# Linear Social Interactions Models

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This talk is based on

## Identification of Social Interactions

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# Background

Harlem is a ruin–many of its ordinary aspects (its crimes, its casual violence, its crumbling buildings with littered areaways, ill-smelling halls and vermin infested rooms) are indistinguishable from the distorted images that appear in dreams, and which, like muggers in a lonely hall, quiver in the waking mind with hidden and threatening significance. Yet this is no dream but the reality of well over four hundred thousand Americans; a reality which for many defines and colors the world. Overcrowded and exploited politically and economically, Harlem is the scene and symbol of the Negro's perpetual alienation in the land of their birth.

Ralph Ellison, *Harlem in Nowhere* (1948)

Not only is the pathology of the ghetto self-perpetuating, but one kind of pathology breeds another. The child born into the ghetto is more likely to come into a world of broken homes and illegitimacy; and this family and social instability is conducive to delinquency, drug addiction, and criminal violence. Neither instability nor crime can be controlled by police vigilance or by reliance on the alleged deterring forces of legal punishment, for the individual crimes are to be understood more as symptoms of the contagious sickness of the community itself rather than as the result of inherent criminal or deliberate viciousness.

Kenneth B. Clark, *Dark Ghetto* (1965)

…changes have taken place in ghetto neighborhoods, and the groups that have been left behind are collectively different than those that lived in these neighborhoods in earlier years. It is true that long-term welfare families and street criminals are distinct groups, but they live and interact in the same depressed community and they are part of the population that has, with the exodus of the more stable working- and middle-class segments, become increasingly isolated socially from mainstream patterns and norms of behavior

William Julius Wilson, *The Truly Disadvantaged* (1993)

## Examples of Facts to Explain

## % of births out of wedlock

## (Child Trends 2005)

White: 25.4%

Black: 69.5%

Hispanic: 47.9%

Asian/Pacific Islander: 16.2%

## High School Smoking Rates by Ethnicity/Gender

## (CDC 2007)

White Males: 23.8%

White Females: 22.5%

Black Males: 14.9%

Black Females: 8.4%

Hispanic Males: 19.7%

Hispanic Females: 14.6%

## Social Interactions: Basic Ideas

1. Individual beliefs, preferences, and opportunities are conditioned by group memberships.

2. Memberships evolve in response to these interactions. Groups stratify along characteristics which affect outcomes.

3. Inequality and poverty result as family dynasties persistently face different interaction environments

This paper is about econometrics of this perspective.

## Outline

1. Linear in Means Models: Theory

2. Linear in Means Models: Identification

3. Social Networks Models

4. Concluding Comments

## Why Linear Models?

Most common class of models in the empirical social interactions literature.

Objective is to highlight identification properties under “ideal” error assumptions, i.e. no self-selection, unobserved group effects.

## 1. Linear in Means Model: Theory

Linear models of social interactions are used to study the joint behavior of individuals who are members of a common group $g$.

The population size of a group is denoted as $n_g$. Our objective is to probabilistically describe the individual choices of each $i$, $\omega_{i,g}$. We index the individual choices by $g$ to emphasize that they are group-specific.

## i. structure

The linear in means model can be derived simply as a Bayes-Nash equilibrium of a game in which each individual's choice is determined by a private benefit and a conformity benefit.

Not surprisingly, the utility functions are quadratic, and the conformity benefit is modeled as linearly decreasing in the quadratic deviation of an individual's choice from the average behavior of all other players. Group membership is exogenous. An individual's realized utility depends upon his own choice and the choices of others.

Utility takes the form

$$u_i\left(\omega_{i,g},\omega_{-i,g}\right)=\theta_{i,g}\omega_{i,g}-\frac{\omega_{i,g}^2}{2}-\frac{\varpi}{2}\left(\omega_{i,g}-\bar{\omega}_{-i,g}\right)^2$$

where $\bar{\omega}_{-i,g}=\dfrac{1}{n_g-1}\sum_{j\neq i}\omega_{j,g}$ is the average choice of the others in $g$. The individual marginal benefit $\theta_{i,g}$ is assumed to linearly decompose as

$$\theta_{i,g}=\chi_0+\chi_1 x_i+\chi_2 y_g+e_i$$

where $x_i$ and $e_i$ are observable and unobservable individual characteristics and $y_g$ is a vector of observable group characteristics.

The determination of individual choices is a game of incomplete information, since each individual, and only that individual, observes $e_i$.

Group characteristics unobservable to individual group members are irrelevant to choices as this model exhibits certainty equivalence in individual choices.

The $e_i$ elements are i.i.d. draws from a distribution on the real line $R$ with mean 0.

