Nonlinear Prediction Filters for Loading-Unloading Dynamics as in Substorms and Sawtooth Events

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The Substorm Problem

- The solar wind driven magnetosphere-ionosphere is a driven-damped complex dynamical system.

- There is a great variety of observational waveforms in the substorm databases suggesting that the appropriate behavior is one of a chaotic system.

- There is global spatial coherence as emphasized by Baker *et al.* (1999) in the correlated measurements from

   (1) the ground-based auroral latitude chain of magnetometers whose wave forms are used to give the classic definition of a substorm event by the AL index.

   (2) the satellite measurements of particle distributions $\delta f_a(x, v, t)$ and electromagnetic fields $\delta \boldsymbol{E}$, $\delta \boldsymbol{B}$ in the geotail plasma that contains the large cross-tail current $I(t)$ flow providing the confinement of the high plasma pressure $p(t)$. The pressure balance is $\langle p \rangle = B_x^2/2\mu_0 = \frac{1}{2}\mu_0 \left(I(t)/L_x\right)^2$.

The WINDMI Substorm Model

Here we investigate two questions concerning the WINDMI description of the substorm dynamics.

Q1: Can the full $d = 6$ energy component system be reduced to the minimal order $d = 3$ system of a deterministic chaotic system while retaining the three regimes of quiet time, quasi-periodic substorms and chaotic dynamics?

- In making the reduction to $d = 3$ the faster evolving energy components are taken analytically to "track" the local fixed point or "equilibrium" values. In making this replacement, the energy and charge conserving features of the full ode system are lost during the short-lived transients.

Q2: What does the minimal model tell us about the intrinsic limits of predictability in the MI system?

Lyapunov Exponents and Lyapunov Fractal Dimension

- The largest Lyapunov exponent, designated as LE, determines the rate of divergence of neighboring trajectories. For a positive LE the reciprocal is the time for one e-folding in the distance between neighboring orbits.

  This divergence time sets the limit of predictability. For typical solar wind conditions we will show that the intrinsic time limit is of order three hours ($L_E^{-1} \sim 3\,\mathrm{hr}$) for moderately strong solar wind driving voltages.

- Determining all three Lyapunov exponents for the minimal model also allows us to determine the Lyapunov dimension (Kaplan and York, 1979) of the chaotic attractor for the system of $D_L \cong 2.1$. We compare this finding from a level of 20 kV ($V_{\mathrm{sw}} = 1$) to a maximum of approximately 200 kV ($V_{\mathrm{sw}} = 10$) with that given earlier for a simple analog model made up of a Rössler system driving a Lorenz system in Horton *et al.* (1999) and Goode *et al.* (2001).

The WINDMI Substorm Model Equations

The WINDMI dynamical model of the Earth's magnetosphere is a set of six first-order ordinary differential equations whose solutions are chaotic for values of the parameter vector $\mathbf{P}$ of interest. Smith *et al.* (2000) expressed these equations in dimensionless form as

$$\frac{dI}{dt} = a_1(V_{\text{sw}} - V) + a_2(V - V_i) \tag{1}$$

$$\frac{dV}{dt} = b_1(I - I_1) - b_2 P^{1/2} - b_3 V \tag{2}$$

$$\frac{dP}{dt} = V^2 - K_{\parallel}^{1/2} P\{1 + \tanh[d_1(I - 1)]\}/2 \tag{3}$$

$$\frac{dK_{\parallel}}{dt} = P^{1/2}V - K_{\parallel} \tag{4}$$

$$\frac{dI_1}{dt} = a_2(V_{\text{sw}} - V) + f_1(V - V_i) \tag{5}$$

$$\frac{dV_i}{dt} = g_1 I_1 - g_2 V_i - g_3 I_1^{1/2} V_i^{3/2} \tag{6}$$

MI System States and Parameters

There are 10 dimensionless parameters to define a parameter vector $\mathbf{P}$ that characterizes the global state of the magnetosphere-ionosphere system. A system that appears to conform closely to the known wave forms of the **classic internally-triggered Type I** substorms is defined by the following parameter values: $\mathbf{P}(S3) = [a_1 = 0.247, a_2 = 0.391, b_1 = 10.8, b_2 = 0.0752, b_3 = 1.06, d_1 = 2200, f_1 = 2.47, g_1 = 1080, g_2 = 4, g_3 = 3.79]$.

