates a chaotic solution. The greater the LLE, the more chaotic the solution is.) For $R < 1.2$ the numerical method becomes unstable.

The partial differential equations were solved using the built-in numerical differential equation solver in *Mathematica*, which uses the method of lines (2). Each equation was evaluated in the spatial direction from $x = 0$ to $x = L$, where L is a fixed prime number, with periodic boundary conditions and in the temporal direction out to absolute time $t = 8000$. This amount of time is long enough for convergent equations to sufficiently approach a fixed point and divergent equations to sufficiently approach infinity so that they may be easily discarded (for they are not chaotic), but short enough to make the computation fast. The calculation of the Lyapunov exponent ignored the first 7000 time units so that the system could reach its attractor, its end-state structure, rather than approaching it, which could give spurious results (transient chaos).

The search for chaotic equations among the 195 candidate equations works as follows. As it turns out, each equation has 2, 3, 4 or 5 terms. Each of those terms is multiplied by a coefficient, which could be any real number. However, with a suitable rescaling of the variables u and t, two of these coefficients can be replaced by ±1, a trick that significantly reduces the number of possible coefficients. Hence, for the equations that have just two terms, the two coefficients are both ±1. For the equations with three terms, the first two coefficients are both ±1 and the third can be any real number; this third coefficient is like a “knob” that we turn. For the equations with four terms, two coefficients are ±1 and two are “knobs.” For the sole equation with five terms (there are not more because all others have complexity greater than 10), two coefficients are ±1 and three are “knobs.”

The question becomes: what values do you try for the “knobs” – that is, the coefficients that can be any real number? The coefficients should be within a few orders of magnitude of each other; if not, then some terms will dominate others. Hence we chose to have each coefficient randomly sample hundreds of values in a uniform logarithmic distribution from 10^{-3} to 10^{2}. (For each coefficient we also tried its negative.)

The initial condition $u(x,0)$ can be varied, as well, for one does not have to always use a sine wave as in Figure 1. In theory, if an equation is chaotic and dissipative, there exists a “basin of attraction,” or region of initial conditions such that the equation is chaotic if and only if the initial condition is in the basin of attraction. Therefore, if one of the 195 candidate equations is chaotic, the computer search must sample that equation using an initial condition *in its basin of attraction*. One form of initial conditions we tried, for example, was

$$u(x,0) = \sin(\frac{2\pi}{LP}x) + V,$$

where L is the spatial length (we tried all primes between 2 and 29), P is the period (we tried 11 roughly evenly spaced values between 0.2 and 10), and V is the vertical offset (we tried 7 roughly evenly spaced values between -1 and 1).

III. Results

We ran this search on a 2 GHz dual core CPU for 10 months and tested more than 10^{8} equations, yet we found no chaotic solutions. To check that the search successfully detects chaos, we tested the Kuramoto-Sivashinsky equation (1.1) with spatial length $L = 19$, and 80 chaotic solutions were found. This strongly suggests that the computer search works, so it is likely that none of the 195 candidate equations are chaotic. Given that thousands of coefficients and initial conditions were tried for each equation (the number varies because the equations have different numbers of “knobs”), it is unlikely that the computer search missed a chaotic equation.

IV. Conclusion

The results of this computer search strongly suggest that the Kuramoto-Sivashinsky equation is the simplest chaotic PDE that contains a single quadratic nonlinearity. Therefore each term in the Kuramoto-Sivashinsky equation might indicate what at bare minimum is needed for chaos:

> “The $\partial^2 u / \partial x^2$ term is a negative viscosity leading to the growth of long wavelength modes, and the $\partial^4 u / \partial x^4$ term is a hyperviscosity that damps the short wavelength modes. The nonlinearity $u\partial u / \partial x$ transports energy from the growing modes to the damped modes.” (7, p. 409)

Since removing terms, reducing the powers, or reducing the derivatives of the Kuramoto-Sivashinsky equation eliminates the chaos, as found in the computer search, these three terms are somehow essential to chaos.

Directions for further study of this topic include expanding the search to equations containing *cubic* nonlinearities. There are 208 equations that have a cubic nonlinearity and a complexity less than that of the Kuramoto-Sivashinsky equation. This search may find better success since quadratic nonlinearities tend to push the system in one direction leading to divergence, whereas cubic nonlinearities can draw the system back toward equilibrium, allowing the system to be locally unstable but globally stable as required for chaos.

V. References

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