For expositional purposes it will be useful to write

$$\theta_{i,g} = \gamma_i + \gamma_g + e_i$$

where

$\gamma_g = \chi_0 + \chi_2 y_g$ is the internally (to the actors) observable group contribution to the marginal utility of $\omega_{i,g}$, and $\gamma_i = \chi_1 x_i$ is the externally (to the econometrician) observable contribution to marginal utility of an individual's characteristics.

## ii. existence of equilibrium

In a Bayes-Nash equilibrium, each individual maximizes expected utility, taking the expectation on $\bar{\omega}_{-i,g}$ with respect to his belief distribution, and all belief distributions will be correct. The first-order condition for individual $i$ is

$$\gamma_g + \gamma_i + e_i - \varpi\left(\omega_{i,g} - E\bar{\omega}_{-i,g}\right) - \omega_{i,g} = 0,$$

and so

$$\omega_{i,g} = \frac{1}{\varpi+1}\gamma_g + \frac{1}{\varpi+1}\gamma_i + \frac{\varpi}{\varpi+1}E\overline{\omega}_{-i,g} + \frac{1}{\varpi+1}e_i =$$

$$\frac{\chi_0}{\varpi+1} + \frac{\chi_1}{\varpi+1}x_i + \frac{\chi_2}{\varpi+1}y_g + \frac{\varpi}{\varpi+1}E\overline{\omega}_{-i,g} + \frac{1}{\varpi+1}e_i$$

Notice that the shock $\frac{1}{\varpi+1}e_i$, has a variance that is affected by the strength of the conformity parameter. In other words, the regression errors in the econometric model are proportional to private utility shocks.

We find an equilibrium by positing a functional form with undetermined coefficients, and then solving for the coefficients to make the beliefs correct. It will be convenient to define $\bar{\gamma}_{-i,g} = \frac{1}{n_g - 1}\sum_{j \neq i} \gamma_j$ to be the mean observable type component in the population. This is simply a sample mean. We suppose that for each individual $j$.

$$\omega_{j,g} = A\gamma_j + B\gamma_g + C\bar{\gamma}_{-j.g} + De_j + F.$$

We derive consistency of beliefs by assuming all individuals other than individual $i$ are choosing according to this functional form, computing the best response for individual $i$, seeing that it is of this linear form, and then solving for the coefficient values such that $A$ through $F$ are common through the entire population. We compute the best response simply by deriving an expression for $\bar{\omega}_{-i,g}$ by substitution. After some algebra one can show that the coefficients fulfill

$$A = \frac{1+\varpi A}{\varpi+1},\ B = \frac{1+\varpi C}{\varpi+1},\ C = \frac{\varpi(B+(n_g-2)C)}{\varpi+1},\ D = \frac{1}{\varpi+1},\ F = \frac{\varpi F}{\varpi+1}$$

Solving these equations gives the values of the undetermined coefficients.

$$\omega_{i,g} = \gamma_g + \frac{n_g - 1 + \varpi}{(\varpi + 1)n_g - 1}\gamma_i + \frac{\varpi(n_g - 1)}{(\varpi + 1)n_g - 1}\overline{\gamma}_{-i,g} + \frac{1}{\varpi + 1}e_i \qquad .$$

This is the exact description of the reduced form for individual choices in the Bayes-Nath equilibrium.

When the population size is large, this is approximately

$$\omega_{i,g} = \gamma_g + \frac{1}{\varpi + 1}\gamma_i + \frac{\varpi}{\varpi + 1}\bar{\gamma}_g + \frac{1}{\varpi + 1}e_i$$

where $\bar{\gamma}_g$ is the group-level average of $\gamma_i$.

Recalling the definitions of the $\gamma$ terms,

$$\omega_{i,g} = \chi_0 + \chi_2 y_g + \frac{\chi_1}{\varpi + 1} x_i + \frac{\varpi \chi_1}{\varpi + 1} \bar{x}_g + \frac{1}{\varpi + 1} e_i$$

where $\bar{x}_g$ is the group mean of the individual characteristics.

This is the reduced form for individual behavior in the linear in mean model.

**iii. uniqueness of Bayes-Nash equilibrium**

Strategies are maps

$$f:\left(\gamma_g,\gamma_i,\overline{\gamma}_{-i,g},e_i\right)\mapsto R.$$

The preceding section demonstrates the existence of a symmetric Bayes-Nash equilibrium with linear strategies. Discrete-choice models of social interaction are replete with multiple equilibria, so one might believe that multiple equilibria may arise here as well. This is not the case.

**Theorem. Uniqueness of equilibrium in the linear in means model**

The Bayes Nash equilibrium strategy for the linear in means model of social interactions model is unique.

Comment: no large population approximation was taken.