General properties of four states $S1, S2, S3, S4$ are discussed in Smith *et al.* (2000).

The bifurcation sequence for the $S3$ state as the solar wind voltage increases from $V_{\text{sw}} = 0$ to 10 ($\simeq 200\,\text{kV}$) is shown in Fig. 1.

Reduction to a Minimal 4D Dynamical Model

A reduced form of these equations results from setting the last two time derivatives to zero and solving for I_1 and V_i, giving

$$V_i = V + a_2(V_{\text{sw}} - V)/f_1 \tag{7}$$

$$I_1 = g_3^2 V_i^3/2g_1^2 + g_2 V_i/g_1 + \left(g_3 V_i^2/2g_1^2\right)\left(4g_1 g_2 + g_3^2 V_i^2\right)^{1/2}. \tag{8}$$

The full solution relaxes to these fixed points in the (nonlinear) $R_1 C_1$-time scale of $1/[g_2 + g_3(I_1 V_1)^{1/2}]$ which for the $S3$ state is short $\lesssim 10^{-1}$. Substituting Eqs. (7) and (8) into Eqs. (1)-(4) yields the first reduced system.

The reduced 4-dimension system was solved numerically, and the results are very similar to the full 6-dimensional case, shown in Fig. 1. Figure 2 shows the largest Lyapunov exponent (base-e) as a function of V_{sw}.

Reduction to a Minimal 3D Dynamo Model

From numerical experiments, it turns out that the solutions are not sensitive to a_2, g_2, and g_3.

Furthermore, Eq. (4) can apparently be eliminated as well, giving $K_\parallel = P^{1/2}V$, for which chaos still occurs. Finally, the cross-tail voltage V in Eq. (3) can be replaced with its equilibrium value of V_{sw}. Putting in all these simplifications gives the following 3-D system:

$$\frac{dI}{dt} = a_1(V_{sw} - V) \tag{9}$$

$$\frac{dV}{dt} = b_1 I - b_2 P^{1/2} - b_3 V \tag{10}$$

$$\frac{dP}{dt} = V_{sw}^2 - P^{5/4}V_{sw}^{1/2}\{1 + \tanh[d_1(I - 1)]\}/2. \tag{11}$$

A further simplification results from replacing the dimensionless pressure P in Eq. (11) with its equilibrium value, $P = (b_1/b_2 - b_3 V_{sw}/b_2)^2$.

Dimensionless Reduced System

Now define new variables,

$$x = d_1(I - 1) \tag{12}$$

$$y = a_1 d_1(V - V_{\text{sw}}) \tag{13}$$

$$z = a_1 b_2 d_1 P^{1/2} + a_1 d_1(b_3 V_{\text{sw}} - b_1) \tag{14}$$

in terms of which the 3-D system above can be written as

$$\frac{dx}{dt} = -y \tag{15}$$

$$\frac{dy}{dt} = c_1 x - b_3 y - z \tag{16}$$

$$\frac{dz}{dt} = -c_2 - c_3 \tanh(x) \tag{17}$$

where

$$c_1 = a_1 b_1,$$

$$c_2 = \frac{1}{4} a_1 d_1 (b_1 - b_3 V_{\text{sw}})^{3/2} (V_{\text{sw}}/b_2)^{1/2} - a_1 b_2^2 d_1 V_{\text{sw}}^2 / 2(b_1 - b_3 V_{\text{sw}}),$$

and

$$c_3 = \frac{1}{4} a_1 d_1 (b_1 - b_3 V_{\text{sw}})^{3/2} (V_{\text{sw}}/b_2)^{1/2}.$$

The condition that $b_3 V_{\text{sw}} < b_1$ is that there be a finite part of the cross-tail current driven by the MHD pressure gradient $j \times B = \nabla p$ condition at the point where I hits the critical current I_c.