Comment: Theorem generalizes to more general networks (Blume, Brock, Durlauf, and Jayaraman (2011)

## 2. Linear in means models of social interactions: econometrics

$$\omega_{i,g} = k + cx_i + dy_g + Jm_{i,g}^e + \varepsilon_i .$$

where $m_{i,g}^e$ denotes the expected average behavior of others in the group, i.e. $m_{i,g}^e = \frac{1}{n_g} \sum_{j \in g} E\left(\omega_j \middle| F_i\right)$

Claims about social interactions are, from the econometric perspective, equivalent to statements about the values of $d$ and $J$.

The statement that social interactions matter is equivalent to the statement that at least some element of the union of the parameters in $d$ and $J$ is nonzero.

The statement that contextual social interactions are present means that at least one element of $d$ is nonzero.

The statement that endogenous social interactions matter means that $J$ is nonzero.

In Manski's original formulation, $y_g = \bar{x}_g$, where $\bar{x}_g = \frac{1}{n_g}\sum_{i \in g} x_i$ denotes the average across $i$ of $x_i$ within a given $g$, which explains the model's name.

$$\forall i,g \; E\left(\varepsilon_i \middle| \bar{x}_g, y_g, i \in g\right) = 0.$$

The inclusion of $i \in g$ means that we take the expectation conditional on membership in the group. This rules out any effect of selection on unobservables. Second we assume that errors are conditionally independent.

Under the Bayes Nash assumption

$$m_g^e = m_g \equiv \frac{k + c\bar{x}_g + dy_g}{1 - J}.$$

The equation says that the individuals' expectation of average behavior in the group equals the average behavior of the group, and this in turn depends linearly on the average of the individual determinants of behavior, $\bar{x}_g$, and the contextual interactions that the group members experience in common.

## Reduced form

Substitution to eliminate $m_g$ provides a reduced form version of the linear in means model in that the individual outcomes are determined entirely by observables and the individual-specific error:

$$\omega_{i,g} = \frac{k}{1-J} + cx_i + \frac{J}{1-J}c\bar{x}_g + \frac{d}{1-J}y_g + \varepsilon_i .$$

Much of the empirical literature has ignored the distinction between endogenous and contextual interactions, and has focused on this reduced form, i.e. focused on the regression

$$\omega_{i,g} = \pi_0 + \pi_1 x_i + \pi_2 y_g + \varepsilon_i .$$

where the parameters $\pi_0, \pi_1, \pi_2$ are taken as the objects of interest in the empirical exercise.  A comparison with the original linear in means model indicates how findings in the empirical literature that end with the reporting of $\pi_0, \pi_1, \pi_2$ inadequately address the task of fully characterizing the social interactions that are present in the data

## Instrumental variables and the reflection problem

It is obvious that if $\bar{\omega}_g$ is projected against the union of elements of a constant, $\bar{x}_g$ and $y_g$, this produces the population mean $m_g$, if this is the information set of each agent. Making this assumption, we can proceed as if $m_g$ is observable. Put differently, our identification arguments rely on the analogy principle which means that one works with population moments to construct identification arguments.

Hence one can study identification of the linear in means model from the vantage point of the standard rank and order conditions of simultaneous equations theory.

Since $y_g$ appears in the reduced form, it will not facilitate identification.

As we shall see, identification via instrumental variables is determined by the informational content of $\bar{x}_g$ relative to $y_g$.

As first recognized by Manski (1993), identification can fail for the linear in means model.

This may be most easily seen under Manski's original assumption that $y_g = \bar{x}_g$. This means that every contextual effect is the average of a corresponding individual characteristic. In this case,

$$m_g = \frac{k + (c + d) y_g}{1 - J}.$$

This means that This linear dependence means that identification fails: Manski (1993) named this failure the reflection problem.

## When Will Identification Hold?

Let $proj\left(a \middle| b,c\right)$ denote the linear projection of the scalar random variable $a$ onto the elements of the random vectors $b$ and $c$.

**Theorem. Identification of the linear in means model of social interactions.**

Using aggregate data, identification of the parameters $\left(k,c,J,d\right)$ requires $proj\left(\bar{\omega}_g \middle| 1, y_g, \bar{x}_g\right) - proj\left(\bar{\omega}_g \middle| 1, y_g\right) \neq 0$.

## What is an Example of an Individual-Level Variable Whose Group Level Analog Does Not Appear in the Linear in Means Model?

Heckman's selection correction!

Following Heckman's original (1979) reasoning, one can think of individuals choosing between groups $g = 1,...,G$ based on an overall individual-specific quality measure for each group:

$$I^*_{i,g} = \gamma_1 x_i + \gamma_2 y_g + \gamma_3 z_{i,g} + \nu_{i,g},$$

where $z_{i,g}$ denotes those observable characteristics that influence $i$'s evaluation of group $g$ but are not direct determinants of $\omega_i$ and $\nu_{i,g}$ denotes an unobservable individual-specific group quality term. Individual $i$ chooses the group with the highest $I^*_{i,g}$. We assume that $\forall i,g$, $E\left(\varepsilon_i \middle| x_i, y_g, z_{i,g}\right) = 0$ and $E\left(\nu_{i,g} \middle| x_i, y_g, z_{i,g}\right) = 0$.

From this vantage point, self selection implies

$$E\left(\varepsilon_i \middle| x_i, \bar{x}_1, y_1, z_{i,1}, \ldots., \bar{x}_G, y_G, z_{i,G}, i \in g\right) \neq 0$$

Notice that this expression includes the characteristics of all groups; this conditioning reflects the fact that the choice of group $g$ depends on characteristics of the groups that were not chosen in addition to the characteristics of the group that was chosen.

The linear in means model, under self-selection, should be written as

$$\omega_{i,g} = cx_i + dy_g + Jm_g + E\left(\varepsilon_i \left| x_i, \bar{x}_1, y_1, z_{i,1}, \ldots, \bar{x}_G, y_G, z_{i,G}, i \in g \right.\right) + \xi_i .$$

where by construction $E\left(\xi_i \left| x_i, \bar{x}_1, y_1, z_{i,1}, \ldots, \bar{x}_G, y_G, z_{i,G}, i \in g \right.\right) = 0$. Notice that the conditioning includes the characteristics of all groups in the choice set; this is natural since the characteristics of those groups not chosen are informative about the errors.

**This is a case where identification can hold under self-selection but fail under random assignment. Idea is being pursued in general network contexts.**

## 3. Social Networks Models

In defining the social interactions thus far we have presumed that interactions are generated by group-specific averages.

While the linear in means model is a good starting point for the study of social interactions, social networks allow for a much richer specification of social relations.

## Background: Graphs and Social Networks

Directed graphs consist of vertices (also known as nodes) and directed edges.

A directed edge is an ordered pair $(i, j)$ of vertices.

A directed graph is a pair $(V, E)$ where $V$ is the set of nodes, and has cardinality $n_V$, and $E$ is the set of edges.

A social network is a graph $(V, E)$ where $V$ is a set of individuals and the directed edges in $E$ signify social influence; $(i, j)$ is in $E$ if and only if $j$ influences $i$.

## Adjacency Matrix

A social network can be represented by an adjacency matrix $A$, also known as a sociomatrix in the mathematical sociology literature.

An adjacency matrix is an $n_V \times n_V$ matrix, with one row and one column for each individual in $V$.

For each pair of individuals $i$ and $j$, $a_{ij} = 1$ if there is an edge from $i$ to $j$, and 0 otherwise. Since the network is supposed to represent social connections, it is natural to assume that no $i$ is connected to himself.

## Undirected Graphs

While social influence can be a one-way relationship, we usually think of some relationships, for example friendship, as being bidirectional.

A bidirectional social network is represented by an undirected graph.

Edges are now undirected, and so there is a path from $i$ to $j$ if and only if there is a path from $j$ to $i$.

## Transitivity

Sociologists allege that social relations like friendship exhibit the property of homophily – loosely but accurately described by the phrase "the friend of my friend is my friend, too." This property is described by the prevalence of transitive triads.

Triads are connected subgraphs consisting of three nodes. Transitivity is the property that the existence of an edge from node $i$ to $j$ and an edge from $j$ to $k$ implies the existence of an edge from $i$ to $k$.

A graph is transitive if it contains no intransitive triads.

## The Linear in Means Model as a Special Case

The linear in means model is specified by assuming $A$ is symmetric, that edges are undirected, and that the graph is transitive.

If this is true, then the graph is the union of completely connected components. The nodes of the component containing $i$ constitute $i$'s group.

## Weighted Adjacency Matrices

The model can be enriched still further by allowing the elements of adjacency matrices to be arbitrary real numbers. In such models, the magnitude of the number $a_{ij}$ measures the degree of influence $j$ has on $i$ and the sign expresses whether that influence is positive or negative.

Throughout this section we will assume that all elements $a_{ij}$ are non-negative, except as noted, and that that the social network version of a contextual effect is a weighted average of the associated individual characteristic. This generalizes the contextual effects in the linear in means model case in which $y_g = \bar{x}_g$.