Properties of the Minimal Model

The system has thus been reduced to one with 6 terms, 4 parameters, and a single nonlinearity. To verify that the approximations are reasonable, the largest Lyapunov exponent for this system is plotted versus V_{sw} in Fig. 3a. Figure 3b shows the Lyapunov dimension D_L and Fig. 3c the bifurcation diagram for the reduced system. The dimension in the chaotic regime is only slightly greater than 2.0 and is consistent with calculations of the correlation dimension.

From the fidelity of the reduced dynamics, we have answered Q1 and shown that the hyperbolic tangent is the important nonlinearity producing the chaos. The 3-D system has a fixed point at $x^* = -\tanh^{-1}(c_2/c_3)$, $y^* = 0$, $z^* = c_1 x^*$ with eigenvalues λ that satisfy

$$\lambda^3 + b_3\lambda^2 + c_1\lambda + c_3 - c_2^2/c_3 = 0.$$

A Hopf bifurcation occurs at $c_1 b_3 = c_3 - c_2^2/c_3$ followed by a period doubling route to chaos.

Lyapunov Exponential Fractal Dimension

For this system, the sum of the Lyapunov exponents is $-b_3$. From the calculated value for the largest exponent and the fact that one exponent must be zero, the entire spectrum can be obtained for the above parameters with $V_{\text{sw}} = 4.8$ as $(0.144, 0, -1.204)$. (The value $V_{\text{sw}} = 4.91$ used for Fig. 1 is in a periodic window.)

The Lyapunov dimension is

$$2 + 0.144/1.204 = 2.12$$

in rough agreement with the calculated correlation dimension of 2.13.

"Jerk" Systems

The simplification can be carried one step further by reducing the system to a "jerk" form (Sprott, 2000a):

$$\frac{d^3x}{dt^3} + b_3\frac{d^2x}{dt^2} + c_1\frac{dx}{dt} = -c_2 - c_3\tanh(x) \qquad (18)$$

for a third-order explicit ODE with a scalar variable. This is a special case of a damped harmonic oscillator driven by a nonlinear memory term, whose solutions have been studied and are known to be chaotic. The period of the dominant frequency of oscillation is $2\pi/c_1^{1/2} = 3.85$, corresponding to a real time of about 1 hour. In Eq. (18), t can be rescaled by $c_1^{1/2}$ to eliminate one of the four coefficients. The dynamical behavior is thus determined by only three parameters $\mathbf{P}_3 = (b_3, c_2, c_3)$. The regions of various dynamical behaviors are plotted in the $b_3 - V_{\text{sw}}$ plane in Fig. 4 with the other parameters as listed above.

Reduction to Alternate Minimal Model

For systems that have a strong unloading response above the critical threshold we can reduce (18) by

$$\frac{1}{2}\Big(1 + \tanh(x)\Big) \simeq e^{2x}$$

$$\frac{d^3x}{dt^3} + b_3 \frac{d^2x}{dt^2} + \frac{dx}{dt} = c_3 - c_2 - 2c_3\, e^{2x}$$

or

$$\boxed{y''' + by'' + y' = a - e^y}$$

The new canonical equation for loading-unloading problems.

Conclusions

1. A wide class of Physics-Engineering problems reduce to 3 ode's describing the storage and release of energy. Substorms and tearing modes are the two classic plasma physics examples.

2. We find that a detailed 6-energy component system of solar wind driven, M-I coupled substorm model can be reduced to a third-order system with two parameters

$$
\begin{aligned}
a &= \text{driving strength} \\
b &= \text{dissipation strength}
\end{aligned}
$$

   while preserving the **qualitative** features of the state-space or bifurcation diagram.

3. Many details of the energy transfers and the shorter time transients are lost in the reduction from 6 odes to 3-odes.

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