## Identification for general social networks models: basic results

For the general linear social networks model, individual outcomes are described by the behavioral equation system

$$\omega_i = k + cx_i + d\sum_{j\neq i} a_{ij}x_j + J\sum_j a_{ij}\omega_j + \varepsilon_i$$

with the error restriction

$$E\left(\varepsilon_i \middle| x_j \forall a_{ij} \neq 0\right) = 0$$

The current identification literature (with the exception of some results I will provide below) for networks assumes that $A$ is known *a priori* to the researcher.

This is a critical assumption in the existing literature which restricts empirical work to contexts in which survey data, for example, can be used to measure network structure.

Notice that the linear in means model is based on *a priori* knowledge of $A$.

The associated reduced form, described in vector notation, is

$$\omega = k\left(I - JA\right)^{-1}\iota + \left(I - JA\right)^{-1}\left(cI + dA\right)x + \left(I - JA\right)^{-1}\varepsilon .$$

where $I$ refers to the $n_V \times n_V$ identity matrix and $\iota$ is a $n_V \times 1$ vector of 1's. (Recall that $n_V$ is the number of individuals in the network.)

A fundamental algebraic result is the following theorem, due to Bramoullé, Djebbari and Fortin (2009).

**Theorem. Identification of social interactions in linear network models.**

For the linear social networks model, assume that $Jc + d \neq 0$ and that for all values of $J$, $(I - JA)^{-1}$ exists.

i. If the matrices $I$, $A$, and $A^2$ are linearly independent, then the parameters $k$, $c$, $d$ and $J$ are identified.

ii. If the matrices $I$, $A$, and $A^2$ are linearly dependent, $\sum_k a_{ik} = \sum_k a_{jk}$, and $A$ has no row in which all entries are 0, then parameters $k$, $c$, $d$ and $J$ are not identified.

The condition that $Jc + d \neq 0$ requires, in the network setting, that endogenous and contextual effects do not cancel out in the reduced form.

This theorem is an algebraic result. This is to say, it does not rely on the specific structure of $A$ which arises from its network context. It applies to any linear system of the which $|J| \cdot ||A|| < 1$ for all possible parameter values $J$.

An interesting feature of this result is that it does not rely on exclusion restrictions.

This actually should not be surprising. Although the number of equations in the system is $n_V$, the size of the population, there are only 4 parameters to estimate.

There are thus many cross-equation and within-equation linear equality constraints: The independence condition describes when these constraints mean that the reduced form coefficients fulfill appropriate rank and order conditions.

## Groups

The analysis of group interaction is the leading case in the econometric literature on networks. It is also appealing from the perspective of existing data sets such as the National Longitudinal Study of Adolescent Health (Add Health). Suppose that the peer relation is symmetric, $j \in P(i)$ if and only if $i \in P(j)$. Suppose too that the peer relation is transitive: If $j \in P(i)$ and $k \in P(j)$, then $k \in P(i)$.

In this case, the graph is the union of a finite number $G$ of completely connected components, that is, groups.

Suppose that component $g$ has $n_g$ members.

We will consider two ways to average over the group. These correspond to ways that Manski and Moffitt defined linear in means effects; we will see that the significance of the difference is overblown.

Exclusive averaging excludes $i$ from $P(i)$. In this case, for $i \in g$,

$$a_{ij} = \begin{cases} \dfrac{1}{n_g - 1} & \text{if } j \neq i \text{ and } j \in g, \\ 0 & \text{otherwise.} \end{cases}$$

Inclusive averaging includes $i$ in $P(i)$. In this case, for $i \in g$,

$$a_{ij} = \begin{cases} \dfrac{1}{n_g} & \text{for } j \in g, \\ 0 & \text{otherwise.} \end{cases}$$

With inclusive averaging, the social networks model is equivalent to our linear in means model, except that realized rather than expected outcomes affect individual outcomes. This difference is inessential since the instrumental variable projections used to replace the endogenous choices of others coincide with equilibrium formulations of beliefs.

Means and realizations, however, represent two distinct theoretical models.

The first is a network version of the incomplete-information.

The second is a complete-information version of the same game.

With exclusive averaging, the subject of Bramoullé *et al*., an additional distinction is that the calculation of group-level contextual effects does not include $i$'s own individual characteristics.

The following theorem, slightly different from Lee (2007) and BDF (2009) for groups immediately follows from the general algebraic theorem.

**Theorem. Identification of social interactions in group structures with different size groups**

i. Suppose that individuals act in groups, and that the $a_{ij}$ are given by exclusive averaging. Then the parameters $k$, $c$, $d$ and $J$ are identified if there are at least two groups of different sizes.

ii. Under inclusive averaging, the model is not identified.

The positive result is similar to Graham's (2008) variance contrast identification strategy, but its source is different. Graham assumes

$$\omega_{i,g} = J\bar{\omega}_{-i,g} + \varepsilon_{i,g}$$

and is based on $\mathrm{var}\left(\bar{\omega}_g\right)$.

Here identification follows the reduced form regression parameters rather than the second moments of the average group outcomes.

Note that in Graham’s case, $Jc + d = 0$ since $c = d = 0$, so his findings are allow identification when individual and contextual effects are absent.

## Back to the Reflection Problem

Why do these identification differ from those usually associated with the linear in means model?

The linear in means model is a large sample approximation.

Recall that the Bayes/Nash model produces

$$\omega_{i,g} = \gamma_g + \frac{n_g - 1 + \varpi}{(\varpi + 1)n_g - 1}\gamma_i + \frac{\varpi(n_g - 1)}{(\varpi + 1)n_g - 1}\overline{\gamma}_{-i,g} + \frac{1}{\varpi + 1}e_i \qquad .$$

not

$$\omega_{i,g} = \gamma_g + \frac{1}{\varpi + 1}\gamma_i + \frac{\varpi}{\varpi + 1}\overline{\gamma}_g + \frac{1}{\varpi + 1}e_i$$

which is the large sample approximation.

This is a case where identification is lost in taking the large sample approximation.

A range of identification results can be developed for different graph structures. The following Theorem is related to results in Bramoullé, Djebbari and Fortin (2009) although the exact theorem itself does not appear there.

**Theorem Nonidentification for weighted averaging implies network transitivity.**

Let $(V,E)$ be a network with weighted adjacency matrix $A$. Assume $Jc+d\neq 0$ and that for all values of $J$, $(I-JA)^{-1}$ exists.

i. If the parameters $k$, $c$, $d$ and $J$ are not identified, then $(V,E)$ is transitive.

ii. If the network is undirected, then $(V,E)$ is the union of groups.

## Is the Failure of Identification Common in Social Networks?

Suppose without loss of generality that the parameter $J$ takes values $[0,1)$ and denote by $S$ the set of all matrices $A$ such that $(I - JA)$ is invertible. If the matrices are $n_V \times n_V$, $S$ is a semi-algebraic set of full dimension in $R^{n_V^2}$.

For a given $R^K$, a semi-algebraic set is a subset of $R^K$ that is defined by a finite sequence of polynomial equalities or inequalities or the finite union of sequences. The set of $n_V \times n_V$ matrices such that $(I - JA)$ is invertible defines a semi-algebraic set contained in $R^{n_V^2}$.

## Theorem. Generic Identifiability of Social Networks Models

The set of all matrices $A \in S$ such that the powers $I$, $A$ and $A^2$ are linearly dependent, is a closed and lower-dimensional (semi-algebraic) subset of $S$.

By lower-dimensional we mean the following.

The set is the union of a finite number of disjoint (semi-algebraic) manifolds, and the highest dimension of these manifolds is less than $n_V^2$.

This theorem is a complement to the McManus (1992) result on the generic identifiability of nonlinear parametric models.

To repeat an earlier point, the social networks context, the key intuition for generic identifiability is that since $A$ is assumed to be known a priori, this knowledge is the equivalent of a large number of coefficient restrictions on the coefficients in the reduced form representation of individual behaviors.

## Unknown Network Structure

All results so far have taken the social network matrix $A$ as known.

This severely restricts the domain of applicability of existing identification results on social networks. We finish this section by considering how identification may proceed when this matrix is unknown.

In order to do this, we believe it is necessary to consider the full implications of the interpretation of linear social interactions models as simultaneous equations systems.

This is evident if one observes that the matrix form of the general social networks model may be written as

$$\left(I - JA\right)\omega = \left(cI + dA\right)x + \varepsilon .$$

where for expositional purposes, the constant term is ignored.

From this vantage point, it is evident that social networks models are special cases of the general linear simultaneous equations system

$$\Gamma \omega = \mathrm{B} x + \varepsilon .$$

Systems of this type, of course, are the focus of the classical identification in econometrics, epitomized in Franklin Fisher's treatise.

One can go further and observe that the assumption that the same network weights apply to both contextual and endogenous social interactions is not well motivated by theory.

From this vantage point it is evident that the distinction between $J$ and $A$ is of interest only when $A$ is known a priori, as is the case both for the linear in means model and the more general social networks framework.

Following the classical literature, one can then think of the presence or absence of identification in terms whether particular sets of restrictions on produce identification.

The simultaneous equations perspective makes clear that the existing results on identification in linear social networks models can be extended to much richer frameworks.

We consider two classes of models in which we interpret all agents $i = 1...n_v$ as arrayed on a circle.

We do this so that agents 1 and $n_V$ are immediate neighbors of one another, thereby allowing us to work with symmetric interaction structures.

First, assume that each agent only reacts to the average behaviors and characteristics of his two nearest neighbors, but is unaffected by anyone else.

In terms of the matrices $\Gamma$ and $B$, one way to model this is to assume that, preserving our earlier normalization, $\forall i$

$$\Gamma_{ii} = 1,\ \Gamma_{i,i-1 \bmod n_v} = \Gamma_{i,i+1 \bmod n_v} = \gamma_1,\ \Gamma_{ij} = 0 \text{ otherwise,}$$

$$B_{ii} = b_0,\ B_{i,i-1 \bmod n_v} = B_{i,i+1 \bmod n_v} = b_1,\ \text{and } B_{ij} = 0 \text{ otherwise.}$$

The model is identified since the nearest neighbor model may be interpreted via the original social networks model via restrictions on $A$.

For our purposes, what is of interest is that identification will still hold if one relaxes the symmetry assumptions so that

$\Gamma_{i,i-1\bmod n_v} = \gamma_{i,-1}$, $\Gamma_{i,i+1\bmod n_v} = \gamma_{i,1}$, $B_{ii} = b_{i,0}$, $B_{i,i-1\bmod n_v} = b_{i,-1}$, and $B_{i,i+1\bmod n_v} = b_{i,1}$.

If these coefficients are nonzero, then the matrices $\Gamma$ and $B$ fulfill the classical rank conditions and one does not need to invoke the social networks identification theorem at all.

Notice that it is not necessary for the interactions parameters to be the same across agents in different positions in the network.

Prior knowledge of $A$ takes the form of the classical exclusion restrictions of simultaneous equations theory. From the vantage point of the classical theory, there is no need to impose equal coefficients across interactions as is conventionally done.

This example may be extended as follows.

Suppose that one is not sure whether or not the social network structure involves connections between agents that are displaced by 2 on the circle, i.e. one wishes to relax the assumption that interactions between agents who are not nearest neighbors are 0.

In other words, we modify the example so that $\forall i$

$$\Gamma_{ii} = 1,\ \Gamma_{i,i-1 \bmod n_v} = \Gamma_{i,i+1 \bmod n_v} = \gamma_1,\ \Gamma_{i,i-2 \bmod n_v} = \gamma_{i,-2},\ \Gamma_{i,i+2 \bmod n_v} = \gamma_{i,2},$$

$$\Gamma_{ij} = 0 \text{ otherwise,}$$

$$B_{ii} = b_{i,0},\ B_{i,i-1 \bmod n_v} = b_{i,-1}, \text{ and } B_{i,i+1 \bmod n_v} = b_{i,1},\ B_{i,i-2 \bmod n_v} = b_{i,-2}, B_{i,i+2 \bmod n_v} = b_{i,2},$$

$$B_{ij} = 0 \text{ otherwise.}$$

If the nearest neighbor coefficients are nonzero, then the coefficients in this model are also identified via classical simultaneity results regardless of the values of the coefficients that link non-nearest neighbors. This is an example in which aspects of the network structure are testable, so that does not need to exactly know $A$ in advance in order to estimate social structure.

The intuition is straightforward, the presence of overlapping network structures between nearest neighbors renders the system overidentified: so that the presence of some other forms of social network structure can be evaluated relative to it.

This form of argument seems important as it suggests ways of uncovering social network structure when individual data are available, and again has yet to be explored.  Of course, not all social network structures are identified for the same reason that without restrictions, the general linear simultaneous equations model is unidentified. What our argument here suggests is that there is much to do in terms of uncovering classes of identified social networks models that are more general than those that have so far been studied.

For a second example, we consider a variation of the model studied by Bramoulle *et al*. which involves geometric weighting of all individuals according to their distance; as before we drop the constant term for expositional purposes.

Specifically, we consider a social networks model

$$\omega_i = k + cx_i + d\sum_{j \neq i} a_{ij}\left(\gamma\right)x_j + J\sum_{j} a_{ij}\left(\gamma\right)\omega_j + \varepsilon_i .$$

The idea is that the weights assigned to the behaviors of others are functions of an underlying parameter $\gamma$.

In vector form, the model is

$$\omega = cx + dA(\gamma)x + JA(\gamma)\omega + \varepsilon.$$

where

$$A(\gamma) = \begin{pmatrix} 0 & \gamma & \gamma^2 & \cdots & \gamma^k & \gamma^k & \gamma^{k-1} & \cdots & \gamma \\ \gamma & 0 & \gamma & \gamma^2 & \cdots & \gamma^k & \gamma^k & \cdots & \gamma^2 \\ & & & & \vdots & & & & \\ \gamma & \gamma^2 & & & \cdots & & & \gamma & 0 \end{pmatrix}$$

As before, $x$ is a scalar characteristic. The parameter space for this model is $\mathcal{P}=\{(c,d,J,\gamma)\in R^{2}\times R_{+}\times[0,1)\}$. The reduced form for this model is

$$\omega=\left(I-JA(\gamma)\right)^{-1}\left(cI+dA(\gamma)\right)x+\left(I-JA(\gamma)\right)^{-1}\varepsilon$$

Denote by $F:\mathcal{P}\rightarrow R^{n_V^2}$ the map

$$F(c,d,J,\gamma)=\left(I-JA(\gamma)\right)^{-1}\left(cI+dA(\gamma)\right)$$

The function $F$ characterizes the mapping of structural model parameters $(c,d,J,\gamma)$ to reduced form parameters.

We will establish what Franklin Fisher calls complete identifiability of the structural parameters from the regression coefficients for the reduced form.

By this he means that each reduced form parameter vector is derived from only a finite number of structural parameter vectors, i.e. that the map from structural parameters to reduced form parameters is finite-to-one.

**Theorem. Identification of the linear social networks model with weights exponentially declining in distance**

Suppose that the number of individuals $n_V$ is at least 4. Then for all $\left(c,d,J,\gamma\right)\in\mathcal{P}$,

i. if $\det I-JA\left(\gamma\right)\neq 0$, $c+d\neq 0$, $\gamma\neq 0$, then the cardinality of $F^{-1}\big(F\left(c,d,J,\gamma\right)\big)$ is no more than $2\left(n_V-1\right)$.

ii. The parameter values $J=d=0$ and $\gamma=0$ are observationally equivalent. In this case, $F\left(c,d,J,\gamma\right)=cI$.

## Outline of Proof

Let $M = F(c,d,J,\gamma)$. These are the population reduced form parameters for the model; our goal is to see how they map back to the structural parameters. We will prove the theorem for $n_V$ odd and equal to $2K+1$. The proof is similar for even $n_V$. By hypothesis, $I - JA(\gamma)$ is non-singular. Thus

$$M = \left(I - JA(\gamma)\right)^{-1}\left(cI + dA(\gamma)\right) =$$

$$\left(I - JA(\gamma)\right)^{-1}\left(cI + dI - dI + dA(\gamma)\right) =$$

$$(c + d)\left(I - JA(\gamma)\right)^{-1} - dI$$

This matrix equation maps observables to unobservables.

In view the definition of $A(\gamma)$,

$$\left(I - JA(\gamma)\right)_{11} = 1$$

and

$$-\frac{\left(I - A(\gamma)\right)_{12}}{\left(I - A(\gamma)\right)_{11}} = J\gamma, \quad \frac{\left(I - A(\gamma)\right)_{13}}{\left(I - A(\gamma)\right)_{12}} = \cdots = \frac{\left(I - A(\gamma)\right)_{1K+1}}{\left(I - A(\gamma)\right)_{1K}} = \gamma$$

Define

$$\mathcal{M} = \left\{ M : \text{for some } \left(c, d, J, \gamma\right) \in \mathcal{P}, F\left(c, d, J, \gamma\right) = M \right\}$$

$$\mathcal{M}_{dJ} = \left\{ M : \text{for some } \left(c, \gamma\right) \in R^{2} \times [0,1), F\left(c, d, J, \gamma\right) = M \right\}$$

These are, respectively, the sets of all possible reduced form matrices and those reduced forms consistent with a particular parameter pair *d, J.*

If $M \in \mathcal{M}_{dJ}$, then

$$\left(dI + M\right)_{11} \neq 0$$

and

$$-\frac{\left(dI + M\right)_{12}^{-1}}{\left(dI + M\right)_{11}^{-1}} = J\frac{\left(dI + M\right)_{13}^{-1}}{\left(dI + M\right)_{12}^{-1}}, \quad \frac{\left(dI + M\right)_{13}^{-1}}{\left(dI + M\right)_{12}^{-1}} = \cdots = \frac{\left(dI + M\right)_{1K+1}^{-1}}{\left(dI + M\right)_{1K}^{-1}}$$

This fact means that for a given reduced form matrix $M \in \mathcal{M}$, there are at most $2(n_V - 1)$ possible values of *d, J* pairs consistent with this equation.

Each of these *d* , *J* pairs is consistent with a unique $(c, \gamma)$, and this proves the theorem.

## Concluding Comments

1. Major outstanding question 1: what more be said when $A$ is unknown?

2. Major outstanding question 2: how does endogenous network formation affect identification?

BBDJ makes progress on these.

Much of what we have done is delineate parts of the assumptions/possibilities frontier for linear social interactions models.

Hopefully, we have made clear that there is still much to be done.